Recursion for Poincaré polynomials of subspace arrangements

Géry Debongnie*

Abstract

A subspace arrangement $\mathcal{A}$ in $\mathbb{C}^m$ is a finite set $\{x_0, \ldots, x_n\}$ of vector subspaces. The complement space $M(\mathcal{A})$ is $\mathbb{C}^m \setminus \bigcup_{x \in \mathcal{A}} x$. When each subspace is an hyperplane, it is also known as an arrangement of hyperplanes. In that case, it is known that the Poincaré polynomials of $M(\mathcal{A})$ is connected to the Poincaré polynomials of the complements of the deleted arrangement $\mathcal{A}' = \mathcal{A} \setminus \{x_0\}$ and of the restricted arrangement $\mathcal{A}'' = \{x_0 \cap y \,|y \in \mathcal{A}'\}$ by the nice formula

$$\mathrm{Poin}(M(\mathcal{A}), t) = \mathrm{Poin}(M(\mathcal{A}'), t) + t\,\mathrm{Poin}(M(\mathcal{A}''), t).$$

In this paper, we prove that for a subspace arrangement, there is a long exact sequence in cohomology which connects $M(\mathcal{A})$ to $M(\mathcal{A}')$ and $M(\mathcal{A}'')$. Using it, we can extend the above formula to arrangements with a geometric lattice, and to some other specific arrangements.

1 Introduction

A subspace arrangement $\mathcal{A}$ in $\mathbb{C}^m$ is a finite set $\{x_0, \ldots, x_n\}$ of vector subspaces. The complement space $M(\mathcal{A})$ is $\mathbb{C}^m \setminus \bigcup_{x \in \mathcal{A}} x$. In general, this is a complicated space. An interesting way to understand this space is to consider the deleted arrangement $\mathcal{A}' = \mathcal{A} \setminus \{x_0\}$ and the restricted arrangement $\mathcal{A}'' = \{x_0 \cap y \,|y \in$

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$\mathcal{A}'\}$. Sometimes, the topology of the space $M(\mathcal{A})$ can be linked to the topology of $M(\mathcal{A}')$ and $M(\mathcal{A}')$.

For example, when $\mathcal{A}$ is an arrangement of hyperplanes, we have the formula (see [3])

$$\operatorname{Poin}(M(\mathcal{A}),t) = \operatorname{Poin}(M(\mathcal{A}'),t) + t\operatorname{Poin}(M(\mathcal{A}''),t).$$

This formula inductively turns the computation of the Betti numbers of $M(\mathcal{A})$ into a simple exercise. In this paper, we shall generalize this formula to some larger classes of arrangements. For those arrangements (which includes the arrangements of hyperplanes), we have :

$$\operatorname{Poin}(M(\mathcal{A}),t) = \operatorname{Poin}(M(\mathcal{A}'),t) + t^{2\operatorname{codim} x_0 - 1}\operatorname{Poin}(M(\mathcal{A}''),t).$$

To obtain this relation, an important step is to prove that there always exists a long exact sequence

$$\begin{aligned}\cdots \to H^q(M(\mathcal{A}'),\mathbb{Q}) \to H^q(M(\mathcal{A}),\mathbb{Q}) \to H^{q-\deg(x_0)}(M(\mathcal{A}''),\mathbb{Q})\\ \to H^{q+1}(M(\mathcal{A}'),\mathbb{Q}) \to H^{q+1}(M(\mathcal{A}),\mathbb{Q}) \to \cdots\end{aligned}$$

where $\deg(x_0) = 2\operatorname{codim} x_0 - 1$.

Our computations are based on a rational model $D(\mathcal{A})$ for the complement space. We describe it in section 2.

The third section is the most technical. Its aim is to prove the existence of the long exact sequence mentioned above. The idea is to consider the injection $j\colon D(\mathcal{A}') \to D(\mathcal{A})$, and to prove that the quotient $\frac{D(\mathcal{A})}{D(\mathcal{A}')}$ is quasi-isomorphic to $D(\mathcal{A}'')$ up to a shift of degree. The long exact sequence follows directly. It also follows that the Euler characteristic of $M(\mathcal{A})$ is always zero (assuming that $\mathcal{A}$ is not empty).

Finally, the last section is about Poincaré polynomials. The two main results, theorems 4.3 and 4.9, state that if $\mathcal{A}$ has a geometric lattice, or if x_0 is a separator (see section 4 for a definition), then

$$\operatorname{Poin}(M(\mathcal{A}),t) = \operatorname{Poin}(M(\mathcal{A}'),t) + t^{\deg x_0}\operatorname{Poin}(M(\mathcal{A}''),t).$$

Note that this is not true in general. A counterexample is given in section 4.

2 Rational model

To each subspace arrangement $\mathcal{A} = \{x_1,\ldots,x_n\}$ in $\mathbb{C}^m$, we associate a rational model $D(\mathcal{A})$ that is a commutative differential graded algebra. This model has been described by Feichtner and Yuzvinsky in [4].

First, let us establish an useful notation : the notation $\{x_{i_1},\ldots,x_{i_r}\}_<$ simply means that $i_1 < \cdots < i_r$.

Now, let us define the model $D(\mathcal{A})$. As a vector space, $D(\mathcal{A}) = \oplus_{\sigma\subseteq\mathcal{A}}\mathbb{Q}[\sigma]$ is the 2^n-dimensional rational vector space generated by the subsets of $\mathcal{A}$. A grading is given by

$$\deg(\sigma) = 2\operatorname{codim}_{\mathbb{C}} \cap\sigma - |\sigma|,$$

where $\cap\sigma$ is the intersection of each subspace of σ and $|\sigma|$ is the number of elements of σ. The multiplication is simply defined by

$$\sigma \cdot \tau = \begin{cases} (-1)^{\operatorname{sgn}\epsilon(\sigma,\tau)}\sigma \cup \tau & \text{if } \deg\sigma + \deg\tau = \deg(\sigma\cup\tau), \\ 0 & \text{otherwise,} \end{cases}$$

where $\sigma, \tau \subseteq \mathcal{A}$ and $\epsilon(\sigma,\tau)$ is the permutation that, applied to $\sigma \cup \tau$ (with the obvious linear order $x_1 < \cdots < x_n$), places elements of τ after elements of σ.

Finally, let us define a differential $d\colon D(\mathcal{A})^\star \to D(\mathcal{A})^{\star+1}$. For $\sigma = \{x_{i_1}, \ldots, x_{i_r}\}_<$, let $J_\sigma = \{j \in [r] \,|\, x_{i_j} \in \sigma \text{ and } \cap(\sigma \setminus \{x_{i_j}\}) = \cap\sigma\}$. Then the differential is defined by the formula

$$d\sigma = \sum_{j \in J_\sigma} (-1)^j (\sigma \setminus \{x_{i_j}\}).$$

3 Deleted and restricted arrangements

3.1 Technical lemmas

Let $\mathcal{A} = \{x_0, x_1, \ldots, x_n\}$ be a subspace arrangement in $\mathbb{C}^m$. Consider the deleted arrangement $\mathcal{A}' = \mathcal{A} \setminus \{x_0\}$ and the restricted arrangement $\mathcal{A}'' = \{x_0 \cap y \mid y \in \mathcal{A}'\}$, viewed as an arrangement in $x_0 \cong \mathbb{C}^{2\dim x_0}$.

In this subsection, we prove that there exists a quasi-isomorphism of degree $1 - 2\operatorname{codim} x_0$

$$\overline{\varphi}\colon \frac{D(\mathcal{A})}{D(\mathcal{A}')} \xrightarrow{\simeq} D(\mathcal{A}'').$$

Let us define an equivalence relation in $\mathcal{A}'$: $x_i \sim x_j$ if and only if $x_0 \cap x_i = x_0 \cap x_j$.

Before constructing a map $D(\mathcal{A})/D(\mathcal{A}') \to D(\mathcal{A}'')$, we construct a map $\varphi\colon D(\mathcal{A}) \to D(\mathcal{A}'')$. For $\sigma \subseteq \mathcal{A}$, we let

$$\varphi(\sigma) = \begin{cases} 0 & \text{if } x_0 \notin \sigma \\ 0 & \text{if } x_0 \in \sigma \text{ and } \exists y \neq z \in \sigma \text{ such that } y \sim z \\ \bigcup_{y \in \sigma\setminus\{x_0\}} \{x_0 \cap y\} & \text{otherwise} \end{cases}$$

This map is not multiplicative but it commutes with the differential, as shown by the following lemma.

Lemma 3.1. *The map $\varphi\colon D(\mathcal{A}) \to D(\mathcal{A}'')$ defined above commutes with the differential.*

Proof. Since φ has odd degree, we must have $\varphi d = -d\varphi$. Let $\sigma \subseteq \mathcal{A}$. If $x_0 \notin \sigma$, then we have $d\varphi(\sigma) = 0 = \varphi(d\sigma)$. If $x_0 \in \sigma$ and there exists $y \neq z \in \sigma$ with $y \sim z$, then $d\varphi(\sigma) = 0$ and

$$d\sigma = d\{x_0, y, z, x_{i_1}, \ldots, x_{i_r}\} = \alpha\,(\sigma \setminus \{x_0\}) + \{x_0, z, x_{i_1}, \ldots, x_{i_r}\} - \{x_0, y, x_{i_1}, \ldots, x_{i_r}\} + \sum_j \alpha_j \left(\sigma \setminus \{x_{i_j}\}\right),$$

where α and the α_j can be 0 or ± 1. So, by definition of φ, we have : $\varphi(d\sigma) = 0$.

Finally, we consider the case where $x_0 \in \sigma$ and there is no $y \neq z \in \sigma$ such that $y \sim z$. In that case, let $\sigma = \{x_0, x_{i_1}, \dots, x_{i_r}\}$ and denote by J the set $J = \{j \mid \vee(\sigma \setminus \{x_{i_j}\}) = \vee\sigma\}$. We have

$$\begin{aligned}
\varphi(d\sigma) &= \varphi\left(\alpha\,(\sigma \setminus \{x_0\}) + \sum_{j\in J}(-1)^{j+1}\sigma \setminus \{x_{i_j}\}\right) \\
&= \sum_{j\in J}(-1)^{j+1}\{x_0 \cap x_{i_1}, \dots, x_0 \cap x_{i_r}\} \setminus \{x_0 \cap x_{i_j}\}, \\
d\varphi(\sigma) &= d\{x_0 \cap x_{i_1}, \dots, x_0 \cap x_{i_r}\} \\
&= \sum_{j\in J}(-1)^{j}\{x_0 \cap x_{i_1}, \dots, x_0 \cap x_{i_r}\} \setminus \{x_0 \cap x_{i_j}\}.
\end{aligned}$$

So, the map φ commutes with the differential d. ■

The map φ is clearly surjective and satisfies $\varphi j = 0$, where j is the injection $D(\mathcal{A}') \hookrightarrow D(\mathcal{A})$. So φ induces a surjective morphism of complexes

$$\overline{\varphi}\colon \frac{D(\mathcal{A})}{D(\mathcal{A}')} \to D(\mathcal{A}'').$$

We will show that this map induces an isomorphism in cohomology. For that, we introduce the set :

$$E = \{(y,z) \in \mathcal{A}' \times \mathcal{A}' \,|y \sim z \text{ and } y \neq z\}.$$

If the set E is empty, then $\overline{\varphi}$ is injective and is an isomorphism. Otherwise, for every $(u,v) \in E$, let $I_{u,v}$ be the vector subspace of $D(\mathcal{A})/D(\mathcal{A}')$ generated by the elements $\{x_0, u, y_{j_1}, \dots, y_{j_r}\} - \{x_0, v, y_{j_1}, \dots, y_{j_r}\}$ and $\{x_0, u, v, y_{j_1}, \dots, y_{j_r}\}$ with $y_i \in \mathcal{A} \setminus \{x_0, u, v\}$. Let $V = \sum_{(u,v)\in E} I_{u,v}$.

The technical part of this section is contained in the following three lemmas.

Lemma 3.2. *In $D(A)/D(A')$, we have $V = \ker\overline{\varphi}$.*

Lemma 3.3. *For every $(u,v) \in E$, $d(I_{u,v}) \subseteq I_{u,v}$, so $I_{u,v}$ is a subcomplex of $D(\mathcal{A})/D(\mathcal{A}')$. Also, the complex $I_{u,v}$ is acyclic.*

Lemma 3.4. *The vector space V is acyclic.*

Lemma 3.4 does not trivially follow from lemma 3.3. There is no reason for a cocycle α to decompose as a sum of cocycles $\alpha_{u,v} \in I_{u,v}$. The problem comes from the fact that $I_{u,v} \cap I_{v,w}$ needs not be empty.

These three lemmas give us the following proposition.

Proposition 3.5. *The map $\overline{\varphi}\colon D(\mathcal{A})/D(\mathcal{A}') \to D(\mathcal{A}'')$ is a quasi-isomorphism.*

Proof. By lemma 3.2, we can factorize $\overline{\varphi}$ as shown in the following diagram.

$$\begin{tikzcd} \frac{D(\mathcal{A})}{D(\mathcal{A}')} \arrow[r, "\overline{\varphi}"] \arrow[d, "p"'] & D(\mathcal{A}'') \\ \frac{D(\mathcal{A})}{D(\mathcal{A}')\oplus V} \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

where $\tilde{\varphi}$ is the quotient map. Since $V = \ker \overline{\varphi}$ and $\overline{\varphi}$ is surjective, $\tilde{\varphi}$ is an isomorphism. By lemma 3.4, the map p is a quasi-isomorphism, which implies that $\overline{\varphi}$ is a quasi-isomorphism as well. ∎

Let us prove these lemmas.

Proof of lemma 3.2. It is clear that $V \subseteq \ker \overline{\varphi}$. Let $u = \sum_{s \in I} \alpha_s \sigma_s \in \ker \overline{\varphi}$, with $\alpha_s \neq 0$ for all $s \in I$ and $\sigma_s \neq \sigma_t$ if $s \neq t$. To prove that $u \in V$, we show that we can substract elements of V from u until we obtain zero.

- If there exists a $s \in I$ such that $\sigma_s = \{x_0, u, v, y_{j_1}, \dots, y_{j_r}\}$ with $x_0 \cap u = x_0 \cap v$ and $u \neq v$, then $\alpha_s \sigma_s \in V$ and we can substract it from u. Let us repeat this operation until we obtain a sum $\sum_{s \in I'} \alpha_s \sigma_s$ without any such σ_s.
- Let $t \in I'$. Then the element σ_t is necessarily of the form $\{x_0, x_{i_1}, \dots, x_{i_r}\}_<$ with $x_0 \cap x_{i_j} \neq x_0 \cap x_{i_k}$ for $j \neq k$. We have

$$0 = \overline{\varphi}\left(\sum_{s \in I'} \alpha_s \sigma_s\right) = \alpha_t \{x_0 \cap x_{i_1}, \dots, x_0 \cap x_{i_r}\} + \sum_{s \in I' \setminus \{t\}} \alpha_s \overline{\varphi}(\sigma_s).$$

  So, there exists $s \in I' \setminus \{t\}$ such that $\overline{\varphi}(\sigma_s) = \overline{\varphi}(\sigma_t)$. This is possible only if $\sigma_s = \{x_0, y_1, \dots, y_r\}$ with $x_0 \cap x_{i_j} = x_0 \cap y_j$. In that case, the difference $\sigma_s - \sigma_t$ can be rewritten as

$$\sum_{j=1}^{r} \left(\{x_0, y_1, \dots, y_{j-1}, y_j, x_{i_{j+1}}, \dots, x_{i_r}\} - \{x_0, y_1, \dots, y_{j-1}, x_{i_j}, x_{i_{j+1}}, \dots, x_{i_r}\}\right),$$

  which is clearly in V. We can substract $\alpha_t(\sigma_t - \sigma_s)$ from the sum and we obtain another sum $\sum_{s \in I''} \alpha_s \sigma_s$ such that $|I''| < |I'|$. This process can be repeated until we obtain an empty sum.

Therefore, $u \in V$ and $V = \ker \overline{\varphi}$. ∎

Proof of lemma 3.3. Let us consider a cocycle

$$\alpha = \sum_i \alpha_i \left(\{x_0, u, y_{i_1}, \dots, y_{i_r}\} - \{x_0, v, y_{i_1}, \dots, y_{i_r}\}\right) + \sum_j \beta_j \{x_0, u, v, z_{j_1}, \dots, z_{j_{r-1}}\},$$

with the $y_{i_k}, z_{j_k} \notin \{x_0, u, v\}$. We want to show that α is a coboundary. Let

$$\begin{aligned}
K_i &= \{k \mid \cap \{x_0, u, y_{i_1}, \ldots, y_{i_r}\} = \cap(\{x_0, u, y_{i_1}, \ldots, y_{i_r}\} \setminus \{y_{i_k}\})\},\\
L_j &= \{k \mid \cap \{x_0, u, v, z_{j_1}, \ldots, z_{j_{r-1}}\} = \\
&\qquad\qquad \cap (\{x_0, u, v, z_{j_1}, \ldots, z_{j_{r-1}}\} \setminus \{z_{j_k}\})\}.
\end{aligned}$$

Since α is a cocycle, its differential is zero and we have

$$\begin{aligned}
d\alpha = &\sum_i \alpha_i \sum_{k \in K_i} (-1)^k \big(\{x_0, u, y_{i_1}, \ldots, y_{i_r}\} \setminus \{y_{i_k}\} \\
&\qquad - \{x_0, v, y_{i_1}, \ldots, y_{i_r}\} \setminus \{y_{i_k}\}\big) \\
&+ \sum_j \beta_j \big(\{x_0, u, z_{j_1}, \ldots, z_{j_{r-1}}\} - \{x_0, v, z_{j_1}, \ldots, z_{j_{r-1}}\}\big) \\
&+ \sum_j \beta_j \sum_{k \in L_j} (-1)^{k+1} \{x_0, u, v, z_{j_1}, \ldots, z_{j_{r-1}}\} \setminus \{z_{j_k}\} = 0.
\end{aligned}$$

Looking at the terms without v and using the relation $d\alpha = 0$, we have

$$\sum_i \alpha_i \sum_{k \in K_i} (-1)^k \{x_0, u, y_{i_1}, \ldots, y_{i_r}\} \setminus \{y_{i_k}\} + \sum_j \beta_j \{x_0, u, z_{j_1}, \ldots, z_{j_{r-1}}\} = 0.$$

But we can add a v into each terms of the previous relation. This gives

$$-\sum_i \alpha_i \sum_{k \in K_i} (-1)^k \{x_0, u, v, y_{i_1}, \ldots, y_{i_r}\} \setminus \{y_{i_k}\} = \sum_j \beta_j \{x_0, u, v, z_{j_1}, \ldots, z_{j_{r-1}}\}.$$

Finally, let $\omega = -\sum_i \alpha_i \{x_0, u, v, y_{i_1}, \ldots, y_{i_r}\}$. Its differential is

$$\begin{aligned}
d\omega = &\sum_i \alpha_i \big(\{x_0, u, y_{i_1}, \ldots, y_{i_r}\} - \{x_0, v, y_{i_1}, \ldots, y_{i_r}\}\big) \\
&\qquad - \sum_i \alpha_i \sum_{k \in K_i} (-1)^k \{x_0, u, v, y_{i_1}, \ldots, y_{i_r}\} \setminus \{y_{i_k}\} = \alpha.
\end{aligned}$$

Hence, α is a coboundary and $I_{u,v}$ is acyclic. ■

Proof of lemma 3.4. Clearly, $d(V) \subseteq V$. Showing that V is acyclic is technical. We'll do it in three steps. First, we choose a finite number of couple $(u_i, v_i) \in E$ that have some nice properties. Then, using these elements, we construct a sequence of vector subspaces $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_r = V$. Finally, we prove by induction that all these subspaces are acyclic.

Without loss of generality, we can assume that $x_1, \ldots, x_n$ are numbered in such a way that equivalent spaces (for $\sim$) are consecutive. Let $\mathcal{A}_1, \ldots, \mathcal{A}_q$ be the equivalence classes on $\mathcal{A}'$:

$$\begin{aligned}
\mathcal{A}_1 &= \{x_{j_1}, \ldots, x_{j_2 - 1}\}_<, \\
\mathcal{A}_2 &= \{x_{j_2}, \ldots, x_{j_3 - 1}\}_<, \\
&\ \ \ldots \\
\mathcal{A}_q &= \{x_{j_q}, \ldots, x_{j_{q+1} - 1}\}_<,
\end{aligned}$$

where $j_1 = 1$ and $j_{q+1} - 1 = n$. Let $\mathcal{A}_i$ be the first class with more than one element and $(u_0, v_0) = (x_{j_i}, x_{j_i+1}) \in E$, $(u_1, v_1) = (u_0, x_{j_i+2}), \dots, (u_\ell, v_\ell) = (u_0, x_{j_{i+1}-1})$. Let $\mathcal{A}_k$ be the next equivalence class with more than 1 element and $(u_{\ell+1}, v_{\ell+1}) = (x_{j_k}, x_{j_k+1}), (u_{\ell+2}, v_{\ell+2}) = (x_{j_k}, x_{j_k+2}), \dots$ We can do this until we have a sequence $(u_i, v_i)_{0 \le i \le r} \subseteq E$ with the following properties : $V = \sum_{i=0}^{r} I_{u_i,v_i}$ and $v_\ell \notin \{u_0, v_0, u_1, v_1, \dots, u_{\ell-1}, v_{\ell-1}\}$.

Let $I_0 = I_{u_0,v_0}$ and $I_j = \sum_{i=0}^{j} I_{u_i,v_i}$. We just defined an increasing sequence

$$I_{u_0,v_0} = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_r = V.$$

Lemma 3.3 shows that I_0 is acyclic. Suppose $I_{\ell-1}$ acyclic and let us show that $I_\ell = I_{\ell-1} + I_{u_\ell,v_\ell}$ is acyclic as well.

Let us start by proving that every element of I_ℓ can be written as a sum $\alpha_1 + \alpha_2 + d\omega$ where $\alpha_1 \in I_{\ell-1}, \alpha_2 \in I_{u_\ell,v_\ell}, \omega \in I_{\ell-1}$ and there is no v_ℓ in any of the terms of α_1. It is sufficient to see that it is true for every element of a generating sequence of $I_{\ell-1}$:

- Let $\sigma = \{x_0, u_i, v_i, \dots\} \in I_{\ell-1}$. If $v_\ell \in \sigma$ and $u_\ell \in \sigma$, then $\sigma = 0 + \sigma + d(0)$. If $v_\ell \in \sigma$ and $u_\ell \notin \sigma$, then $d(\sigma \cup \{u_\ell\}) = \pm\sigma \pm (\sigma \cup \{u_\ell\}) \setminus \{v_\ell\} + S$ where S is a sum with $\{u_\ell, v_\ell\}$ in each term, which means that S is in I_{u_ℓ,v_ℓ}. It shows that we have the required decomposition $\sigma = \pm(\sigma \cup \{u_\ell\}) \setminus \{v_\ell\} \pm S \pm d(\sigma \cup \{u_\ell\})$, for an appropriate choice of signs.

- Let $\alpha = \{x_0, u_i, y_{j_1}, \dots, y_{j_s}\} - \{x_0, v_i, y_{j_1}, \dots, y_{j_s}\} \in I_{\ell-1}$. If we have $v_\ell \in \{y_{j_1}, \dots, y_{j_s}\}$ and $u_\ell \notin \{y_{j_1}, \dots, y_{j_s}\}$, then consider $\omega = \{x_0, u_i, y_{j_1}, \dots, y_{j_s}\} \cup \{u_\ell\} - \{x_0, v_i, y_{j_1}, \dots, y_{j_s}\} \cup \{u_\ell\} \in I_{\ell-1}$. As in the first case, we obtain a proper decomposition from $d\omega$.

Now, we can show that every cocycle in I_ℓ is a coboundary. Let $\alpha \in I_\ell$ such that $d\alpha = 0$. Since $I_\ell = I_{\ell-1} + I_{u_\ell,v_\ell}$, we can write $\alpha = \alpha_1 + \alpha_2 + d\omega$ with $\alpha_1 \in I_{\ell-1}, \alpha_2 \in I_{u_\ell,v_\ell}, \omega \in I_\ell$ and no v_ℓ in any term of α_1. The vector α_2 is a sum

$$\alpha_2 = \sum_i \gamma_i \left(\{x_0, u_\ell, y_{i_1}, \dots, y_{i_s}\} - \{x_0, v_\ell, y_{i_1}, \dots, y_{i_s}\}\right) + \sum_j \beta_j \{x_0, u_\ell, v_\ell, z_{j_1}, \dots, z_{j_{s-1}}\}.$$

For appropriate sets K_i and L_j (as in lemma 3.3), its differential is

$$\begin{aligned} d\alpha_2 = &\sum_i \gamma_i \sum_{k \in K_i} (-1)^k \big(\{x_0, u_\ell, y_{i_1}, \dots, y_{i_s}\} \setminus \{y_{i_k}\} \\ &\qquad - \{x_0, v_\ell, y_{i_1}, \dots, y_{i_s}\} \setminus \{y_{i_k}\}\big) \\ &+ \sum_j \beta_j \big(\{x_0, u_\ell, z_{j_1}, \dots, z_{j_{s-1}}\} - \{x_0, v_\ell, z_{j_1}, \dots, z_{j_{s-1}}\}\big) \\ &+ \sum_j \beta_j \sum_{k \in L_j} (-1)^{k+1} \{x_0, u_\ell, v_\ell, z_{j_1}, \dots, z_{j_{s-1}}\} \setminus \{z_{j_k}\}. \end{aligned}$$

But there is no term in α_1 containing v_ℓ and $d\alpha_1 + d\alpha_2 = 0$. So, looking at the terms with u_ℓ and v_ℓ in $d\alpha_2$, we deduce that

$$\sum_j \beta_j \sum_{k \in L_j} (-1)^{k+1} \{x_0, u_\ell, v_\ell, z_{j_1}, \ldots, z_{j_{s-1}}\} \setminus \{z_{j_k}\} = 0.$$

Hence the terms in $d\alpha_2$ with only v_ℓ are such that

$$\sum_i \gamma_i \sum_{k \in K_i} (-1)^k \{x_0, v_\ell, y_{i_1}, \ldots, y_{i_s}\} \setminus \{y_{i_k}\} + \sum_j \beta_j \{x_0, v_\ell, z_{j_1}, \ldots, z_{j_{s-1}}\} = 0.$$

And this relation stays true if we replace v_ℓ by u_ℓ. So, we just proved that $d\alpha_2 = 0$, which implies that $d\alpha_1 = 0$ as well. By the induction hypothesis, there exists a $\omega_1 \in I_{\ell-1}$ such that $d\omega_1 = \alpha_1$ and by lemma 3.3, there exists a $\omega_2 \in I_{u_\ell, v_\ell}$ such that $d\omega_2 = \alpha_2$. So, $\alpha = d(\omega_1 + \omega_2 + \omega)$. The vector subspace I_ℓ is acyclic and the proof by induction is complete. ∎

3.2 A long exact sequence in cohomology

With the results proved in the previous section, it is fairly easy to prove the following theorem.

Theorem 3.6. *Let $\mathcal{A} = \{x_0, \ldots, x_n\}$ be a subspace arrangement in $\mathbb{C}^m$, $\mathcal{A}' = \mathcal{A} \setminus \{x_0\}$, $\mathcal{A}'' = \{x_0 \cap y \,|y \in \mathcal{A}'\}$. Then, there exists a long exact sequence in cohomology :*

$$\begin{aligned} \cdots \to H^q(M(\mathcal{A}'), \mathbb{Q}) \to H^q(M(\mathcal{A}), \mathbb{Q}) \to H^{q-\deg(x_0)}(M(\mathcal{A}''), \mathbb{Q}) \\ \to H^{q+1}(M(\mathcal{A}'), \mathbb{Q}) \to H^{q+1}(M(\mathcal{A}), \mathbb{Q}) \to \cdots \end{aligned}$$

Proof. From the short exact sequence

$$0 \to D(\mathcal{A}') \to D(\mathcal{A}) \to D(\mathcal{A})/D(\mathcal{A}') \to 0,$$

we obtain a long exact sequence connecting the cohomology of those complexes. By proposition 3.5, we have a quasi-isomorphism $\overline{\varphi}\colon D(\mathcal{A})/D(\mathcal{A}') \to D(\mathcal{A}'')$. From this, we obtain the promised long exact sequence. ∎

3.3 Euler characteristic

The long exact sequence in cohomology of theorem 3.6 is quite powerful. In this subsection, we give a first application : the Euler characteristic of $M(\mathcal{A})$ is always zero for any non-empty arrangements.

Corollary 3.7. *Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{C}^m$. Then, the Euler characteristic of $M(\mathcal{A})$ is 1 if $\mathcal{A}$ is empty, and 0 otherwise.*

Proof. If $\mathcal{A}$ is empty, then $M(\mathcal{A}) = \mathbb{C}^m$ is contractible and $\chi(M(\mathcal{A})) = 1$. If $|\mathcal{A}| = 1$, then $M(\mathcal{A}) = \mathbb{C}^m \setminus \{x_0\}$ for some vector subspace x_0. So, $M(\mathcal{A})$ has the homotopy type of an odd-dimensional sphere and $\chi(M(\mathcal{A})) = 0$.

By the universal coefficient theorems (in homology and cohomology), we have : $b_i(M(\mathcal{A})) = \dim H^i(M(\mathcal{A}), \mathbb{Q})$, where $b_i(M(\mathcal{A}))$ is the ith Betti number of $M(\mathcal{A})$. The long exact sequence in cohomology from theorem 3.6 gives us the following formula : $\chi(M(\mathcal{A})) = \chi(M(\mathcal{A}')) + (-1)^{\deg(x_0)}\chi(M(\mathcal{A}'')) = \chi(M(\mathcal{A}')) - \chi(M(\mathcal{A}''))$. Now, let us prove by induction on $|\mathcal{A}|$ that $\chi(M(\mathcal{A})) = 0$. Suppose that this is true for arrangements with n or less subspaces. Let $\mathcal{A}$ be an arrangement with $n+1$ subspaces and $x_0 \in \mathcal{A}$. Since $|\mathcal{A}'| = |\mathcal{A}| - 1$ and $1 \leq |\mathcal{A}''| < |\mathcal{A}|$, by induction, $\chi(M(\mathcal{A})) = \chi(M(\mathcal{A}')) - \chi(M(\mathcal{A}'')) = 0 - 0 = 0$. ■

Note that this is only true for complex subspace arrangements. Clearly, if $\mathcal{B}$ is the real arrangement with one point in $\mathbb{R}^3$, then $M(\mathcal{B}) \simeq S^2$ and its Euler characteristic is 2.

4 Poincaré polynomials

4.1 Inductive formulas

In subsection 3.3, we actually proved a formula to compute inductively the Euler characteristic of the complement spaces :

$$\chi(M(\mathcal{A})) = \chi(M(\mathcal{A}')) + (-1)^{\deg(x_0)}\chi(M(\mathcal{A}'')).$$

In practice, it turns out that this formula is irrelevant since the Euler characteristic is zero anyway. But the idea is interesting, and can (sometimes) be applied to a finer topological invariant : the sequence of Betti numbers, or more precisely, the Poincaré polynomial

$$\mathrm{Poin}(M(\mathcal{A}), t) = \sum_{i=0}^{\infty} b_i(M(\mathcal{A}))t^i.$$

In some cases, we have a nice inductive formula.

Proposition 4.1. *Let $\mathcal{A} = \{x_0, \ldots, x_n\}$ be a subspace arrangement in $\mathbb{C}^m$, $\mathcal{A}' = \mathcal{A} \setminus \{x_0\}$, and $\mathcal{A}'' = \{x_0 \cap y \,|y \in \mathcal{A}'\}$. Then the following conditions are equivalent :*

1. $\mathrm{Poin}(M(\mathcal{A}), t) = \mathrm{Poin}(M(\mathcal{A}'), t) + t^{\deg(x_0)}\,\mathrm{Poin}(M(\mathcal{A}''), t)$,
2. *the long exact sequence of theorem 3.6 splits into short exact sequences*

$$0 \to H^q(M(\mathcal{A}'), \mathbb{Q}) \to H^q(M(\mathcal{A}), \mathbb{Q}) \to H^{q-\deg(x_0)}(M(\mathcal{A}''), \mathbb{Q}) \to 0,$$

3. *the injective map $D(\mathcal{A}') \to D(\mathcal{A})$ induces an injective map $H^\star(M(\mathcal{A}'), \mathbb{Q}) \to H^\star(M(\mathcal{A}), \mathbb{Q})$.*

Proof. It is an easy consequence from the long exact sequence in cohomology, from theorem 3.6. ■

Proposition 4.1 does not hold in general. Consider for instance the arrangement $\mathcal{A} = \{h_0, h_1, h_2\}$ in $\mathbb{C}^5$ where h_0, h_1 and h_2 are defined by the equations $z_1 = z_5 = 0$, $z_1 = z_2 = z_3 = 0$ and $z_3 = z_4 = z_5 = 0$, respectively.

Using the model $D(\mathcal{A})$, we compute the Poincaré polynomials of $M(\mathcal{A}), M(\mathcal{A}')$ and $M(\mathcal{A}'')$:

$$\begin{aligned}
\operatorname{Poin}(M(\mathcal{A}), t) &= 1 + t^3 + 2t^5 + 2t^6, \\
\operatorname{Poin}(M(\mathcal{A}'), t) &= 1 + 2t^5 + t^8, \\
\operatorname{Poin}(M(\mathcal{A}''), t) &= 1 + 2t^3 + t^4.
\end{aligned}$$

So, they don't satisfy the relationship described in proposition 4.1, with $\deg(x_0) = \deg(h_0) = 3$.

However, Orlik and Terao proved in [3] that the arrangements of hyperplanes always satisfy the properties of proposition 4.1. In the next two subsections, we will prove that some classes of arrangements have that property as well, namely the arrangements with a separator and the arrangements with a geometric lattice.

4.2 Arrangements with a separator

The following definition is a generalization of the concept of separator found in [3]. This is probably the simplest condition for the long exact sequence to split.

Definition 4.2. Let $\mathcal{A} = \{x_0, \ldots, x_n\}$ be a subspace arrangement. We say that x_i is a *separator* if $\cap\{x_0, \ldots, x_n\} \neq \cap(\{x_0, \ldots, x_n\} \setminus \{x_i\})$.

As shown by the next theorem, deleted and restricted arrangements with respect to a separator have the desired property.

Theorem 4.3. *Let $\mathcal{A} = \{x_0, \ldots, x_n\}$ be a subspace arrangement in $\mathbb{C}^m$, $\mathcal{A}' = \mathcal{A} \setminus \{x_0\}$, and $\mathcal{A}'' = \{x_0 \cap y \,|y \in \mathcal{A}'\}$. If x_0 is a separator, then the long exact sequence of theorem 3.6 splits and we have*

$$\operatorname{Poin}(M(\mathcal{A}), t) = \operatorname{Poin}(M(\mathcal{A}'), t) + t^{\deg(x_0)} \operatorname{Poin}(M(\mathcal{A}''), t).$$

Proof. It is sufficient to show that the injective map $j\colon D(\mathcal{A}') \to D(\mathcal{A})$ induces an injective map $j^\star\colon H^\star(D(\mathcal{A}')) \to H^\star(D(\mathcal{A}))$. Let us define a map $k\colon D(\mathcal{A}) \to D(\mathcal{A}')$ of cochain complexes by $k(\sigma) = 0$ if $x_0 \in \sigma$ and $k(\sigma) = \sigma$ if $x_0 \notin \sigma$. This map commutes with the differential :

- If $x_0 \notin \sigma$, then $dk(\sigma) = kd(\sigma)$,
- if $x_0 \in \sigma$, then $dk(\sigma) = 0$ and $kd(\sigma) = \alpha(\sigma \setminus \{x_0\})$, with $\alpha = 0$ if and only if $\cap(\sigma \setminus \{x_0\}) \neq \vee\sigma$, which is the case because x_0 is a separator.

So, the map k is a map of chain complexes and induces a map $k^\star\colon H^\star(D(\mathcal{A})) \to H^\star(D(\mathcal{A}'))$. Obviously, we have $k^\star j^\star = \mathrm{id}$, which implies that $j^\star$ is injective. Since the sequence of theorem 3.6 is exact, it splits as required. ∎

4.3 Arrangements with a geometric lattice

To each subspace arrangement $\mathcal{A}$, we can associate an important combinatorial tool : its lattice $L(\mathcal{A})$, which is the set of all intersections of elements of $\mathcal{A}$.

The set $L(\mathcal{A})$ is ordered by reverse inclusion : $x \leq y$ if and only if $x \supseteq y$. We define on the poset $L(\mathcal{A})$ two operations, $\vee$ and $\wedge$, which makes it a lattice :

$$x \vee y = x \cap y,$$
$$x \wedge y = \cap\{z \in L(\mathcal{A}) \,|\, x \cup y \subset z\} \in L(\mathcal{A}).$$

When we have a lattice, it is useful to consider the rank function, $\mathrm{rk}\colon L(\mathcal{A}) \to \mathbb{N}$.

Definition 4.4. The *rank* $\mathrm{rk}(x)$ is the length r of a maximal chain $\mathbb{C}^m < x_1 < \cdots < x_r = x$ of maximal length. The rank of $\mathbb{C}^m$ is $\mathrm{rk}(\mathbb{C}^m) = 0$.

For the rest of this section, we will be particularly interested in arrangements with a geometric lattice.

Definition 4.5. The lattice $L(\mathcal{A})$ is *geometric* if, for every $x, y \in L(\mathcal{A})$, we have :

$$\mathrm{rk}(x) + \mathrm{rk}(y) \geq \mathrm{rk}(x \wedge y) + \mathrm{rk}(x \vee y).$$

It is well known that arrangements of hyperplanes have a geometric lattice (see [3] for a proof).

Let us mention two interesting combinatorial results. First, Goresky and MacPherson described the linear cohomology structure of the complement space.

Theorem 4.6 (Goresky-MacPherson)**.** *Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{C}^m$. Then*

$$\widetilde{H}^i(M(\mathcal{A})) \simeq \bigoplus_{x \in L(\mathcal{A}) \setminus \{\mathbb{C}^m\}} \widetilde{H}_{2\,\mathrm{codim}(x)-2-i}(\Delta(0,x)),$$

where $\Delta(0,x)$ denotes the order complex of the interval $[\mathbb{C}^m, x]$ in $L(\mathcal{A})$.

On the other hand, Folkman (theorem 4.1 in [1]) proves that if L is a geometric lattice with 0 and 1 as smallest and largest element, then the homology of its order complex $\Delta(L)$ is

$$H_i(\Delta(L)) = \begin{cases} 0 & \text{if } i \neq \mathrm{rk}\, L - 2, \\ \mathbb{Z}^{|\mu(0,1)|} & \text{if } i = \mathrm{rk}\, L - 2, \end{cases}$$

where $\mu\colon L \times L \to \mathbb{Z}$ is the Mœbius function, defined recursively by the relations :

$$\begin{cases} \mu(x,x) = 1 & \text{if } x \in L, \\ \sum_{z \in [x,y]} \mu(x,z) = 0 & \text{if } x, y, z \in L \text{ and } x < y, \\ \mu(x,y) = 0 & \text{otherwise.} \end{cases}$$

As a consequence, we have the following proposition :

Proposition 4.7. *If $\mathcal{A}$ is a subspace arrangement in $\mathbb{C}^m$ with a geometric lattice $L(\mathcal{A})$, then*

$$\operatorname{Poin}(M(\mathcal{A}),t) = \sum_{x\in L(\mathcal{A})} \mu(0,x)(-t)^{2\operatorname{codim} x - \operatorname{rk}(x)}.$$

Proof. It is almost a direct consequence of theorem 4.6 and the result of Folkman. The only thing we need to check is the sign of $\mu(0,x)$. But a simple property of the Mœbius function is that $(-1)^{\operatorname{rk}(x)}\mu(0,x) \geq 0$ (see for example theorem 2.47 in [3]). ∎

The previous proposition is the key to prove a deletion-restriction formula for arrangements with a geometric lattice. But we need first an interpretation of the Mœbius function in the case of subspace arrangements.

Lemma 4.8. *Let $\mathcal{A}$ be a subspace arrangement. For $u, v \in L(\mathcal{A})$ with $u \leq v$, let $\mathcal{A}_u = \{x \in \mathcal{A} \,|\, u \subseteq x\}$ and $S(u,v) = \{\mathcal{B} \subseteq \mathcal{A} \,|\, \mathcal{A}_u \subseteq \mathcal{B}$ and $T(\mathcal{B}) = v\}$ (where $T(\mathcal{B})$ denotes the maximal element of $L(\mathcal{B})$). Then*

$$\mu(u,v) = \sum_{\mathcal{B}\in S(u,v)} (-1)^{|\mathcal{B}\setminus\mathcal{A}_u|}.$$

Proof. This lemma is the same as lemma 2.35 in [3] except that $\mathcal{A}$ is not necessarily an arrangement of hyperplanes. The proof of [3] is still valid in this case. ∎

Now, a simple adaptation of the proofs in [3] gives the desired result.

Theorem 4.9. *Let $\mathcal{A} = \{x_0, \ldots, x_n\}$ be a subspace arrangement in $\mathbb{C}^m$, $\mathcal{A}' = \mathcal{A} \setminus \{x_0\}$, and $\mathcal{A}'' = \{x_0 \cap y \,|\, y \in \mathcal{A}'\}$. If the lattice $L(\mathcal{A})$ is geometric, then we have*

$$\operatorname{Poin}(M(\mathcal{A}),t) = \operatorname{Poin}(M(\mathcal{A}'),t) + t^{\deg(x_0)} \operatorname{Poin}(M(\mathcal{A}''),t)$$

and the long exact sequence of theorem 3.6 splits.

Proof. First, let us notice that for any arrangement $\mathcal{A}$, we have

$$\begin{aligned}
\operatorname{Poin}(M(\mathcal{A}),t) &= \textstyle\sum_{x\in L(\mathcal{A})} \mu(0,x)(-t)^{2\operatorname{codim} x-\operatorname{rk}(x)} \\
&= \textstyle\sum_{x\in L(\mathcal{A})} \left(\sum_{\mathcal{B}\in S(0,x)} (-1)^{|\mathcal{B}|}(-t)^{2\operatorname{codim} x-\operatorname{rk}(x)}\right) \\
&= \textstyle\sum_{\mathcal{B}\subseteq\mathcal{A}} (-1)^{|\mathcal{B}|}(-t)^{2\operatorname{codim} T(\mathcal{B})-\operatorname{rk} T(\mathcal{B})}.
\end{aligned}$$

The last equality come from the fact that every subarrangement $\mathcal{B} \subseteq \mathcal{A}$ occurs in a unique $S(0,x)$. Now, separate the sum over $\mathcal{B} \subseteq \mathcal{A}$ into two sums R' and R'' with R' the sum over those $\mathcal{B}$ which do not contain x_0 and R'' the sum over those $\mathcal{B}$ which contain x_0. Clearly, by using the previous equality, we have $R' = \operatorname{Poin}(M(\mathcal{A}'),t)$. So we proved that $\operatorname{Poin}(M(\mathcal{A}),t)$ is equal to the sum

$\mathrm{Poin}(M(\mathcal{A}'),t) + R''$. Now, let us compute R''.

$$\begin{aligned}
R'' &= \textstyle\sum_{x_0 \in \mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|} (-t)^{2\,\mathrm{codim}\, T(\mathcal{B}) - \mathrm{rk}(\mathcal{B})} \\
&= \textstyle\sum_{v \in L(\mathcal{A}'')} \sum\limits_{\mathcal{B} \in S(x_0, v)} (-1)^{|\mathcal{B}|} (-t)^{2\,\mathrm{codim}\, v - \mathrm{rk}(v)} \\
&= -\textstyle\sum_{v \in L(\mathcal{A}'')} \sum\limits_{\mathcal{B} \in S(x_0, v)} (-1)^{|\mathcal{B} \setminus \mathcal{A}_{x_0}|} (-t)^{2\,\mathrm{codim}\, v - \mathrm{rk}(v)} \\
&= -\textstyle\sum_{v \in L(\mathcal{A}'')} \mu(x_0, v) (-t)^{2(\mathrm{codim}\, v - \mathrm{codim}\, x_0) - (\mathrm{rk}(v) - 1) + 2\,\mathrm{codim}(x_0) - 1} \\
&= t^{2\,\mathrm{codim}\, x_0 - 1}\, \mathrm{Poin}(M(\mathcal{A}''), t) = t^{\deg(x_0)}\, \mathrm{Poin}(M(\mathcal{A}''), t).
\end{aligned}$$

The third line comes from lemma 4.8 and the last line comes from the fact that $(\mathrm{codim}\, v - \mathrm{codim}\, x_0)$ is equal to the codimension of v, seen in the ambient space of $\mathcal{A}''$, and that $\mathrm{rk}(v) - 1$ is the rank of v in $L(\mathcal{A}'')$. This complete the proof of the fact that $\mathrm{Poin}(M(\mathcal{A}),t) = \mathrm{Poin}(M(\mathcal{A}'),t) + t^{\deg(x_0)}\,\mathrm{Poin}(M(\mathcal{A}''),t)$. ∎

4.4 Additional results

When we suppose that the arrangement $\mathcal{A}$ has a geometric lattice *and* that there is a subspace x_0 which is a separator, we have some stronger results. First, let us prove the following lemma.

Lemma 4.10. *Let $\mathcal{A} = \{x_0, \ldots, x_n\}$ be a subspace arrangement with a geometric lattice such that x_0 is a separator and $1 \le i_1 < \ldots < i_{r+1} \le n$. Then the following conditions are equivalent :*

1. $\vee\{x_{i_1}, \ldots, x_{i_r}\} = \vee\{x_{i_1}, \ldots, x_{i_r}, x_{i_{r+1}}\}$,
2. $\vee\{x_0, x_{i_1}, \ldots, x_{i_r}\} = \vee\{x_0, x_{i_1}, \ldots, x_{i_r}, x_{i_{r+1}}\}$.

Proof. Clearly, the condition (1) implies the condition (2). Suppose that we have

$$\vee\{x_0, x_{i_1}, \ldots, x_{i_r}\} = \vee\{x_0, x_{i_1}, \ldots, x_{i_r}, x_{i_{r+1}}\}.$$

Since x_0 is a separator, $\vee\{x_0, \ldots, x_n\} \neq \vee(\{x_0, \ldots, x_n\} \setminus \{x_0\})$. An easy consequence is that $\vee\{x_0, x_{i_1}, \ldots, x_{i_r}\} \neq \vee\{x_{i_1}, \ldots, x_{i_r}\}$. Since the lattice is geometric, we can obtain the maximal chain (longest strictly increasing sequence between two elements) $\vee\{x_{i_1}, \ldots, x_{i_r}\} < \vee\{x_0, x_{i_1}, \ldots, x_{i_r}\}$ from the maximal chain $\mathbb{C}^m < x_0$. Therefore, we have the two following chains of inequalities :

$$\begin{aligned}
\vee\{x_{i_1}, \ldots, x_{i_r}\} < \vee\{x_0, x_{i_1}, \ldots, x_{i_r}\} &= \vee\{x_0, x_{i_1}, \ldots, x_{i_{r+1}}\}, \\
\vee\{x_{i_1}, \ldots, x_{i_r}\} \le \vee\{x_{i_1}, \ldots, x_{i_{r+1}}\} &< \vee\{x_0, x_{i_1}, \ldots, x_{i_{r+1}}\}.
\end{aligned}$$

Since the lattice is geometric, all maximal chains between two fixed elements have the same length (see [2] for a proof). The first line is a maximal chain of length 2, so the second line has the same length and we have the equality $\vee\{x_{i_1}, \ldots, x_{i_r}\} = \vee\{x_{i_1}, \ldots, x_{i_{r+1}}\}$. ∎

Lemma 4.11. *Let $\mathcal{A}$ be a subspace arrangement with a geometric lattice and $\mathcal{A}', \mathcal{A}''$ the deleted and restricted arrangements with respect to $x_0 \in \mathcal{A}$. If x_0 is a separator, then there exists an isomorphism $H^\star(M(\mathcal{A}');\mathbb{Q}) \to H^\star(M(\mathcal{A}'');\mathbb{Q})$ of $\mathbb{Z}_2$-graded vector spaces.*

Proof. Let us define a map $\theta\colon D(\mathcal{A}') \to D(\mathcal{A}'')$ by sending the element $\{x_{i_1}, \ldots, x_{i_r}\}$ to $\{x_0 \cap x_{i_1}, \ldots, x_0 \cap x_r\}$. It is obviously a surjection. If $x_0 \cap x_i = x_0 \cap x_j$, then $x_0 \cap x_i = x_0 \cap x_i \cap x_j$, and by lemma 4.10, $x_i = x_i \cap x_j$, which is impossible (we always suppose that $\mathcal{A}$ does not have two subspaces x and y such that $x \subset y$). So, the map θ is a bijection.

To show that θ commutes with the differential, consider $\sigma = \{x_{i_1}, \ldots, x_{i_r}\}$ and let

$$J_1 = \{j \,|1 \leq j \leq r \text{ and } \vee \sigma = \vee(\sigma \setminus \{x_{i_j}\})\},$$
$$J_2 = \{j \,|1 \leq j \leq r \text{ and } \vee (\sigma \cup \{x_0\}) = \vee(\sigma \cup \{x_0\} \setminus \{x_{i_j}\})\}.$$

We have :

$$\theta(d\sigma) = \sum_{j \in J_1} (-1)^j \{x_0 \cap x_{i_1}, \ldots, x_{i_r} \cap x_{i_r}\} \setminus \{x_0 \cap x_{i_j}\},$$
$$d(\theta(\sigma)) = \sum_{j \in J_2} (-1)^j \{x_0 \cap x_{i_1}, \ldots, x_{i_r} \cap x_{i_r}\} \setminus \{x_0 \cap x_{i_j}\}.$$

Lemma 4.10 shows that $J_1 = J_2$, so θ commutes with the differential. Therefore, the map in cohomology $H^\star\theta\colon H^\star(M(\mathcal{A}');\mathbb{Q}) \to H^\star(M(\mathcal{A}'');\mathbb{Q})$ is an isomorphism of vector spaces.

Finally, let $\sigma = \{x_{i_1}, \ldots, x_{i_r}\} \in D(\mathcal{A}')$. It is mapped to $\{x_0 \cap x_{i_1}, \ldots, x_0 \cap x_{i_r}\} \in D(\mathcal{A}'')$. We have :

$$\begin{aligned} \deg \sigma - \deg \theta(\sigma) &= (2m - 2\dim \vee\sigma - r) - (2\dim x_0 - 2\dim(x_0 \cap \vee\sigma) - r) \\ &= 2m - 2\dim(x_0 + \vee\sigma) = 2\operatorname{codim}(x_0 + \vee\sigma) \end{aligned}$$

So, θ decreases the degree by an even number. ∎

Notice that when the subspace x_0 is also an hyperplane, the map θ does not change the degree. So, we have a stronger result.

Proposition 4.12. *Let $\mathcal{A}$ be a subspace arrangement with a geometric lattice and $\mathcal{A}', \mathcal{A}''$ the deleted and restricted arrangements with respect to $x_0 \in \mathcal{A}$. If x_0 is an hyperplane and a separator, then*

$$\operatorname{Poin}(M(\mathcal{A}), t) = (1 + t)\operatorname{Poin}(M(\mathcal{A}''), t).$$

Proof. Let $\sigma \subseteq \mathcal{A}'$. Since $x_0 \cap \vee\sigma \neq \vee\sigma$, we have $\vee\sigma \not\subset x_0$. But x_0 is an hyperplane, so $\operatorname{codim}(x_0 + \vee\sigma) = 0$. Therefore, the map $H^\star\theta$ of lemma 4.11 preserves the degree and is an isomorphism of graded vector spaces. This implies that $\operatorname{Poin}(M(\mathcal{A}'), t) = \operatorname{Poin}(M(\mathcal{A}''), t)$. Using theorem 4.3, we obtain directly the desired relation. ∎

An interesting example is when $\mathcal{A}$ is an arrangements of n hyperplanes in general position (meaning that $\operatorname{codim} \cap_{x\in\mathcal{A}} x = \sum_{x\in\mathcal{A}} \operatorname{codim} x$). In that case, the lattice $L(\mathcal{A})$ is geometric and it is not hard to see that every subspace is a separator, in every restricted arrangement. So, by proposition 4.12, we can see that

$$\operatorname{Poin}(M(\mathcal{A}), t) = (1+t)^n.$$

It directly implies that $M(\mathcal{A})$ satisfies Poincaré duality. It can be shown that $M(\mathcal{A})$ has in this case the homotopy type of a product of n spheres S^1.

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UCL, Département de mathématique
Chemin du Cyclotron, 2
B-1348 Louvain-la-neuve
Belgium

Current address:School of Mathematics
University of Manchester, Oxford Road
Manchester M13 9PL
email:gery.debongnie@uclouvain.be