# Occasionally weakly compatible mappings and fixed points

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**Abstract**

In the present paper we show that contractive condition employed by Al-Thagafi and Shahzad in particular and contractive conditions in general, do not constitute a proper setting for studying common fixed points of occasionally weakly compatible mappings. Further, we improve the results of Al-Thagafi and Shahzad by employing a proper setting.

## 1 Introduction and preliminaries

In a recent work Al-Thagafi and Shahzad [2] introduced the notion of occasionally weakly compatible mappings and employed the new notion to prove fixed point theorems under contractive conditions. Two selfmappings $I$ and $T$ of a subset $D$ of a metric space $(X, d)$ are called occasionally weakly compatible (owc in short) if $Ix = Tx$ and $ITx = TIx$ for some $x \in D$. A point $x$ satisfying $Ix = Tx$ is called a coincidence point of $I$ and $T$. Thus, if $I$ and $T$ are owc mappings such that $Ix = Tx$, $ITx = TIx$ for some $x$ then both $x$ and $Ix(= Tx)$ are coincidence points of $I$ and $T$.
Now suppose that $I$ and $T$ satisfy a typical contractive condition. If $I$ and $T$ have a common fixed point, say $z$, then $z = Iz = Tz$, $ITz = TIz = z$ and $I$ and $T$ are, therefore, owc mappings. On the other hand, if $I$ and $T$ are owc mappings such that $Ix = Tx$ and $ITx = TIx$ for some $x$ then, since some contractive conditions

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exclude the existence of two coincidence points $x, y$ for $I$ and $T$ such that $Ix \neq Iy$, we get $Ix = IIx (= TIx)$. This means that $Ix = Tx$ is a common fixed point of $I$ and $T$. Therefore, under many contractive conditions existence of a common fixed point and occasional weak compatibility are equivalent conditions and, consequently, proving existence of fixed points by assuming owc is often equivalent to proving the existence of fixed points by assuming the existence of fixed points.

In view of this, proving fixed point theorems for owc mappings under contractive condition often reduces to a redundant exercise. We thus see that contractive conditions do not provide an ideal setting for the application of the concept of owc and for proper applications of the notion of owc one should look to mappings satisfying nonexpansive condition, Lipschitz type condition or some other general condition. Moreover, owc mappings can be divided in two categories:

1. Mappings commuting at all the coincidence points, and
2. Mappings commuting on a proper subset of the set of coincidences.

In the first case the mappings are obviously pointwise $R-$weakly commuting [6] or, equivalently, weakly compatible [4]. In the second case the mappings are noncompatible. Therefore, a proper setting for the application of owc should allow the existence of multiple fixed points or multiple coincidence points with distinct functional values and the classes of mappings that allow such possibility include:

(a) Noncompatible mappings satisfying nonexpansive or Lipschitz type condition,

(b) Weakly compatible mappings [4] satisfying nonexpansive or Lipschitz type condition and $(E.A.)$ property [1].

Before proceeding further, we recall some relevant concepts and results.

**Definition 1.1[3].** Two selfmaps $I$ and $T$ of a metric space $(X, d)$ are called compatible iff $\lim_n d(ITx_n, TIx_n) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_n Ix_n = \lim_n Tx_n = t$ for some $t$ in $X$.

It is clear from the above definition that $I$ and $T$ will be noncompatible if there exists a sequence $\{x_n\}$ in $X$ such that $\lim_n Ix_n = \lim_n Tx_n = t$ for some $t$ in $X$ but $\lim_n d(ITx_n, TIx_n)$ is either non-zero or non-existent.

**Definition 1.2[6].** Two selfmaps $I$ and $T$ of a metric space $(X, d)$ are called pointwise $R$-weakly commuting on $X$ if given $x$ in $X$ there exists $R > 0$ such that $d(ITx, TIx) \leq Rd(Ix, Tx)$.

**Definition 1.3[4].** A pair $(I, T)$ of selfmappings of a nonempty set $X$ is said to be weakly compatible if the mappings commute at their coincidence points, i.e., $Ix = Tx$ for some $x \in X$ implies $ITx = TIx$.

**Definition 1.4[6,7].** Two selfmappings $I$ and $T$ of a metric space $(X, d)$ are called reciprocally continuous iff $\lim_n ITx_n = It$ and $\lim_n TIx_n = Tt$ whenever $\{x_n\}$ is a sequence such that $\lim_n Ix_n = \lim_n Tx_n = t$ for some $t$ in $X$.

**Definition 1.5[1].** A pair $(I,T)$ of selfmappings of a metric space $(X,d)$ is said to satisfy the property (E.A.) if there exist a sequence $\{x_n\}$ in $X$ such that

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t, \quad \text{for some } t \in X.$$

Clearly, a pair of noncompatible (as well as nontrivial compatible) mappings satisfies the property (E.A.).

Let $I$ and $T$ be selfmaps of a subset $D$ of a metric space $(X,d)$. For every $x,y \in D$, define

$$\phi_{I,T}(x,y) := \max\{d(Ix,Iy), d(Ix,Tx), d(Iy,Ty), d(Ix,Ty), d(Iy,Tx)\}.$$

In a recent work Al-Thagafi and Shahzad [2] proved the following theorem:

**Theorem 1.1.** Let $D$ be a subset of a metric space $(X,d)$, $I$ and $T$ selfmaps of $D$, $\overline{T(D)} \subseteq I(D)$, and $\overline{T(D)}$ is complete. Suppose that $I$ and $T$ are occasionally weakly compatible and $d(Tx,Ty) \leq k\phi_{(I,T)}(x,y)$ for all $x,y \in D$ and some $k \in [0,1[$. Then $I$ and $T$ have a unique common fixed point.

We first observe that Theorem 1.1 does not require the lengthy proof given by Al-Thagafi and Shahzad [2] and it also does not require many of the conditions assumed in the theorem. In fact, it is to be observed that under the contractive condition

$$d(Tx,Ty) \leq k\phi_{(I,T)}(x,y) \qquad k \in [0,1[, \quad x,y \in D \tag{1.1}$$

of Theorem 1.1, assumption of owc and the existence of a unique common fixed point are equivalent conditions. To see this, first suppose that $I$ and $T$ have a unique common fixed point $z$ then, as already discussed above, $I$ and $T$ are owc. On the other hand, if $I$ and $T$ are owc mappings then there exists a $u$ in $X$ such that $Iu = Tu$ and $ITu = TIu (= TTu = IIu)$. Condition (1.1) now straight away implies that $Tu = TTu = ITu$ and $Tu$ is a unique common fixed point of $I$ and $T$. We thus see that under the contractive condition (1.1) of Theorem 1.1 assumption of owc and the existence of a unique common fixed point are equivalent conditions. The same will be true for many other contractive conditions, e.g,

$$d(Tx,Ty) \leq \varphi(\phi_{(I,T)}(x,y)); \quad x,y \in D, \tag{1.2}$$

where $\varphi : \Re_+ \to \Re_+$ is such that $\varphi(t) < t$ for each $t > 0$. This shows that contractive conditions do not provide a nontrivial setting for the application of owc. There can be two possible approaches to remedy the situation and improve the results of Al-Thagafi and Shahzad [2]:

(c) To replace the contractive condition by more general conditions that may hold for mappings satisfying contractive as well as nonexpansive and Lipschitz type conditions. We adopt this approach in the next theorem (Theorem 1.2) for improving Theorem 1.1 of Al-Thagafi and Shahzad [2].

(d) To introduce a condition that is weaker than owc but which can be used with contractive conditions also. In the present paper we introduce such a notion and demonstrate that the new notion is a necessary condition for the existence of common fixed points.

**Theorem 1.2.** Let $I$ and $T$ be occasionally weakly compatible selfmappings of a metric space $(X,d)$ satisfying

(i) $d(Tx,T^2x) \neq \max\{d(Ix,ITx),d(Ix,Tx),d(ITx,TTx),d(Ix,TTx),d(ITx,Tx)\}$, whenever $Tx \neq T^2x$.

Then I and T have a common fixed point.

*Proof.* Since $I$ and $T$ are owc, there exists a point $u$ in $X$ such that $Iu = Tu$ and $ITu = TIu$. This in turn yields $IIu = ITu = TIu = TTu$. If $Tu \neq T^2u$ then using (i) we get $d(Tu,T^2u) \neq \max\{d(Iu,ITu),d(Iu,Tu),d(ITu,TTu),d(Iu,TTu), d(ITu,Tu)\} = d(Iu,T^2u)$, a contradiction. Hence $Tu = TTu = ITu$ and $Tu$ is a common fixed point of $I$ and $T$.

As a particular case of the above theorem we get the following corollary:

**Corollary 1.1.** Let $I$ and $T$ be occasionally weakly compatible selfmappings of a metric space $(X,d)$ satisfying

(ii) $d(Tx,T^2x) < \max\{d(Ix,ITx),d(Ix,Tx),d(ITx,TTx),d(Ix,TTx),d(ITx,Tx)\}$, whenever $Tx \neq T^2x$.

Then I and T have a common fixed point.

We now give an example which illustrates the above theorem.

**Example 1.1.** Let $X = [0,1]$ and $d$ be the usual metric on $X$. Define self mappings $I$, and $T$ on $X$ as follows

$$I(x) = \frac{(1-x)}{3},$$

$$T(x) = \frac{(\sqrt{5-4(2x-1)^2}-1)}{4}.$$

Then $I$ and $T$ satisfy all the conditions of the above theorem and have two coincidence points $x = 1, x = 1/4$ and a common fixed point $x = 1/4$. It may be verified in this example that $I$ and $T$ are owc maps. $I$ and $T$ are owc since they commute at one of their coincidence points $x = 1/4$. It is also easy to verify that $I$ and $T$ satisfy the condition $d(Tx,T^2x) < \max\{d(Ix,ITx),d(Ix,Tx),d(ITx,TTx), d(Ix,TTx),d(ITx,Tx)\}$. Furthermore, $I$ and $T$ are noncompatible, let us consider the sequence $\left\{x_n = 1-\frac{1}{n}\right\}$. Then $\lim_{n\to\infty} Ix_n = 0$, $\lim_{n\to\infty} Tx_n = 0$, $\lim_{n\to\infty} ITx_n = \frac{1}{3}$, $\lim_{n\to\infty} TIx_n = 0$. Hence $I$ and $T$ are noncompatible.

Next, we introduce a new commutativity notion which is a proper generalization of nontrivial compatibility as well as owc.

**Definition 1.6.** Two selfmappings $I$ and $T$ of a metric space $(X,d)$ will be defined to be conditionally compatible iff whenever the set of sequences $\{x_n\}$ satisfying $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n$ is nonempty, there exists a sequence $\{y_n\}$ such that $\lim_{n\to\infty} Iy_n = \lim_{n\to\infty} Ty_n = t(say)$ and $\lim_{n\to\infty} d(ITy_n,TIy_n) = 0$.

If $I$ and $T$ are nontrivially compatible then they are obviously conditionally compatible but, as shown in Example 1.2, the converse is not true. Thus the new notion is a proper generalization of nontrivial compatibility.

**Example 1.2.** Let $X = [2, 20]$ and $d$ be the usual metric on $X$. Define self mappings $I$ and $T$ on $X$ as follows

$$Ix = 2 \quad \text{if } x = 2 \text{ or } x > 5, \quad Ix = 4 \quad \text{if } 2 < x \leq 5,$$
$$T2 = 2, Tx = 4 \quad , \text{if } 2 < x \leq 5, \quad Tx = \frac{(x+1)}{3} \quad \text{if } x > 5.$$

In this example $I$ and $T$ are conditionally compatible but not compatible. To see this let us consider the constant sequence $\{y_n = 2\}$ then $\lim_{n\to\infty} Iy_n = 2$, $\lim_{n\to\infty} Ty_n = 2, \lim_{n\to\infty} ITy_n = 2, \lim_{n\to\infty} TIy_n = 2$ and $\lim_{n\to\infty} d(ITy_n, TIy_n) = 0$. If we consider the sequence $\left\{x_n = 5 + \frac{1}{n}\right\}$ then $\lim_{n\to\infty} Ix_n = 2, \lim_{n\to\infty} Tx_n = (2 + \frac{1}{3n}) \to 2, \lim_{n\to\infty} ITx_n = 4, \lim_{n\to\infty} TIx_n = 2$ and $\lim_{n\to\infty} d(ITx_n, TIx_n) = 2$. Thus $I$ and $T$ are conditionally compatible but not compatible.

It is also relevant to mention here that if $I$ and $T$ are occasionally weakly compatible then they are obviously conditionally compatible, but, the converse is not true in general.

**Example 1.3.( Example 1.2** [5]**).** Let $X = [0, \infty)$ equipped with the usual metric $d$. Define selfmappings $I$ and $T$ on $X$ as follows

$$I(x) = x^2 \qquad \text{and} \qquad T(x) = \begin{cases} x + 2 & \text{if } x \in [0, 4] \cup (9, \infty), \\ x + 12 & \text{if } x \in (4, 9]. \end{cases}$$

In this example $I$ and $T$ are conditionally compatible but not owc (see Example 1.2 [5]).

It may be pointed that the notion of pointwise R-weak commutativity [6] and the equivalent notion of weak compatibility [4] imply commutativity at coincidence points but do not help in establishing the existence of coincidence points whereas the new notion is useful in establishing the existence of coincidence points.

**Theorem 1.3.** Let $I$ and $T$ be conditionally compatible selfmappings of a metric space $(X, d)$ satisfying

(iii) $d(x, Tx) \neq \max\{d(x, Ix), d(Tx, Ix)\}$, whenever the right-hand side is nonzero.

If $I$ and $T$ are noncompatible and reciprocally continuous then $I$ and $T$ have a common fixed point.

*Proof.* Since $I$ and $T$ are noncompatible, there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$ but either $\lim_{n\to\infty} d(ITx_n, TIx_n)$ is either non-zero or the limit does not exist. Also, since $I$ and $T$ are conditionally compatible and $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$, there exists a sequence $\{z_n\}$ in $X$ satisfying $\lim_{n\to\infty} Iz_n = \lim_{n\to\infty} Tz_n = u(say)$, such that $\lim_{n\to\infty} d(ITz_n, TIz_n) = 0$. Reciprocal continuity of $I$ and $T$ implies that $\lim_{n\to\infty} ITz_n = Iu$ and

$\lim_{n\to\infty} TIz_n = Tu$. The last three limits together imply $Iu = Tu$. If $u \neq Tu$ then using (iii) we get $d(u, Tu) \neq \max\{d(u, Iu), d(Iu, Tu)\} = d(u, Tu)$, a contradiction. Hence $u = Iu = Tu$ and $u$ is a common fixed point of $I$ and $T$.

Example 1.1 illustrates the above theorem also if $I$ and $T$ are replaced by their restrictions on $(0, 1)$ and $x \in (0, 1)$.

We now employ the notion of conditional compatibility to obtain a common fixed point theorem under the $\varphi-$contractive condition (1.2) which contains the condition (1.1) of Al-Thagafi and Shahzad [2] as a particular case.

**Theorem 1.4.** Let $I$ and $T$ be reciprocally continuous noncompatible selfmappings of a metric space $(X, d)$ satisfying

(iv) $d(Tx, Ty) \leq \varphi(\max\{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), d(Ix, Ty), d(Iy, Tx)\})$.

Where $\varphi : \Re_+ \to \Re_+$ is such that $\varphi(t) < t$ for each $t > 0$. If $I$ and $T$ are conditionally compatible then $I$ and $T$ have a unique common fixed point.

*Proof.* Since $I$ and $T$ are noncompatible, there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$ but either $\lim_{n\to\infty} d(ITx_n, TIx_n) \neq 0$ or the limit does not exist. Also, since $I$ and $T$ are conditionally compatible and $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$, there exists a sequence $\{z_n\}$ in $X$ satisfying $\lim_{n\to\infty} Iz_n = \lim_{n\to\infty} Tz_n = u(say)$, such that $\lim_{n\to\infty} d(ITz_n, TIz_n) = 0$. Reciprocal continuity of $I$ and $T$ implies that $\lim_{n\to\infty} ITz_n = Iu$ and $\lim_{n\to\infty} TIz_n = Tu$. The last three limits together imply $Iu = Tu$. If $u \neq Tu$ then using (iv) we get $d(Tz_n, Tu) \leq \varphi(\max\{d(Iz_n, Iu), d(Iz_n, Tz_n), d(Iu, Tu), d(Iz_n, Tu), d(Iu, Tz_n)\})$. On letting $n \to \infty$ we get $d(u, Tu) \leq \varphi((d(u, Tu))) < d(u, Tu)$, a contradiction. Hence $u = Iu = Tu$ and $u$ is a common fixed point of $I$ and $T$.
Uniqueness of the common fixed point theorem follows easily.

Theorem 1.4 can be generalized further if we use the property $(E.A.)$[1] instead of the notion of noncompatibility. We do so in our next theorem.

**Theorem 1.5.** Let $I$ and $T$ be reciprocally continuous conditionally compatible selfmappings of a metric space $(X, d)$ satisfying

(v) $d(Tx, Ty) \leq \varphi(\max\{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), d(Ix, Ty), d(Iy, Tx)\})$.

Where $\varphi : \Re_+ \to \Re_+$ is such that $\varphi(t) < t$ for each $t > 0$. If $I$ and $T$ are conditionally compatible and satisfy the $(E.A.)$ property, then $I$ and $T$ have a unique common fixed point.

*Proof.* Since $I$ and $T$ satisfy the property $(E.A.)$, there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} Ix_n \to t$ and $\lim_{n\to\infty} Tx_n \to t$ for some $t \in X$. Rest of the proof can be completed on the similar lines as has been done in Theorem 1.4.

**Remark 1.1.** Suppose $I$ and $T$ are selfmappings of a metric space $(X, d)$ having a common fixed point, say $z$ then $z = Iz = Tz$ and $TIz = ITz = Tz = Iz = z$. If we consider the constant sequence $\{x_n = z\}$ then $\lim_{n\to\infty} Ix_n = z$ $\lim_{n\to\infty} Tx_n = z$,$\lim_{n\to\infty} ITx_n = Iz = z$ , $\lim_{n\to\infty} TIx_n = Tz = z$ and $\lim_{n\to\infty} d(ITx_n, TIx_n) =$

$d(z,z) = 0$, i.e., $I$ and $T$ are conditionally compatible. This shows that existence of a common fixed point implies conditional compatibility, that is, conditional compatibility is a necessary condition for the existence of a common fixed point of given mappings $I$ and $T$.

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