Inner invariant means on locally compact topological semigroups*

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Abstract

Let $\mathcal{S}$ be a locally compact semigroup and $M_a(\mathcal{S})$ be its semigroup algebra. In this paper, we investigate inner invariant means on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ of all $M_a(\mathcal{S})$-measurable complex-valued bounded functions on $\mathcal{S}$ and its closed subspace $C_b(\mathcal{S})$, the space of all bounded continuous complex-valued functions on $\mathcal{S}$. We also study topological inner invariant means on certain closed subspaces X of $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ and their relation with inner invariant means on X.

1 Introduction

Throughout this paper, $\mathcal{S}$ denotes a *locally compact semigroup*; i.e., a semigroup with a locally compact Hausdorff topology whose binary operation is jointly continuous. The space of all bounded complex regular Borel measures on $\mathcal{S}$ is denoted by $M(\mathcal{S})$. This space with the convolution multiplication $*$ and the total variation norm defines a Banach algebra. The space of all measures $\mu \in M(\mathcal{S})$ for which the maps $s \longmapsto \delta_s * |\mu|$ and $s \longmapsto |\mu| * \delta_s$ from $\mathcal{S}$ into $M(\mathcal{S})$ are weakly continuous is denoted by $M_a(\mathcal{S})$ (or $\widetilde{L}(\mathcal{S})$ as in [2]), where δ_s denotes the Dirac measure at s. It is well-known that $M_a(\mathcal{S})$ is a closed two-sided L-ideal of $M(\mathcal{S})$; see [2].

*This research was in part supported by a grant from IPM (No. 81430032), the Institute for Studies in Theoretical Physics and Mathematics.

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Received by the editors November 2006 - In revised form in October 2007.

Communicated by F. Bastin.

2000 *Mathematics Subject Classification* : Primary 43A07, Secondary 43A10, 43A15, 43A20.

Key words and phrases : Inner invariant mean; locally compact semigroup; topological inner invariant means.

Denote by $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ the set of all complex-valued bounded functions g on $\mathcal{S}$ that are $M_a(\mathcal{S})$-measurable; that is, μ-measurable for all $\mu \in M_a(\mathcal{S})$. We identify functions in $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ that agree μ-almost everywhere for all $\mu \in M_a(\mathcal{S})$. For every $g \in L^\infty(\mathcal{S}; M_a(\mathcal{S}))$, define

$$\| g \|_\infty = \sup\{ \|g\|_{\infty,|\mu|} : \mu \in M_a(\mathcal{S}) \},$$

where $\|.\|_{\infty,|\mu|}$ denotes the essential supremum norm with respect to $|\mu|$. Observe that $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ with the complex conjugation as involution, the pointwise operations and the norm $\|.\|_\infty$ is a commutative C^*-algebra. Let X be a subspace of $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ which is *left and right translations invariant*; that is, ${}_sg$ and g_s are in X for all $g \in X$ and $s \in \mathcal{S}$, where

$$({}_sg)(t) = g(st) \quad \text{and} \quad (g_s)(t) = g(ts)$$

for all $t \in \mathcal{S}$. A linear functional F on X is called *inner invariant* whenever

$$F({}_sg) = F(g_s)$$

for all $s \in \mathcal{S}$ and $g \in X$. Recall that a bounded linear functional m with norm one on X is said to be a *mean* if $m(g) \geq 0$ for all $g \in X$ with $g \geq 0$.

The study of inner invariant means was initiated by Effros [10] and pursued by Akemann [1], H. Choda and M. Choda [5], M. Choda [6, 7] for discrete groups, Lau and Paterson [16] and [17], Losert and Rindler [19], Yuan [28] for locally compact groups, and by Ling [18] and the authors [21] for discrete semigroups.

In this paper, we investigate inner invariant means on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ and its closed subspace $C_b(\mathcal{S})$ of all bounded continuous complex-valued functions on $\mathcal{S}$. We also study topological inner invariant means on certain closed subspaces X of $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ and their relation with inner invariant means on X.

2 Topological inner invariant means

Given any $\mu \in M_a(\mathcal{S})$ and $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$, define the complex-valued functions $g \circ \mu$ and $\mu \circ g$ on $\mathcal{S}$ by

$$(g \circ \mu)(s) = \mu({}_sg) \quad \text{and} \quad (\mu \circ g)(s) = \mu(g_s)$$

for all $s \in \mathcal{S}$. It is clear that

$$(g \circ \mu)(s) = (\delta_x * \mu)(g) \quad \text{and} \quad (\mu \circ g)(s) = (\mu * \delta_x)(g)$$

and so $g \circ \mu$ and $\mu \circ g$ are in $C_b(\mathcal{S})$ with

$$\|g \circ \mu\|_\infty \leq \|g\|_\infty \, \|\mu\| \quad \text{and} \quad \|\mu \circ g\|_\infty \leq \|g\|_\infty \, \|\mu\|.$$

A closed subspace X of $L^\infty(\mathcal{S}; M_a(\mathcal{S}))$ is called *topologically invariant* if

$$X \circ M_a(\mathcal{S}) \subseteq X \quad \text{and} \quad M_a(\mathcal{S}) \circ X \subseteq X.$$

Let $LUC(\mathcal{S})$ (resp. $RUC(\mathcal{S})$) be the space of all *left* (resp. *right*) *uniformly continuous functions* on $\mathcal{S}$; recall that a function $f \in C_b(\mathcal{S})$ is called left (resp. right) uniformly continuous if the mapping $s \longmapsto {}_sf$ (resp. $s \longmapsto f_s$) from $\mathcal{S}$ into $C_b(\mathcal{S})$ is $\|.\|_\infty$-continuous. Also, a function $f \in C_b(\mathcal{S})$ is called *uniformly continuous* if f is in

$$UC(\mathcal{S}) := LUC(\mathcal{S}) \cap RUC(\mathcal{S}).$$

It follows from the equalities

$${}_s(f \circ \mu) = {}_sf \circ \mu \quad \text{and} \quad (\mu \circ f)_s = \mu \circ f_s$$

for all $s \in \mathcal{S}, f \in C_b(\mathcal{S})$ and $\mu \in M_a(S)$ that

$$LUC(\mathcal{S})) \circ M_a(\mathcal{S}) \subseteq LUC(\mathcal{S}) \quad \text{and} \quad M_a(\mathcal{S}) \circ RUC(\mathcal{S}) \subseteq RUC(\mathcal{S}).$$

Before we state the following lemma which is needed in the sequel, let us recall that $\mathcal{S}$ is called *foundation semigroup* if $\bigcup\{\text{supp}(\mu) : \mu \in M_a(\mathcal{S})\}$ is dense in $\mathcal{S}$. Foundation semigroups form a large class of locally compact semigroups which includes locally compact groups and discrete semigroups as elementary examples; as another example, consider the semigroup $\mathcal{S} := [0, 1]$ with the usual topology of the real line and the operation $xy = \min\{x + y, 1\}$ defines a compact foundation semigroup with identity; indeed,

$$M_a(\mathcal{S}) = L^1([0, 1]) \oplus \mathbb{C}\,\delta_1.$$

Moreover, the additive semigroup $\mathcal{S} := \mathbb{R}^+$ of all non-negative real numbers with the usual topology defines a non-compact foundation semigroup with identity; indeed,

$$M_a(\mathcal{S}) = L^1(\mathbb{R}^+).$$

Also, the multiplicative semigroup $\mathcal{S} := \{0, 1, 1/2, 1/3, ...\}$ with the restriction of the usual topology of the real line defines a compact foundation semigroup with identity; indeed,

$$M_a(\mathcal{S}) = \ell^1(\mathcal{S} \setminus \{0\}).$$

Lemma 2.1. *Let $\mathcal{S}$ be a foundation semigroup with identity. If X and Y are closed subspaces of $L^\infty(\mathcal{S}; M_a(\mathcal{S}))$ such that $UC(\mathcal{S}) \subseteq X \subseteq LUC(\mathcal{S})$ and $UC(\mathcal{S}) \subseteq Y \subseteq RUC(\mathcal{S})$. Then*

$$X \subseteq M_a(\mathcal{S}) \circ X \subseteq LUC(\mathcal{S}) \quad \textit{and} \quad Y \subseteq Y \circ M_a(\mathcal{S}) \subseteq RUC(\mathcal{S}).$$

In particular, $UC(\mathcal{S})$, $LUC(\mathcal{S})$ and $RUC(\mathcal{S})$ are topologically invariant.

Proof. Let $f \in Y$ and $\mu \in M_a(\mathcal{S})$. It follows from the hypothesis that the map $x \mapsto \mu * \delta_x$ from $\mathcal{S}$ into $M_a(\mathcal{S})$ is norm continuous; see [9], Theorem 5.6.1. This together with

$$(f \circ \mu)_x = f \circ (\mu * \delta_x) \quad (x \in \mathcal{S})$$

imply that $f \circ \mu \in RUC(\mathcal{S})$. It follows that $Y \circ M_a(\mathcal{S}) \subseteq RUC(\mathcal{S})$, in particular,

$$RUC(\mathcal{S}) \circ M_a(\mathcal{S}) \subseteq RUC(\mathcal{S}),$$

and thus $RUC(\mathcal{S})$ is topologically invariant.

On the other hand, for every $\varepsilon > 0$, there is a neighbourhood U of the identity element of $\mathcal{S}$ such that

$$\|f_x - f\|_\infty < \varepsilon \quad (x \in U).$$

Since $\mathcal{S}$ is foundation, there exists a probability measure e_0 in $M_a(\mathcal{S})$ with supp $(e_0) \subseteq U$. Then

$$\|f \circ e_0 - f\|_\infty \leq \varepsilon.$$

Now, let $(e_\gamma)_{\gamma\in\Gamma}$ be an approximate identity for $M_a(\mathcal{S})$ bounded by one; see [12], Lemma 2.1. Then for each $\gamma \in \Gamma$ we have

$$\begin{aligned}
\|f \circ e_\gamma - f\|_\infty &\leq \|f \circ e_\gamma - (f \circ e_0) \circ e_\gamma\|_\infty \\
&+ \|(f \circ e_0) \circ e_\gamma - f \circ e_0\|_\infty + \|f \circ e_0 - f\|_\infty \\
&\leq \|f - f \circ e_0\|_\infty \\
&+ \|f \circ (e_0 * e_\gamma - e_0)\|_\infty + \|f \circ e_0 - f\|_\infty \\
&\leq 2\,\varepsilon + \|f\|_\infty\, \|e_0 * e_\gamma - e_0\|.
\end{aligned}$$

It follows that $\|f \circ e_\gamma - f\|_\infty \to 0$. This together with the Cohen factorization theorem imply that $Y \subseteq Y \circ M_a(\mathcal{S})$; see [11], Theorem 32.5. The proof of the other inclusions are similar. ∎

Let us point out that the second dual $M_a(\mathcal{S})^{**}$ of $M_a(\mathcal{S})$ is a Banach algebra with the first Arens product $\odot$ defined by the equations

$$\begin{aligned}
(F \odot H)(f) &= F(Hf), \\
(Hf)(\mu) &= H(f\mu), \\
(f\mu)(\nu) &= f(\mu * \nu)
\end{aligned}$$

for all $F, H \in M_a(\mathcal{S})^{**}, f \in M_a(\mathcal{S})^*$, and $\mu, \nu \in M_a(\mathcal{S})$. In the case where, $\mathcal{S}$ is a foundation semigroup with identity, $M_a(\mathcal{S})^*$ can be identified with $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$; in fact, the equation

$$\tau(g)(\mu) := \mu(g) = \int_{\mathcal{S}} f\, d\mu$$

defines an isometric isomorphism τ of $L^\infty(\mathcal{S}; M_a(\mathcal{S}))$ into the continuous dual space $M_a(\mathcal{S})^*$ of $M_a(\mathcal{S})$; see Proposition 3.6 of Sleijpen [27]. Moreover, for each $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ and $\mu \in M_a(\mathcal{S})$,

$$\tau(g \circ \mu) = \mu\, \tau(g) \qquad \text{and} \quad \tau(\mu \circ g) = \tau(g)\, \mu.$$

Let X be a topologically invariant closed subspace of $L^\infty(\mathcal{S}; M_a(\mathcal{S}))$ containing the constant functions and m be a mean on X; i.e., $\|m\| = m(1) = 1$. Recall from [14] that m is *topological inner invariant* on X whenever

$$\mu \odot m = m \odot \mu \qquad (\mu \in M_a(\mathcal{S}));$$

or equivalently

$$m(\mu \circ g) = m(g \circ \mu) \qquad (\mu \in M_a(\mathcal{S}), g \in X.)$$

The notion of topological inner invariant means was introduced and studied by the second author [23] for a large class of Banach algebras known as Lau algebras. The subject of Lau algebras originated with the paper [15] published in 1983 by Lau in which he referred to them as F-algebras. Later on, in his useful monograph, Pier [25] introduced the name Lau algebra. Let us remark from [22] that $M_a(\mathcal{S})$ is a Lau algebra for all foundation semigroups S with identity; in this case, any mixed identity with norm one in $M_a(\mathcal{S})^{**}$ is a topological inner invariant mean on $L^\infty(S, M_a(\mathcal{S}))$.

Proposition 2.2. *Let $\mathcal{S}$ be a foundation semigroup with identity. Then any topological inner invariant mean on $UC(\mathcal{S})$, $LUC(\mathcal{S})$, or $RUC(\mathcal{S})$ is inner invariant.*

Proof. Let m be a topological inner invariant mean on $LUC(\mathcal{S})$. By Lemma 2.1, for each f in $LUC(\mathcal{S})$ we have $f = \mu \circ g$ for some $\mu \in M_a(\mathcal{S})$ and $g \in LUC(\mathcal{S})$. Since for each $s \in \mathcal{S}$,

$$\begin{aligned} {}_s(\mu \circ g) &= (\mu * \delta_s) \circ g \\ g \circ (\mu * \delta_s) &= g_s \circ \mu \\ \mu \circ g_s &= (\mu \circ g)_s \end{aligned}$$

we conclude

$$\begin{aligned} m({}_s f) &= m({}_s(\mu \circ g)) \\ &= m((\mu * \delta_s) \circ g) \\ &= m(g \circ (\mu * \delta_s)) \\ &= m(g_s \circ \mu) \\ &= m(\mu \circ g_s) \\ &= m((\mu \circ g)_s) \\ &= m(f_s). \end{aligned}$$

That is, m is inner invariant on $LUC(\mathcal{S})$. Similar arguments hold for $RUC(\mathcal{S})$ and $UC(\mathcal{S})$. ■

As a consequence of Proposition 2.2 we have the following improvement of Theorem 3.1 of [20] from locally compact groups to a large class of locally compact semigroups; see also [13] and [14].

Corollary 2.3. *Let $\mathcal{S}$ be a foundation semigroup with identity and m be a mean on $UC(\mathcal{S})$. Then m is inner invariant if and only if it is topological inner invariant.*

Proof. The "if" part follows from Proposition 2.2. To prove the converse, let m be an inner invariant mean on $UC(\mathcal{S})$. Then there is a net $(m_\gamma)_{\gamma\in\Gamma}$ in $UC(\mathcal{S})^*$ such that $m_\gamma \to m$ in the weak* topology of $UC(\mathcal{S})^*$ and

$$m_\gamma = \sum_{i=1}^{n_\gamma} c_{i,\gamma}\, \delta_{s_{i,\gamma}} \quad (\gamma \in \Gamma),$$

where $s_{i,\gamma} \in \mathcal{S}$, $c_{i,\gamma}$ are complex numbers with

$$\sum_{i=1}^{n_\gamma} |c_{i,\gamma}| \leq 1;$$

see for example [8], page 417, Theorem 10. Now, let $f \in UC(\mathcal{S})$ and $\mu \in M_a(\mathcal{S})$ be a measure with compact support C. Then the sets

$$\{{}_sf : s \in C\} \qquad \text{and} \quad \{f_s : s \in C\}$$

are norm compact in $UC(\mathcal{S})$, and therefore

$$m_\gamma({}_sf) \to m({}_sf) \qquad \text{and} \quad m_\gamma(f_s) \to m(f_s)$$

uniformly on C by the Makey-Arens theorem. We thus have

$$\begin{aligned}
m(f \circ \mu) &= \lim_\gamma m_\gamma(f \circ \mu) \\
&= \lim_\gamma \sum_{i=1}^{n_\gamma} c_{i,\gamma} \delta_{s_{i,\gamma}}(f \circ \mu) \\
&= \lim_\gamma \sum_{i=1}^{n_\gamma} c_{i,\gamma} \int_{\mathcal{S}} \delta_{s_{i,\gamma}}(f_s)\, d\mu(s) \\
&= \lim_\gamma \int_{\mathcal{S}} m_\gamma(f_s)\, d\mu(s) \\
&= \int_{\mathcal{S}} m(f_s)\, d\mu(s) \\
&= \int_{\mathcal{S}} m({}_sf)\, d\mu(s) \\
&= \lim_\gamma \int_{\mathcal{S}} m_\gamma({}_sf)\, d\mu(s) \\
&= \lim_\gamma \sum_{i=1}^{n_\gamma} c_{i,\gamma} \int_{\mathcal{S}} \delta_{s_{i,\gamma}}({}_sf)\, d\mu(s) \\
&= \lim_\gamma \sum_{i=1}^{n_\gamma} c_{i,\gamma}\, \delta_{s_{i,\gamma}}(\mu \circ f) \\
&= \lim_\gamma m_\gamma(\mu \circ f) \\
&= m(\mu \circ f).
\end{aligned}$$

Since measures with compact supports are norm dense in $M_a(\mathcal{S})$, it follows that m is topological inner invariant on $UC(\mathcal{S})$. ∎

Let us remark that an element $E \in M_a(\mathcal{S})^{**}$ is called a *mixed identity* if

$$\mu \odot E = E \odot \mu \qquad (\mu \in M_a(\mathcal{S})).$$

It well-known from [3], page 146, that an element $E \in M_a(\mathcal{S})^{**}$ is a mixed identity with norm one if and only if it is a weak* cluster point of an approximate identity bounded by one in $M_a(\mathcal{S})$; see [14], Theorem 2.3, for other descriptions of mixed identities with norm one in $M_a(\mathcal{S})^{**}$.

Moreover, note that if $\mathcal{S}$ is a foundation semigroup with identity, then for every $F \in M_a(\mathcal{S})^{**}$ and $n \in LUC(\mathcal{S})^*$, the functional $F \odot n$ can be defined as an element of $M_a(\mathcal{S})^{**}$ in a way similar to the first Arens product; this is because that $\mu \circ g \in LUC(\mathcal{S})$ for all $\mu \in M_a(\mathcal{S})$ and $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$; see Lemma 2.1 of [12].

The next proposition should be compared with the corresponding result concerning topological left invariant means; see Theorem 4.2.4 of [9]. It should be noted that the standard technic used in [9] does not work in our setting.

Proposition 2.4. *Let $\mathcal{S}$ be a foundation semigroup with identity. If m is a topological inner invariant mean on $LUC(\mathcal{S})$, then $E \odot m$ is a topological inner invariant mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ for all mixed identities E in $M_a(\mathcal{S})^{**}$ with norm one.*

Proof. Let $E \in M_a(\mathcal{S})^{**}$ be a mixed identity with norm one, and (e_γ) be an approximate identity for $M_a(\mathcal{S})$ bonded by one such that e_γ converges to E in the weak* topology of $M_a(\mathcal{S})^{**}$. Then for $\mu \in M_a(\mathcal{S})$ and $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ we have

$$\|e_\gamma \circ (\mu \circ g) - \mu \circ g\|_\infty = \|(e_\gamma * \mu - \mu) \circ g\|_\infty \to 0$$

and

$$\|\mu \circ (e_\gamma \circ g) - \mu \circ g\|_\infty = \|(\mu * e_\gamma - \mu) \circ g\|_\infty \to 0.$$

Now, suppose that m is a topological inner invariant mean on $LUC(\mathcal{S})$. Then

$$\lim_\gamma m(e_\gamma \circ (\mu \circ g)) = \lim_\gamma m(\mu \circ (e_\gamma \circ g)).$$

Since $M_a(\mathcal{S}) \circ L^\infty(\mathcal{S}, M_a(\mathcal{S})) \subseteq LUC(\mathcal{S})$, it follows that $e_\gamma \circ g \in LUC(\mathcal{S})$ for all γ and thus

$$\begin{aligned} m(\mu \circ (e_\gamma \circ g)) &= m((e_\gamma \circ g) \circ \mu) \\ &= m(e_\gamma \circ (g \circ \mu)) \\ &= (e_\gamma \odot m)(g \circ \mu). \end{aligned}$$

This shows that

$$\begin{aligned} \lim_\gamma (e_\gamma \odot m)(\mu \circ g) &= \lim_\gamma m(e_\gamma \circ (\mu \circ g)) \\ &= \lim_\gamma m(\mu \circ (e_\gamma \circ g)) \\ &= \lim_\gamma (e_\gamma \odot m)(g \circ \mu). \end{aligned}$$

Since $e_\gamma \odot m$ converges to $E \odot m$ in the weak* topology of $M_a(\mathcal{S})^{**}$, we get

$$\begin{aligned} (E \odot m)(\mu \circ g) &= \lim_\gamma (e_\gamma \odot m)(\mu \circ g) \\ &= \lim_\gamma (e_\gamma \odot m)(g \circ \mu) \\ &= (E \odot m)(g \circ \mu). \end{aligned}$$

This implies that $E \odot m$ is a topological inner invariant mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ and the proof is complete. ■

The following result is of independent interest.

Proposition 2.5. *Let $\mathcal{S}$ be a foundation semigroup with identity. If m is a topological inner invariant mean on $C_b(\mathcal{S})$, then $m(\mu \circ g) = m(g \circ \mu)$ for all $\mu \in M_a(\mathcal{S})$ and $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. In particular, any extension of m to a mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ is topological inner invariant.*

Proof. Let $\mu, \nu \in M_a(\mathcal{S})$ and $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. Then $\nu \circ g,\ \ g \circ \mu \in C_b(\mathcal{S})$, and hence we have

$$\begin{aligned} m((\nu * \mu) \circ g) &= m(\mu \circ (\nu \circ g)) \\ &= m((\nu \circ g) \circ \mu) \\ &= m(\nu \circ (g \circ \mu) \\ &= m((g \circ \mu) \circ \nu) \\ &= m(g \circ (\nu * \mu)). \end{aligned}$$

Now, let (e_γ) be an approximate identity for $M_a(\mathcal{S})$. Then for each γ,

$$\lim_\gamma m((e_\gamma * \mu) \circ g) = \lim_\gamma m(g \circ (e_\gamma * \mu)).$$

Also,

$$\|(e_\gamma * \mu) \circ g - \mu \circ g\|_\infty \to 0$$

and

$$\|g \circ (e_\gamma * \mu) - g \circ \mu\|_\infty \to 0.$$

It follows that

$$\begin{aligned} m(\mu \circ g) &= \lim_\gamma m((e_\gamma * \mu) \circ g) \\ &= \lim_\gamma m(g \circ (e_\gamma * \mu)) \\ &= m(g \circ \mu). \end{aligned}$$

Therefore, if M is an extension of m from $C_b(\mathcal{S})$ to a mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$, then $M(\mu \circ g) = M(g \circ \mu)$. Thus M defines a topological inner invariant mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. ■

3 Inner invariant means on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$

In this section we shall be concerned with the inner invariant means on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ for a locally compact semigroup $\mathcal{S}$. Before, we give our first result, let us recall that a subset A of $\mathcal{S}$ is called $M_a(\mathcal{S})$-measurable if it is μ-measurable for all $\mu \in M_a(\mathcal{S})$.

Lemma 3.1. *Let $\mathcal{S}$ be a left (resp. right) cancellative locally compact semigroup such that xA (resp. Ax) are $M_a(\mathcal{S})$-measurable for all $M_a(\mathcal{S})$-measurable subset A of $\mathcal{S}$. Then the space of inner invariant functionals on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ is the linear span of inner invariant means.*

Proof. Let F be an inner invariant functional on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. We have to show that F is a linear span of some inner invariant means on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. Without loss of generality we may assume that F is nonzero and self-adjoint. In view of 1.14.3 of [26], there are unique positive functionals F^+ and F^- on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ such that

$$F = F^+ - F^- \quad \text{and} \quad \|F\| = \|F^+\| + \|F^-\|.$$

The result then will follow if we show that F^+ and F^- are inner invariant functionals. This is because that if F^+ (resp. F^-) is nonzero, then the mean $F^+(1)^{-1}F^+$ (resp. $F^-(1)^{-1}F^-$) is inner invariant.

To this end, let $s \in \mathcal{S}$, and $s.F$ be the linear functional on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ defined by

$$(s.F)(g) = F({}_sg) \quad (g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))).$$

Then there are unique positive functionals $(s.F)^+$ and $(s.F)^-$ on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ such that

$$s.F = (s.F)^+ - (s.F)^- \quad \text{and} \quad \|s.F\| = \|(s.F)^+\| + \|(s.F)^-\|.$$

We show that $(s.F)^+ = s.F^+$ and $(s.F)^- = s.F^-$. By uniqueness and that $s.F = s.F^+ - s.F^-$ we only need to prove that

$$\|(s.F)^+\| = \|s.F^+\| \quad \text{and} \quad \|(s.F)^-\| = \|s.F^-\|.$$

Using the fact that

$$\begin{aligned}
\|(s.F)^+\| = (s.F)^+(1) &= \sup\{F({}_sg) : g \in L^\infty(\mathcal{S}, M_a(\mathcal{S})), \ 0 \le g \le 1\} \\
\|(s.F)^-\| = (s.F)^-(1) &= -\inf\{F({}_sg) : g \in L^\infty(\mathcal{S}, M_a(\mathcal{S})), \ 0 \le g \le 1\}
\end{aligned}$$

and

$$\begin{aligned}
\|s.F^+\| = F^+(1) &= \sup\{F(g) : g \in L^\infty(\mathcal{S}, M_a(\mathcal{S})), \ 0 \le g \le 1\} \\
\|s.F^-\| = F^-(1) &= -\inf\{F(g) : g \in L^\infty(\mathcal{S}, M_a(\mathcal{S})), \ 0 \le g \le 1\}.
\end{aligned}$$

By the hypothesis it suffices to show that the two sets

$$\{F({}_sg) : g \in L^\infty(\mathcal{S}, M_a(\mathcal{S})), \ 0 \le g \le 1\}$$

and

$$\{F(g) : g \in L^\infty(\mathcal{S}, M_a(\mathcal{S})), \ 0 \le g \le 1\}$$

are the same. To see this, let $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ with $0 \le g \le 1$. If $\mathcal{S}$ is left cancellative, then for each $s \in \mathcal{S}$ and $t \in s\mathcal{S}$, let $s^{-1}t$ denote the unique element y of $\mathcal{S}$ for which $t = sy$. Now, we may define $g' : \mathcal{S} \longrightarrow \mathbb{C}$ for each $t \in \mathcal{S}$ by

$$g'(t) = \begin{cases} g(s^{-1}t) & t \in s\mathcal{S} \\ 0 & t \notin s\mathcal{S} \end{cases}$$

Then g' is well-defined. Moreover, for any open subset V of $\mathbb{C}$ we have

$$(g')^{-1}(V) = \begin{cases} s\,(\mathcal{S} \cap g^{-1}(V)) & 0 \notin V \\ s\,(\mathcal{S} \cap g^{-1}(V)) \cup (\mathcal{S} \setminus s\mathcal{S}) & 0 \in V \end{cases}$$

Thus g' is $M_a(\mathcal{S})$-measurable by the hypothesis and that $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. It follows that

$$g' \in L^\infty(\mathcal{S}, M_a(\mathcal{S})), \quad \text{with} \quad 0 \le g' \le 1 \quad \text{and} \quad g =_s g'.$$

In the case where $\mathcal{S}$ is right cancellative, in a similar way it can be proved that there is

$$g' \in L^\infty(\mathcal{S}, M_a(\mathcal{S})), \quad \text{with} \quad 0 \le g' \le 1 \quad \text{and} \quad g = g'_s.$$

In both cases, since F is inner invariant, we have $F({}_sg') = F(g'_s)$.

By a similar argument we have $(F.s)^+ = F^+.s$ and $(F.s)^- = F^-.s$ for all $s \in \mathcal{S}$, where $F.s$ is the linear functional on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ defined by

$$(F.s)(g) = F(g_s) \quad (g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))).$$

Therefore

$$s.F^+ = (s.F)^+ = (F.s)^+ = F^+.s,$$

and

$$s.F^- = (s.F)^- = (F.s)^- = F^-.s.$$

That is F^+ and F^- are inner invariant as required. ∎

In the next theorem, we denote by $\mathcal{H}(\mathcal{S})$ (resp. $\mathcal{H}_{\mathbb{R}}(\mathcal{S})$) the complex (resp. real) linear span of functions of the form ${}_sg - g_s$ for some $s \in \mathcal{S}$ and complex-valued (resp. real-valued) functions $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$.

Theorem 3.2. *Let $\mathcal{S}$ be a locally compact semigroup and consider the following statements.*

(a) *There is an inner invariant mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$.*

(b) $\sup\{h(s) : s \in \mathcal{S}\} \ge 0$ *for all $h \in \mathcal{H}_{\mathbb{R}}(\mathcal{S})$.*

(c) $\inf\{\|1-h\|_\infty : h \in \mathcal{H}(\mathcal{S})\} = 1$.

(d) *$\mathcal{H}(\mathcal{S})$ is not norm dense in $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$.*

Then (a)$\Longleftrightarrow$(b)$\Longleftrightarrow$(c) $\Longrightarrow$(d). *If $\mathcal{S}$ is as in Lemma 3.1, then* (a)-(d) *are equivalent.*

Proof. (a)$\Longrightarrow$(b). If m is an inner invariant mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$, then for each $h \in \mathcal{H}_{\mathbb{R}}(\mathcal{S})$ we have

$$\sup\{h(s) : s \in \mathcal{S}\} \ge m(h) = 0.$$

(b)$\Longrightarrow$(c). Suppose on the contrary that $\inf\{\|1-h\|_\infty : h \in \mathcal{H}(\mathcal{S})\} < 1$. Then

$$\sup\{-\mathrm{Re}\, h(s) : s \in \mathcal{S}\} < 0$$

for some $h \in \mathcal{H}(\mathcal{S})$. This together with that $-\mathrm{Re}\, h \in \mathcal{H}_{\mathbb{R}}(\mathcal{S})$ contradict (b). Now, (c) follows from that $0 \in \mathcal{H}(\mathcal{S})$.

The implications (c)$\Longrightarrow$(d) and (c)$\Longrightarrow$(a) follow from the fact that by the Hahn-Banach theorem, there is $n \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))^*$ with norm one such that $n(\mathcal{H}(\mathcal{S})) = \{0\}$, and

$$n(1) = \inf\{\|1-h\|_\infty : h \in \mathcal{H}(\mathcal{S})\}.$$

The rest of the proof follows at once from Lemma 3.1. ∎

In the following result, let $L^\infty(\mathcal{G})$ be the usual Lebesgue space of all essentially bounded measurable functions on a locally compact group $\mathcal{G}$, and note that $L^\infty(\mathcal{G}) = L^\infty(\mathcal{G}, M_a(\mathcal{G}))$.

Corollary 3.3. *Let $\mathcal{G}$ be a locally compact group. Then the following statements are equivalent*

(a) *There is an inner invariant mean on $L^\infty(\mathcal{G})$.*
(b) $\sup\{h(x) : x \in \mathcal{G}\} \geq 0$ *for all $h \in \mathcal{H}_{\mathbb{R}}(\mathcal{G})$.*
(c) $\inf\{\|1-h\|_\infty : h \in \mathcal{H}(\mathcal{G})\} = 1$.
(d) *$\mathcal{H}(\mathcal{G})$ is not norm dense in $L^\infty(\mathcal{G})$.*

Before we give the next result, let us recall that a family $(A_\gamma)_{\gamma\in D}$ of sets is upward directed if D is a directed set and $A_\gamma \subseteq A_\beta$ when $\gamma \leq \beta$.

Proposition 3.4. *Let $(\mathcal{S}_\gamma)_{\gamma\in D}$ be an upward directed family of locally compact subsemigroups of a locally compact semigroup $\mathcal{S}$. If for each $\gamma \in D$, there exists an inner invariant mean on $L^\infty(\mathcal{S}_\gamma, M_a(\mathcal{S}_\gamma))$, then there exists an inner invariant mean on $L^\infty(\cup_{\gamma\in D}\mathcal{S}_\gamma, M_a(\cup_{\gamma\in D}\mathcal{S}_\gamma))$.*

Proof. By Theorem 3.2, we only need to note that if $h \in \mathcal{H}_{\mathbb{R}}(\cup_{\gamma\in D}S_\gamma)$, then $h \in \mathcal{H}_{\mathbb{R}}(\mathcal{S}_\gamma)$ for some $\gamma \in D$. ∎

As an immediate consequence of Proposition 3.4, we obtain

Corollary 3.5. *Let $\mathcal{S}$ be a locally compact semigroup. If there is an inner invariant mean on $L^\infty(\mathcal{S}_0, M_a(\mathcal{S}_0))$ for all finitely generated subsemigroups $\mathcal{S}_0$ of $\mathcal{S}$, then there is an inner invariant mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$.*

Let $\mathcal{S}_0$ be a subset of a locally compact semigroup $\mathcal{S}$. We say that a mean m on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ is *inner $\mathcal{S}_0$-invariant* if $m({}_xg) = m(g_x)$ for all $x \in \mathcal{S}_0$ and $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$.

Proposition 3.6. *Suppose that $\mathcal{S}_0$ is a closed subsemigroup of a locally compact semigroup $\mathcal{S}$. Then there exists an inner invariant mean on $L^\infty(\mathcal{S}_0, M_a(\mathcal{S}_0))$ if and only if there is an inner $\mathcal{S}_0$-invariant mean m on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ with $m(\chi_{\mathcal{S}_0}) = 1$.*

Proof. Suppose that n is an inner invariant mean on $L^\infty(\mathcal{S}_0, M_a(\mathcal{S}_0))$. Since $g|_{\mathcal{S}_0}$, the restriction of g to $\mathcal{S}_0$ belongs to $L^\infty(\mathcal{S}_0, M_a(\mathcal{S}_0))$ for all g in $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$, the map

$$m : g \longmapsto n(g|_{\mathcal{S}_0})$$

defines a mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. Moreover, $m(\chi_{\mathcal{S}_0}) = 1$ trivially, and also m is inner $\mathcal{S}_0$-invariant. Indeed, for each $t \in \mathcal{S}_0$ and $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ we have

$$\begin{aligned} m({}_tg - g_t) &= n(({}_tg - g_t)|_{\mathcal{S}_0}) \\ &= n({}_t(g|_{\mathcal{S}_0}) - (g|_{\mathcal{S}_0})_t). \end{aligned}$$

Conversely, suppose m is an inner $\mathcal{S}_0$-invariant mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ with $m(\chi_{\mathcal{S}_0}) = 1$. For every $f \in L^\infty(\mathcal{S}_0, M_a(\mathcal{S}_0))$, let $\widetilde{f} : \mathcal{S} \longrightarrow \mathbb{C}$ be the function which is equal to f on $\mathcal{S}_0$ and zero on $\mathcal{S} \setminus \mathcal{S}_0$. Since the restriction of μ to $\mathcal{S}_0$ is in $M_a(\mathcal{S}_0)$ for all $\mu \in M_a(\mathcal{S})$, it follows easily that $\widetilde{f}$ is $M_a(\mathcal{S})$-measurable. That is $\widetilde{f} \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. Thus the linear functional

$$n : f \longmapsto m(\widetilde{f})$$

defines a mean on $L^\infty(\mathcal{S}_0, M_a(\mathcal{S}_0))$. Furthermore, $\widetilde{(f_s)} = \widetilde{f}_s$ on $\mathcal{S}_0$ for all $s \in \mathcal{S}_0$ and $f \in L^\infty(\mathcal{S}_0, M_a(\mathcal{S}_0))$, and therefore,

$$|\widetilde{(f_s)} - \widetilde{f}_s| \leq \|\widetilde{(f_s)} - \widetilde{f}_s\|_\infty \, \chi_{\mathcal{S} \setminus \mathcal{S}_0}.$$

It follows that $n(\widetilde{(f_s)}) = n(\widetilde{f}_s)$. Similarly, $n(\widetilde{({}_s f)}) = n({}_s\widetilde{f})$. That is n is an inner invariant mean on $L^\infty(\mathcal{S}_0, M_a(\mathcal{S}_0))$ as required. ∎

4 Inner invariant means on $C_b(\mathcal{S})$

Let $\mathcal{S}$ be a locally compact semigroup. In the case where $\mathcal{S}$ has an identity e, $C_b(\mathcal{S})$ has always an inner invariant mean; in fact, δ_e is an inner invariant mean on $C_b(\mathcal{S})$. However, this is not true in general; for example, consider a left zero semigroup with at least two elements. In this section, we study the existence of inner invariant means on $C_b(\mathcal{S})$. Before we give our first result of this section, let us remind that a mean m on $C_b(\mathcal{S})$ is called *two-sided invariant* if

$$m({}_s f) = m(f_s) = m(f) \qquad (s \in \mathcal{S}, f \in C_b(\mathcal{S})).$$

Proposition 4.1. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two locally compact semigroups. If $C_b(\mathcal{S}_1)$ has a two-sided invariant mean and $C_b(\mathcal{S}_2)$ has an inner invariant mean, then $C_b(\mathcal{S}_1 \times \mathcal{S}_2)$ has an inner invariant mean.*

Proof. Let m be a two-sided invariant mean on $C_b(\mathcal{S}_1)$ and n be an inner invariant mean on $C_b(\mathcal{S}_2)$. For each $f \in C_b(\mathcal{S}_1 \times \mathcal{S}_2)$, define the function $f_2 \in C_b(\mathcal{S}_2)$ by

$$f_2(t) = m(f_1^t) \qquad (t \in \mathcal{S}_2),$$

where $f_1^t \in C_b(\mathcal{S}_1)$ is defined by

$$f_1^t(s) = f(s,t) \qquad (s \in \mathcal{S}_1).$$

It follows that

$$({}_{(x,y)}f\,)_1^t = {}_x(f_1^{yt}) \quad \text{and} \quad (f_{(x,y)})_1^t = (f_1^{ty})_x$$

for all $x \in \mathcal{S}_1$ and $y, t \in \mathcal{S}_2$. Moreover,

$$\begin{aligned} ({}_{(x,y)}f)_2(t) &= m(({}_{(x,y)}f\,)_1^t) \\ &= m({}_x(f_1^{yt})) \\ &= m(f_1^{yt}) \\ &= {}_y(f_2)(t), \end{aligned}$$

and

$$\begin{aligned}(f_{(x,y)})_2(t) &= m(f_{(x,y)})_1^t) \\ &= m((f_1^{ty})_x) \\ &= m(f_1^{ty}) \\ &= (f_2)_y(t).\end{aligned}$$

Therefore

$$({}_{(x,y)}f)_2 = {}_y(f_2) \quad \text{and} \quad (f_{(x,y)})_2 = (f_2)_y.$$

Now, define the mean M on $C_b(\mathcal{S}_1 \times \mathcal{S}_2)$ by $M(f) = n(f_2)$ for all $f \in C_b(\mathcal{S}_1 \times \mathcal{S}_2)$. Then for each $x \in \mathcal{S}_1$ and $y \in \mathcal{S}_2$,

$$\begin{aligned}M({}_{(x,y)}f) &= n(({}_{(x,y)}f)_2) \\ &= n({}_y(f_2)) \\ &= n((f_2)_y) \\ &= n((f_{(x,y)})_2) \\ &= M(f_{(x,y)}).\end{aligned}$$

That is M is an inner invariant mean on $C_b(\mathcal{S}_1 \times \mathcal{S}_2)$. ∎

Corollary 4.2. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be non-trivial locally compact semigroups with identities e_1 and e_2 respectively. Suppose that $C_b(\mathcal{S}_1)$ has a two-sided invariant mean and $C_b(\mathcal{S}_2)$ has an inner invariant mean. Then there is an inner invariant mean on $C_b(\mathcal{S}_1 \times \mathcal{S}_2)$ not equal to $\delta_{(e_1,e_2)}$.*

Proof. Let m be a two-sided invariant mean on $C_b(\mathcal{S}_1)$, and M be the inner invariant mean on $C_b(\mathcal{S}_1 \times \mathcal{S}_2)$ defined as in the proof of Proposition 4.1. Since $C_b(\mathcal{S}_1)$ separates the points of $\mathcal{S}_1$, $m(g) \neq g(e_1)$ for some $g \in C_b(\mathcal{S}_1)$. Now, define the function $f \in C_b(\mathcal{S}_1 \times \mathcal{S}_2)$ by $f(s_1, s_2) = g(s_1)$ for all $s_1 \in \mathcal{S}_1$ and $s_2 \in \mathcal{S}_2$. Then

$$M(f) = m(g) \neq g(e_1) = f(e_1, e_2).$$

Therefore $M \neq \delta_{(e_1,e_2)}$ as required. ∎

Let $\mathcal{S}_0$ be a subset of a locally compact semigroup $\mathcal{S}$. We say that a mean m on $C_b(\mathcal{S})$ is *inner $\mathcal{S}_0$-invariant* if $m({}_sg) = m(g_s)$ for all $s \in \mathcal{S}_0$ and $g \in C_b(\mathcal{S})$.

Proposition 4.3. *Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are two locally compact semigroups and θ is a continuous homomorphism from $\mathcal{S}_1$ into $\mathcal{S}_2$. If there is an inner invariant mean on $C_b(\mathcal{S}_1)$, then there is an inner $\theta(\mathcal{S}_1)$-invariant mean m on $C_b(\mathcal{S}_2)$.*

Proof. First note that if $s_1 \in \mathcal{S}_1$, then

$$({}_{s_2}g - g_{s_2}) \circ \theta ={}_{s_1} (g \circ \theta) - (g \circ \theta)_{s_1},$$

where $s_2 = \theta(s_1)$. Indeed, for each $s \in \mathcal{S}_1$ we have

$$\begin{aligned}[({}_{s_2}g - g_{s_2}) \circ \theta](s) &= g(\theta(s_1)\theta(s)) - g(\theta(s)\theta(s_1)) \\ &= g(\theta(s_1 s)) - g(\theta(s s_1)) \\ &= (g \circ \theta)(s_1 s) - (g \circ \theta)(s s_1) \\ &= {}_{s_1}(g \circ \theta)(s) - (g \circ \theta)_{s_1}(s).\end{aligned}$$

Now, suppose n is an inner invariant mean on $C_b(\mathcal{S}_1)$. Then the mean

$$m : g \longmapsto n(g \circ \theta)$$

is inner $\theta(\mathcal{S}_1)$-invariant on $C_b(\mathcal{S}_2)$. In fact, for each $s_2 \in \mathcal{S}_2$ with $s_2 = \theta(s_1)$ for some $s_1 \in \mathcal{S}_1$, we have

$$m({}_{s_2}g - g_{s_2}) = n({}_{s_1}(g \circ \theta) - (g \circ \theta)_{s_1}) = 0.$$

This establishes the proof. ■

Let $\mathcal{C}$ be a congruence relation on $\mathcal{S}$; that is, an equivalence relation such that $x\ \mathcal{C}\ y$ implies both $xs\ \mathcal{C}\ ys$ and $sx\ \mathcal{C}\ sy$ $(x, y, s \in \mathcal{S})$. We denote by $\mathcal{S}/\mathcal{C}$ the semigroup of all equivalence classes $x/\mathcal{C}$ $(x \in \mathcal{S})$ induced by $\mathcal{C}$ with the usual operation

$$(x/\mathcal{C})\ (y/\mathcal{C}) = xy/\mathcal{C} \quad (x, y \in \mathcal{S}).$$

The quotient space $\mathcal{S}/\mathcal{C}$ endowed with the quotient topology is in general not a locally compact semigroup in our sense; see [4], pages 46-50. Observe that if $\mathcal{C}$ is a congruence relation on a locally compact semigroup $\mathcal{S}$ such that $\mathcal{S}/\mathcal{C}$ is a locally compact semigroup, then the natural map $\phi : S \longmapsto S/\mathcal{C}$ is a continuous homomorphism.

Corollary 4.4. *Let $\mathcal{C}$ be a closed congruence relation such that $\mathcal{S}/\mathcal{C}$ is a locally compact semigroup. If there is an inner invariant mean on $C_b(\mathcal{S})$, then there is an inner invariant mean on $C_b(\mathcal{S}/\mathcal{C})$.*

Proof. The canonical map $s \longmapsto s/\mathcal{C}$ from $\mathcal{S}$ onto $\mathcal{S}/\mathcal{C}$ is a continuous homomorphism. So the result follows from Proposition 4.3 ■

Corollary 4.5. *Let $\mathcal{S}$ be a σ-compact locally compact semigroup and $\mathcal{S}_0$ be a closed ideal of $\mathcal{S}$. If there is an inner invariant mean on $C_b(\mathcal{S})$, then there is an inner invariant mean on $C_b(\mathcal{S}/\mathcal{S}_0)$.*

Proof. From Theorem 1.57 of [4], it follows that $\mathcal{S}/\mathcal{S}_0$ is a locally compact semigroup. So, the result follows from Proposition 4.3 and Corollary 4.4. ■

Let $\{\mathcal{S}_i : i \in I\}$ be a family of locally compact semigroups. *The full direct product* $\Pi_{i \in I}\mathcal{S}_i$ of $\{\mathcal{S}_i : i \in I\}$ is the set of all functions ϕ defined on I with $\phi(i) \in \mathcal{S}_i$ for $i \in I$. Note that $\Pi_{i \in I}\mathcal{S}_i$ equipped with the binary operation $(\phi, \psi) \mapsto \phi \cdot \psi$ defined by

$$(\phi \cdot \psi)(i) = \phi(i)\ \psi(i) \quad (i \in I)$$

is a semigroup. Moreover, if every $\mathcal{S}_i$ is a locally compact semigroup, then $\Pi_{i \in I}\mathcal{S}_i$ together with the product topology is also a locally compact semigroup.

Corollary 4.6. *Suppose that $\{\mathcal{S}_i : i \in I\}$ is a family of locally compact semigroups. If there is an inner invariant mean on $C_b(\Pi_{i \in I}\mathcal{S}_i)$, then for each $i \in I$, there is an inner invariant mean on $C_b(\mathcal{S}_i)$.*

Proof. Set $\mathcal{S} := \Pi_{i\in I}\mathcal{S}_i$ and note that for each $i \in I$, the projection map $\phi \longmapsto \phi(i)$ from $\mathcal{S}$ onto $\mathcal{S}_i$ is a continuous homomorphism. So, the result follows from Proposition 4.3. ∎

Acknowledgments. The authors would like to thank the referee of the paper for invaluable comments and wish to thank the Institute for Theoretical Physics and Mathematics, IPM. Also, the second author wishes to thank the Research Center for Mathematical Analysis and the Center of Excellence for Mathematics at the Isfahan University of Technology for their kind support.

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