Amenability and essential amenability of certain convolution Banach algebras on compact hypergroups

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Abstract

In this paper we investigate amenability, and essential amenability of the convolution Banach algebra $A(K)$ for a compact hypergroup K together with their applications to convolution Banach algebras $L^p(K)$ $(1 < p \leq \infty)$, and $C(K)$.

Introduction

In [2] F. Ghahramani and R. J. Loy proved that for a compact group G, the convolution Banach algebras $(L^p(G), *)$ $(1 < p < \infty)$ are essentially amenable. Their given proof heavily depends on the amenability of the group algebra $L^1(G)$ (see Theorem 7.1 and Corollary 7.1(1),(2) of [2]). In the present paper by a quite different technique we generalize this result to compact hypergroups. Note that we do not know whether $L^1(K)$ is amenable for a compact hypergroup K. Vrem ([9]) gave a definition of $A(K)$ for a compact hypergroup K and proved that $A(K)$ is a Banach algebra with convolution product. This Banach algebra plays a key role throughout the paper.

The organization of this paper is as follows. The preliminaries and notations are given in section 1. In section 2 we state and prove a basic result on essential amenability of general Banach algebras that is needed for the rest of the paper. In the main theorem of this section (Theorem 2.1) we introduce a class of essentially

Received by the editors August 2007 - In revised form in November 2007.

Communicated by A. Valette.

Key words and phrases : Hypergroup, Fourier space, Banach algebra, Amenability, Essential amenability.

amenable Banach algebras. In section 3, through a different technique, we generalize Lemma 28.1 of [4] from compact groups to compact hypergroups. Indeed we prove that for an infinite compact hypergroup K, $\widehat{K}$ is infinite. As an application we prove that $L^1(K)$ for a locally compact hypergroup K is contractible if and only if K is finite. Furthermore, we prove that the convolution Banach algebra $A(K)$ on a compact hypergroup K is essentially amenable. Moreover this Banach algebra is amenable if and only if K is finite. In section 4 we prove that for a compact hypergroup K, the convolution Banach algebras $L^p(K)$ $(1 < p \leq \infty)$, and $C(K)$ are essentially amenable. Also, we prove such Banach algebras are amenable if and only if K is finite.

1 Preliminaries

For a Banach algebra A, an A-bimodule will always refer to a *Banach A-bimodule* X, that is a Banach space which is algebraically an A-bimodule, and for which there is a constant $C_X \geq 0$ such that $\|a.x\|, \|x.a\| \leq C_X\|a\|\|x\|$ $(a \in A, x \in X)$. A bounded linear map $D : A \to X$ is called an X-*derivation*, if for each $a, b \in A$, $D(ab) = D(a).b + a.D(b)$. For every $x \in X$, we define ad_x^A by $ad_x^A(a) = a.x - x.a$ $(a \in A)$. It is easily seen that ad_x^A is a derivation. Derivations of this form are called *inner derivations*.

Let A be a Banach algebra and X be a Banach A-bimodule. Then the Banach space X^* with the *dual* module multiplications given by

$$(fa)(x) = f(ax), (af)(x) = f(xa) \quad (a \in A,\ f \in X^*,\ x \in X),$$

defines a Banach A-bimodule called the *dual* Banach A-bimodule X^*.

A Banach algebra A is called *amenable* if for each Banach A-bimodule X, every continuous derivation from A into X^* is inner. An A-bimodule X is *neo-unital* if $X = A.X.A$. Recall from [2], a Banach algebra A is called *essentially amenable* if for any neo-unital A-bimodule X, every continuous derivation $D : A \to X^*$ is inner (See also [6]). A Banach algebra A is called *contractible* if for each Banach A-bimodule X, every continuous derivation $D : A \to X$ is inner.

Throughout this paper K is a (measured) locally compact hypergroup with involution $x \mapsto \bar{x}$ and the identity e as defined by Jewett ([5]). By the term measured we mean that K admits a left Haar measure ω_K. Let $M(K)$ be the space of all bounded regular Borel measures on K. For $1 \leq p \leq \infty$, let $L^p(K) = L^p(K, \omega_K)$. For $x, y \in K$ we define $x * y$ as the set $\operatorname{supp}(\varepsilon_x * \varepsilon_y)$. For Borel functions f and g, at least one of which is σ-finite, we define the convolution $f * g$ on K by $(f * g)(x) = \int_K f(x * y)g(\bar{y})d\omega_K(y)$ $(x \in K)$, where $f(x * y) = \int_K f d(\varepsilon_x * \varepsilon_y)$.

Let K be a compact hypergroup. By Theorem 1.3.28 of [1], K admits a left Haar measure. Throughout the present paper we use the normalized Haar measure ω_K on the compact hypergroup K (i.e. $\omega_K(K) = 1$). If $\pi \in \widehat{K}$ (where $\widehat{K}$ is the set of equivalence classes of continuous irreducible representations of K, c.f. [1], 11.3 of [5], and [9]), then by Theorem 2.2 of [9], π is finite dimensional. Furthermore by the proof of Theorem 2.2 of [9], there exists a constant c_π such that for each $\xi \in H_\pi$ with $\|\xi\| = 1$

$$\int_K |\langle \pi(x)\xi, \xi \rangle|^2 \, d\omega_K(x) = c_\pi.$$

Let $k_\pi = c_\pi^{-1}$. By Theorem 2.6 of [9], $k_\pi \geq d_\pi$. Moreover if K is a group then $k_\pi = d_\pi$. For each $\pi \in \widehat{K}$, let H_π be the representation space of π and $d_\pi = \dim H_\pi$. The $*$-algebra $\prod_{\pi\in\widehat{K}} \mathcal{B}(H_\pi)$ ($\mathcal{B}(H_\pi)$ is the space of all linear operators on H_π) will denoted by $\mathfrak{E}(\widehat{K})$; scalar multiplication, addition, multiplication, and the adjoint of an element are defined coordinate-wise. Let $E = (E_\pi)$ be an element of $\mathfrak{E}(\widehat{K})$. We define $\|E\|_p := \left(\sum_{\pi\in\widehat{K}} k_\pi \|E_\pi\|_{\varphi_p}^p\right)^{\frac{1}{p}}$ $(1 \leq p < \infty)$, and $\|E\|_\infty = \sup_{\pi\in\widehat{K}} \|E_\pi\|_{\varphi_\infty}$ (recall from Definition D.37 and Theorem D.40 of [4] that for $E_\pi \in \mathcal{B}(H_\pi)$, $\|E_\pi\|_{\varphi_\infty} = \max_{1\leq i\leq d_\pi} |\lambda_i|$, and $\|E_\pi\|_{\varphi_p} = \left(\sum_{i=1}^{d_\pi} |\lambda_i|^p\right)^{\frac{1}{p}}$ $(1 \leq p < \infty)$, where $(\lambda_1, \ldots, \lambda_{d_\pi})$ is the sequence of eigenvalues of the operator $|E_\pi|$, written in any order). For $1 \leq p \leq \infty$, $\mathfrak{E}_p(\widehat{K})$ is defined as the set of all $E \in \mathfrak{E}(\widehat{K})$ for which $\|E\|_p < \infty$. By Theorems 28.25, 28.27, and 28.32(v) of [4], the spaces $(\mathfrak{E}_p(\widehat{K}), \|.\|_p)$ $(1 \leq p \leq \infty)$ are Banach algebras. Let $\mu \in M(K)$. The set of all $E \in \mathfrak{E}(\widehat{K})$ such that $\{\pi \in \widehat{K} : E_\pi \neq 0\}$ is finite is denoted by $\mathfrak{E}_{00}(\widehat{K})$. The *Fourier transform* of μ at $\pi \in \widehat{K}$ is denoted by $\widehat{\mu}(\pi)$ and defined as the operator $\widehat{\mu}(\pi) = \int_K \pi(\bar{x})\, d\mu(x)$ on H_π. Define $\widehat{\mu} \in \mathfrak{E}(\widehat{K})$ by $\widehat{\mu}_\pi = \widehat{\mu}(\pi) \in \mathcal{B}(H_\pi)$ (for more details see Theorem 3.2 of [9]). If $\pi \in \widehat{K}$, $\mathfrak{T}_\pi(K)$ is defined as the set of all finite complex linear combinations of functions of the form $x \mapsto \langle \pi(x)(\xi), \eta\rangle$, where $\xi, \eta \in H_\pi$. Define $\mathfrak{T}(K) = \bigcup_{\pi\in\widehat{K}} \mathfrak{T}_\pi(K)$. Functions in $\mathfrak{T}(K)$ are called *trigonometric polynomials* on K. Clearly $\{\widehat{f} : f \in \mathfrak{T}(K)\} = \mathfrak{E}_{00}(\widehat{K})$ (see also Theorem 28.39 of [4] for the case of groups). If $f \in L^1(K)$, and $\sum_{\pi\in\widehat{K}} k_\pi \|\widehat{f}(\pi)\|_{\varphi_1} < \infty$, we say f has an *absolutely convergent Fourier series*. The set of all functions with absolutely convergent Fourier series is denoted by $A(K)$ and called *the Fourier space* of K. For $f \in A(K)$ we define $\|f\|_{\varphi_1} = \|\widehat{f}\|_1$. By Proposition 4.2 of [9], $A(K)$ with the convolution product is a Banach algebra and isometrically isomorphic with $\mathfrak{E}_1(\widehat{K})$. Moreover each function $f \in A(K)$ can be regarded as the continuous function $\sum_{\pi\in\widehat{K}} k_\pi \mathrm{tr}(\widehat{f}(\pi)\pi(x))$. Also $\|f\|_\infty \leq \|f\|_{\varphi_1}$. However, $A(K)$ may not form a Banach algebra under point-wise product (see Example 4.12 of [9]).

2 Essential amenability of a class of Banach algebras

The following is the main result of this section.

Theorem 2.1. *Let $(\mathfrak{A}, \|.\|_{\mathfrak{A}})$ be a Banach algebra. Suppose that there exists a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that*

(i) *There is a norm $\|.\|_{\mathfrak{B}}$ on $\mathfrak{B}$ such that $(\mathfrak{B}, \|.\|_{\mathfrak{B}})$ is a Banach algebra and there exists $C > 0$ such that for each $b \in \mathfrak{B}$, $\|b\|_{\mathfrak{A}} \leq C\|b\|_{\mathfrak{B}}$.*

(ii) *there exists $m \in \mathbb{N}$ such that $\mathfrak{A}^m = \{\prod_{i=1}^m a_i : a_1, \ldots, a_m \in \mathfrak{A}\} \subseteq \mathfrak{B}$.*

(iii) *$(\mathfrak{B}, \|.\|_{\mathfrak{B}})$ is essentially amenable.*

Then $\mathfrak{A}$ is essentially amenable.

Proof. Let X be a neo-unital Banach $\mathfrak{A}$-bimodule, and D be a derivation from $\mathfrak{A}$ into X^*. For each $b \in \mathfrak{B}$, $\|b.x\|_X \leq C_X\|b\|_{\mathfrak{A}}\|x\|_X \leq CC_X\|b\|_{\mathfrak{B}}\|x\|_X$. Similarly $\|x.b\|_X \leq CC_X\|b\|_{\mathfrak{B}}\|x\|_X$. Hence X is a $\mathfrak{B}$-bimodule. Now

$$X = \mathfrak{A}.X.\mathfrak{A} = \mathfrak{A}.(\mathfrak{A}.X.\mathfrak{A}).\mathfrak{A} = (\mathfrak{A}.\mathfrak{A}).X.(\mathfrak{A}.\mathfrak{A}) = \mathfrak{A}^2.X.\mathfrak{A}^2,$$

and so by induction $X = \mathfrak{A}^m.X.\mathfrak{A}^m$. Therefore

$$X = \mathfrak{A}^m.X.\mathfrak{A}^m \subseteq \mathfrak{B}.X.\mathfrak{B} \subseteq X,$$

and so X is a neo-unital Banach $\mathfrak{B}$-bimodule. Since

$$\|D(b)\| \leq \|D\|\|b\|_{\mathfrak{A}} \leq C\|D\|\|b\|_{\mathfrak{B}} \quad (b \in \mathfrak{B}),$$

so the mapping

$$\overline{D} : \mathfrak{B} \to X^*, b \mapsto D(b),$$

is a continuous derivation. By essential amenability of $\mathfrak{B}$, $\overline{D}$ is inner. Hence there exists $\xi \in X^*$ such that $\overline{D} = ad_\xi^{\mathfrak{B}}$. Define $\widetilde{D} = D - ad_\xi^{\mathfrak{A}}$. Clearly $\widetilde{D} \in \mathcal{Z}(\mathfrak{A}, X^*)$ and $\widetilde{D}(\mathfrak{B}) = \{0\}$. Let $a \in \mathfrak{A}$ and $x \in X$. Since $X = \mathfrak{A}^m.X.\mathfrak{A}^m$, so there exists $b \in \mathfrak{A}^m$ and $y \in X$ such that $x = b.y$. Now, since $b, ab \in \mathfrak{A}^m$, we have

$$\widetilde{D}(a)(x) = \widetilde{D}(a)(b.y) = \left(\widetilde{D}(a).b\right)(y) = \left(\widetilde{D}(ab) - a.\widetilde{D}(b)\right)(y) = 0.$$

Hence $\widetilde{D} = 0$ and so $D = ad_\xi^{\mathfrak{A}}$. Therefore $\mathfrak{A}$ is essentially amenable. ∎

Corollary 2.2. *Let A be a Banach algebra and B be a closed subalgebra of A containing $A.A$. If B is essentially amenable, then so is A.*

3 Amenability and essential amenability of the convolution Banach algebra $A(K)$ on the compact hypergroup K

Before starting the first result of this section, we note that by Lemma 28.1 of [4] a compact group G is finite if and only if $\widehat{G}$ is finite. In the following lemma, by a different technique, we generalize this result to compact hypergroups.

Lemma 3.1. *A compact hypergroup K is finite if and only if $\widehat{K}$ is finite.*

Proof. If K is finite, then clearly $\widehat{K}$ is finite. Conversely, if $\widehat{K}$ is finite, then $\mathfrak{C}_1(\widehat{K})$ is finite-dimensional. Since $\widehat{\mathfrak{T}(K)} = \mathfrak{C}_{00}(\widehat{K})$, so $\mathfrak{T}(K)$ is finite-dimensional. By Theorem 2.13 of [9], $\mathfrak{T}(K)$ is uniformly dense in $C(K)$. Since each finite dimensional subspace of a Banach space is closed, it follows that $\mathfrak{T}(K) = C(K)$. Therefore $C(K)$ is finite-dimensional. Now, by the comment on page 57 of [7] K is finite. ∎

As an application of the above lemma, we have the following result.

Proposition 3.2. *Let K be a locally compact hypergroup. Then $L^1(K)$ is contractible if and only if K is finite.*

Proof. If K is a finite hypergroup, then $L^1(K) = \ell^1(K) = C(K) = A(K)$, and $\widehat{K}$ is finite. Hence

$$\ell^1(K) \cong \widehat{A(K)} = \mathfrak{C}_1(\widehat{K}) \cong \ell^\infty - \oplus_{\pi\in\widehat{K}}\mathbb{M}_{d_\pi}(\mathbb{C}),$$

and so by Exercise 4.1.3 of [8], $L^1(K)$ is contractible.

Suppose $L^1(K)$ is contractible. By Examples C.1.2(c) and 3.1.12(b) of [8], the Banach space $L^1(K)$ has the approximation property (c.f. Definition C.1.1(i) of [8]). Now, by Theorem 4.1.5 of [8], $L^1(K)$ is finite-dimensional. Since $A(K) \subseteq L^1(K)$, so $A(K)$ is finite-dimensional. Hence by Lemma 4.1, K is finite. ■

The following theorem is adapted from Theorem 2.3 of [6].

Theorem 3.3. *If K is compact, then the convolution Banach algebra $A(K)$ is essentially amenable. Moreover the convolution Banach algebra $A(K)$ is amenable if and only if K is finite.*

Proof. By Proposition 4.2 of [9], the mapping $f \mapsto \widehat{f}$ is an isometric algebra isomorphism from the convolution Banach algebra $A(K)$ onto $\mathfrak{C}_1(\widehat{K})$.

Clearly $\mathfrak{C}_0(\widehat{K}) = c_0 - \oplus_{\pi\in\widehat{K}}\mathcal{B}(H_\pi)$, where $\mathcal{B}(H_\pi)$ is equipped with the norm $\|.\|_{\varphi_\infty}$. By Remark D.42 of [3] and Example 2.3.16 of [8], for each $\pi \in \widehat{K}$, the Banach algebra $B(H_\pi)$ with the norm $\|.\|_{\varphi_\infty}$ is 1-amenable. So by Corollary 2.3.19 of [8], $\mathfrak{C}_0(\widehat{K})$ is amenable. For each finite subset F of $\widehat{K}$ define E_F by

$$(E_F)_\pi = \begin{cases} id_{H_\pi} & \text{for } \pi \in F \\ 0 & \text{otherwise,} \end{cases}$$

where id_{H_π} is the identity operator of $B(H_\pi)$. It is easy to show that $(E_F)_F$ is an approximate identity for both Banach algebras $\mathfrak{C}_0(\widehat{K})$ and $\mathfrak{C}_1(\widehat{K})$. By Theorems 28.32(ii,iii) of [3] and 7.1 of [2], $\mathfrak{C}_1(\widehat{K})$ is essentially amenable. So $(A(K), *)$ is essentially amenable.

If $(E_\alpha)_\alpha$ is an approximate identity for $\mathfrak{C}_1(\widehat{K})$, then for each finite subset F of I

$$\begin{aligned} \mathrm{Card}(F) &\leq \sum_{\pi\in F} k_\pi \|id_{H_\pi}\|_{\varphi_1} \\ &= \|E_F\|_1 = \lim_\alpha \|E_F E_\alpha\|_1 \\ &= \lim_\alpha \left\| \Big((E_F)_\pi (E_\alpha)_\pi \Big)_\pi \right\|_1 \\ &\leq \liminf_\alpha \|E_\alpha\|_1. \end{aligned}$$

So for infinite set I, $\lim_\alpha \|E_\alpha\|_1 = \infty$, and hence $\mathfrak{C}_1(\widehat{K})$ does not have a bounded approximate identity. Therefore by Proposition 2.2.1 of [8], this Banach algebra is not amenable. Now, $\mathfrak{C}_1(\widehat{K})$ is amenable if and only if $\widehat{K}$ is finite. By Lemma 3.1, $\widehat{K}$ is finite if and only if K is finite. ■

4 Essential amenability of certain convolution Banach algebras on compact hypergroups

We start this section with the following lemmas.

Lemma 4.1. *Let K be a compact hypergroup. Then the following statements are equivalent:*

(i) *K is finite.*
(ii) *$A(K)$ is finite dimensional.*
(iii) *The convolution Banach algebra $A(K)$ has an identity.*
(iv) *$A(K) * A(K) = A(K)$.*
(v) *The linear span of $A(K) * A(K)$ is equal to $A(K)$.*

Proof. (i)⇒(ii)⇒(iii)⇒(iv)⇒(v) is obvious.

(v)⇒(i): If K is infinite, then by Lemma 3.1 $\widehat{K}$ is infinite. Let $E \in \mathfrak{E}(\widehat{K})$ be such that

$$\sum_{\pi\in\widehat{K}} k_\pi \|E_\pi\|_{\varphi_1} < \infty, \text{ and } \sum_{\pi\in\widehat{K}} k_\pi \|E_\pi\|_{\varphi_1}^{\frac{1}{2}} = \infty.$$

For example let $\{\pi_n\}_{n\in\mathbb{N}}$ be an infinite countable set of distinct elements of $\widehat{K}$, and define $E \in \mathfrak{E}(\widehat{K})$ as follows: $E_{\pi_n} = \frac{1}{k_{\pi_n} d_{\pi_n} n^2} id_{H_{\pi_n}}$ for $n \in \mathbb{N}$ and $E_\pi = 0$ for all other π's in $\widehat{K}$. Since $E \in \mathfrak{E}_1(\widehat{K})$, so there exists a unique $f \in A(K)$ such that $\widehat{f} = E$. If $E = \sum_{i=1}^m E^{1,i}E^{2,i}$ for some $m \in \mathbb{N}$ and $E^{1,i}, E^{2,i} \in \mathfrak{E}(\widehat{K})$ $(1 \le i \le m)$, then

$$\begin{aligned}
\sum_{i=1}^m \Big(\|E^{1,i}{}_\pi\|_{\varphi_1} + \|E^{2,i}{}_\pi\|_{\varphi_1}\Big) &\ge 2\sum_{i=1}^m \Big(\|E^{1,i}{}_\pi\|_{\varphi_1}\|E^{2,i}{}_\pi\|_{\varphi_1}\Big)^{\frac{1}{2}} \\
&\ge 2\Big(\sum_{i=1}^m \|E^{1,i}{}_\pi\|_{\varphi_1}\|E^{2,i}{}_\pi\|_{\varphi_1}\Big)^{\frac{1}{2}} \\
&\ge 2\Big\|\sum_{i=1}^m E^{1,i}{}_\pi E^{2,i}{}_\pi\Big\|_{\varphi_1}^{\frac{1}{2}} = 2\|E_\pi\|_{\varphi_1}^{\frac{1}{2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{i=1}^m \Big(\|E^{1,i}\|_1 + \|E^{2,i}\|_1\Big) &= \sum_{i=1}^m \Big(\sum_{\pi\in\widehat{K}} k_\pi\|E^{1,i}{}_\pi\|_{\varphi_1} + \sum_{\pi\in\widehat{K}} k_\pi\|E^{2,i}{}_\pi\|_{\varphi_1}\Big) \\
&= \sum_{\pi\in\widehat{K}} k_\pi\Big(\sum_{i=1}^m(\|E^{1,i}{}_\pi\|_{\varphi_1} + \|E^{2,i}{}_\pi\|_{\varphi_1})\Big) \\
&\ge 2\sum_{\pi\in\widehat{K}} k_\pi \|E_\pi\|_{\varphi_1}^{\frac{1}{2}} = \infty,
\end{aligned}$$

and so for some $1 \le i \le m$, and $j = 1,2$, $E^{j,i} \notin \mathfrak{E}_1(\widehat{K})$). Therefore E is not in the linear span of $\mathfrak{E}_1(\widehat{K})\mathfrak{E}_1(\widehat{K})$, and so f is not in the linear span of $A(K) * A(K)$. ∎

Lemma 4.2. *Let K be a compact hypergroup. Then the following statements are valid:*

(i) *For* $1 < p < 2$, $L^p(K) * L^p(K) \subseteq L^{\frac{p}{2-p}}(K)$.

(ii) *For* $2 \leq p \leq \infty$, $L^p(K) * L^p(K) \subseteq L^2(K) * L^2(K) = A(K)$.

(iii) $C(K) * C(K) \subseteq L^2(K) * L^2(K) = A(K)$.

(iv) *For* $f \in A(K)$, *and* $1 \leq p \leq \infty$, $\|f\|_p \leq \|f\|_\infty \leq \|f\|_{\varphi_1}$.

Proof. (i): Since for each $x, y \in K$, $\varepsilon_x * \varepsilon_y$ is a probability measure, so for $f \in L^p(K)$

$$\begin{aligned} |f(x*y)|^p &= \left| \int_K f d(\varepsilon_x * \varepsilon_y) \right|^p \leq \left(\int_K |f| d(\varepsilon_x * \varepsilon_y) \right)^p \\ &\leq \int_K |f|^p d(\varepsilon_x * \varepsilon_y) = |f|^p(x*y), \end{aligned}$$

and hence $\int_K |f(x*y)|^p dy \leq \|f\|_p^p$. Now, an exact method as the proof of Theorem 20.18 of [3], proves (i).

(ii),(iii): Since K is compact, so for each $2 \leq p \leq \infty$, $C(K) \subseteq L^p(K) \subseteq L^2(K)$, and by Theorem 4.9 of [9], $A(K) = L^2(K) * L^2(K)$. Hence (ii),(iii) are valid.

(iv): Since $\omega_K(K) = 1$ and $A(K) \subseteq L^p(K)$ for $1 \leq p \leq \infty$, so $\|f\|_p \leq \|f\|_\infty$ for every $f \in A(K)$. By Proposition 4.2 of [9], $\|f\|_\infty \leq \|f\|_{\varphi_1}$ $(f \in A(K))$. ∎

Theorem 4.3. *Let K be a compact hypergroup and $\mathfrak{A}$ be any of Banach spaces $(L^p(K), *)$ $(1 < p \leq \infty)$, and $(C(K), *)$. Then $\mathfrak{A}$ is essentially amenable. Moreover $\mathfrak{A}$ is amenable if and only if K is finite.*

Proof. Let $1 < p < 2$ and $\mathfrak{A} = L^p(K)$. There is $m \in \mathbb{N}$ such that

$$1 + \frac{1}{2^{m+1} - 1} \leq p < 1 + \frac{1}{2^m - 1}.$$

So $\frac{p}{2^{m-1} - (2^{m-1} - 1)p} < 2 \leq \frac{p}{2^m - (2^m - 1)p}$. Now, by induction and using Lemma 4.2(i) we have

$$\mathfrak{A}^{2^m} \subseteq L^{\frac{p}{2^m - (2^m - 1)p}}(K) \subseteq L^2(K).$$

Hence

$$\mathfrak{A}^{2^{m+1}} = \mathfrak{A}^{2^m} * \mathfrak{A}^{2^m} \subseteq L^2(K) * L^2(K) = A(K).$$

If $\mathfrak{A}$ is any of Banach spaces $(L^p(K), *)$ $(2 \leq p \leq \infty)$, and $(C(K), *)$, then by Lemma 3.4(ii),(iii)

$$\mathfrak{A}^2 = \mathfrak{A} * \mathfrak{A} \subseteq L^2(K) * L^2(K) = A(K).$$

Let $\mathfrak{B} = (A(K), *)$. By Lemma 4.2(iv), $\|f\|_{\mathfrak{A}} \leq \|f\|_{\mathfrak{B}}$ $(f \in \mathfrak{B})$. Since by Theorem 3.3 $\mathfrak{B}$ is essentially amenable, from Theorem 2.1 it follows that $\mathfrak{A}$ is essentially amenable.

If $\mathfrak{A}$ is amenable, then by Proposition 2.2.1 of [8] and Cohn's Factorization Theorem $\mathfrak{A} * \mathfrak{A} = \mathfrak{A}$. Hence for each $m \in \mathbb{N}$, $\mathfrak{A}^m = \mathfrak{A}$. But in the first paragraph we proved that there exists $m \in \mathbb{N}$ such that $\mathfrak{A}^m \subseteq A(K)$. Therefore $\mathfrak{A} \subseteq A(K)$. Clearly $A(K) \subseteq \mathfrak{A}$. Hence $A(K) = \mathfrak{A}$, and so

$$A(K) * A(K) = \mathfrak{A} * \mathfrak{A} = \mathfrak{A} = A(K).$$

Now, from Lemma 4.1, K is finite.

Conversely, if K is finite, then $\mathfrak{A}$ is an essentially amenable Banach algebra with the identity ε_e. Hence by Proposition 2.1.5 of [8], $\mathfrak{A}$ is amenable. ∎

Acknowledgements. The authors would like to express their deep gratitude to the referees for their careful reading of the earlier version of the manuscript and several insightful comments. The first author also wishes to thank both The Center of Excellence for Mathematics and The Research Affairs (Research Project No. 850709) of the University of Isfahan for their financial supports. The second author wishes to thank the University of Bu-Ali Sina for moral support.

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