Unicity of meromorphic functions related to their derivatives*

Qi Han Pei-Chu Hu

Abstract

In this paper, we shall study the unicity of meromorphic functions defined over non-Archimedean fields of characteristic zero such that their valence functions of poles grow slower than their characteristic functions. If f is such a function, and f and a linear differential polynomial $P(f)$ of f, whose coefficients are meromorphic functions growing slower than f, share one finite value a CM, and share another finite value b ($\neq a$) IM, then $P(f) = f$.

1 Introduction.

In 1929, R. Nevanlinna studied the unicity of meromorphic functions in $\mathbb{C}$. The five value theorem due to R. Nevanlinna states that if two non-constant meromorphic functions f and g in $\mathbb{C}$ share five distinct complex numbers a_j IM (ignoring multiplicity), which means

$$f^{-1}(a_j) = g^{-1}(a_j),\ j = 1, 2, ..., 5$$

in the sense of sets, then it follows that $f = g$. The four value theorem of R. Nevanlinna states that if two non-constant meromorphic functions f and g in $\mathbb{C}$ share four distinct complex numbers a_j CM (counting multiplicity), which means

$$f^{-1}(a_j) = g^{-1}(a_j),\ j = 1, 2, ..., 4$$

in the sense of counting multiplicities, then f is some Möbius transformation of g.

*The work of authors was partially supported by NSFC of China: Project No. 10371064.

2000 *Mathematics Subject Classification :* Primary 12J25, Secondary 46S10.

Key words and phrases : uniqueness of meromorphic functions, value sharing, Nevanlinna theory, non-Archimedean analysis.

In 1977, replacing the function g by the first derivative f' of f, L. Rubel and C. C. Yang proved that if f is an entire function in $\mathbb{C}$ such that f and f' share only two distinct finite complex numbers a, b CM, then $f = f'$. Further, it has been generalized to IM value sharing assumptions by E. Mues and N. Steinmetz, and independently by G. G. Gundersen when $ab \neq 0$. Afterwards, there are a lot of researches along this direction. For example, G. Frank etc. proved that a meromorphic function f in $\mathbb{C}$ and its m-th derivative $f^{(m)}$ are equal if they share two distinct finite complex numbers CM. E. Mues and M. Reinders, G. Frank and X. H. Hua, and P. Li continuously obtained that a meromorphic function f in $\mathbb{C}$ is equal to a linear differential polynomial $P(f)$ of f if f and $P(f)$ share three distinct finite complex numbers IM. In particular, when f in $\mathbb{C}$ is entire, C. A. Bernstein, C. D. Chang and B. Q. Li, and P. Li and C. C. Yang also obtained the relationship $f = P(f)$ if and only if f and $P(f)$ share two distinct finite complex numbers CM (see, e.g., [1], [8], [10], [11] or [12]).

Let κ be an algebraically closed field of characteristic zero, complete for a non-trivial non-Archimedean absolute value $|\cdot|$. Let f and g be two non-constant meromorphic functions on κ. A unicity theorem (cf. [7]) states that if f and g share two distinct values a, b CM, then there exists some non-zero constant $c \in \kappa$ such that

$$c = \frac{f-a}{f-b} \cdot \frac{g-b}{g-a}. \tag{1}$$

To determine f and g completely, we need other conditions to determine the constant c. For example, $c = 1$ if there exists a point $z_0 \in \kappa$ such that $f(z_0) = g(z_0)$ $(\neq a, b)$. In this paper, we replace the function g by a linear differential polynomial of f with the following expression

$$P(f) = b_{-1} + b_0 f + b_1 f' + \cdots + b_m f^{(m)}, \tag{2}$$

where $m \geq 1$ is an integer, and b_i are meromorphic functions in κ with $b_m(z) \not\equiv 0$ such that their characteristic functions grow slower than that of f, that is

$$T(r, b_i) = o(T(r, f)),\ \ i = -1, 0, 1, ..., m \ \text{ as } \ r \to \infty. \tag{3}$$

Now, our main theorem states

Theorem 1.1. *Let f be a non-constant meromorphic function on κ satisfying*

$$\bar{N}(r, f) = o(T(r, f)). \tag{4}$$

If f and $P(f)$ share a finite value a CM, and share another finite value b $(\neq a)$ IM, then $P(f) = f$.

Conversely, it is easy to show that the condition $P(f) = f$ implies the relation (4). A natural question is that, under the condition (4), if f and $P(f)$ share two distinct finite values a, b IM, whether the relation $P(f) = f$ still holds or not. Further, we have the following

Corollary 1.2. *Let f be a transcendental entire function on κ, or more generally, a transcendental meromorphic function on κ having finitely many poles. If f and $P(f)$ share a finite value a CM, and share another finite value b $(\neq a)$ IM, then $P(f) = f$.*

2 Preliminaries.

In this section, we recall some basic notations and information related to our proofs of Theorem 1.1 and other results. Let κ be stated as in the previous section, and let $\mathcal{A}(\kappa)$ be the ring of entire functions on κ. Then each $f \in \mathcal{A}(\kappa)$ can be given by a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with coefficients in κ such that for any $z \in \kappa$, we have $|a_n z^n| \to 0$ as $n \to \infty$. For a positive real r, the *maximum term* of f is defined to be

$$\mu(r, f) = \max_{n \geq 0} |a_n| r^n.$$

Let $n\Big(r, \frac{1}{f}\Big)$ denote the *counting function* of zeros of f, which is the number of zeros (counting multiplicities) of f in the disc $\kappa[0; r] = \{z \in \kappa \mid |z| \leq r\}$. The following fact is fundamental

$$n\left(r, \frac{1}{f}\right) = \max_{n \geq 0}\Big\{n \Big| |a_n| r^n = \mu(r, f)\Big\}.$$

Fix a real number $\rho_0 > 0$. For $r > \rho_0$, define the *valence function* of zeros of f by

$$N\left(r, \frac{1}{f}\right) = \int_{\rho_0}^{r} \frac{n\Big(t, \frac{1}{f}\Big)}{t} dt.$$

Then we have the following *Jensen Formula*

$$N\left(r, \frac{1}{f}\right) = \log \mu(r, f) - \log \mu(\rho_0, f).$$

We also denote the number of distinct zeros of f in $\kappa[0; r]$ by $\bar{n}\Big(r, \frac{1}{f}\Big)$ and define the *refined valence function* to be

$$\bar{N}\left(r, \frac{1}{f}\right) = \int_{\rho_0}^{r} \frac{\bar{n}\Big(t, \frac{1}{f}\Big)}{t} dt.$$

Let $n_{k)}\Big(r, \frac{1}{f}\Big)$ $\Big($resp. $n_{(k}\Big(r, \frac{1}{f}\Big)\Big)$ denote the number of zeros of f in $\kappa[0; r]$ with multiplicities no more (resp. less) than k and define $N_{k)}\Big(r, \frac{1}{f}\Big)$ $\Big($resp. $N_{(k}\Big(r, \frac{1}{f}\Big)\Big)$ as above; $\bar{n}_{k)}\Big(r, \frac{1}{f}\Big)$ $\Big($resp. $\bar{n}_{(k}\Big(r, \frac{1}{f}\Big)\Big)$ and thus $\bar{N}_{k)}\Big(r, \frac{1}{f}\Big)$ $\Big($resp. $\Big(\bar{N}_{(k}\Big(r, \frac{1}{f}\Big)\Big)$ are similarly defined.

The fractional field of $\mathcal{A}(\kappa)$ is denoted by $\mathcal{M}(\kappa)$. An element f in the field $\mathcal{M}(\kappa)$ will be called a *meromorphic function* on κ. Next, let f be a non-constant meromorphic function in κ. Since the greatest common factors of any two elements in $\mathcal{A}(\kappa)$ exist, there exist $f_0, f_1 \in \mathcal{A}(\kappa)$ with $f = \frac{f_0}{f_1}$ such that f_0 and f_1 have no common factors in the ring $\mathcal{A}(\kappa)$. We can uniquely extend μ to a meromorphic function f by defining

$$\mu(r, f) = \frac{\mu(r, f_0)}{\mu(r, f_1)}.$$

Define the *compensation function* of f by

$$m(r,f)=\max\Big\{0,\log\mu(r,f)\Big\}.$$

As usual, we define the *characteristic function* of f by

$$T(r,f)=m(r,f)+N(r,f),$$

where $N(r,f)=N\left(r,\frac{1}{f_1}\right)$ is the valence function of poles of f. Then, the first main theorem (cf. [3] or [7]) claims

$$T(r,f)=m\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{f-a}\right)+O(1) \tag{5}$$

for any $a\in\kappa$. Further, we have the basic formula (cf. [7])

$$T(r,f)=\max\left\{N\left(r,\frac{1}{f-a}\right),N\left(r,\frac{1}{f-b}\right)\right\}+O(1) \tag{6}$$

for any two distinct values $a,b\in\kappa\cup\{\infty\}$.

The lemma of the logarithmic derivative now states that for any positive integer $k>0$,

$$\mu\left(r,\frac{f^{(k)}}{f}\right)\le\frac{1}{r^k},$$

which further means

$$m\left(r,\frac{f^{(k)}}{f}\right)\le k\log^+\frac{1}{r}=O(1). \tag{7}$$

The Jensen formula can be generalized into the following form (cf. [7])

$$T\left(r,\frac{A_f}{B_f}\right)=\max(p,q)T(r,f)+O\left(\sum_{i=0}^{p}T(r,u_i)+\sum_{j=0}^{q}T(r,v_j)\right), \tag{8}$$

where $A_f=\sum\limits_{i=0}^{p}u_if^i$ and $B_f=\sum\limits_{j=0}^{q}v_jf^j$ are two coprime polynomials of f of degrees p and q, respectively, and $u_i,v_j\in\mathcal{M}(\kappa)$ for all $i=0,1,\dots,p$ and $j=0,1,\dots,q$.

The second main theorem (cf. [3] or [7]) states that for q distinct finite values $a_1,a_2,\dots,a_q$ of κ,

$$\begin{aligned}(q-1)T(r,f)\ &\le\ N(r,f)+\sum_{i=1}^{q}N\left(r,\frac{1}{f-a_i}\right)-N_{\mathrm{Ram}}(r,f)-\log r+O(1)\\ &\le\ \bar N(r,f)+\sum_{i=1}^{q}\bar N\left(r,\frac{1}{f-a_i}\right)-\log r+O(1),\end{aligned}$$

where $N_{\mathrm{Ram}}(r,f)$ is defined by

$$N_{\mathrm{Ram}}(r,f)=2N(r,f)-N(r,f')+N\left(r,\frac{1}{f'}\right),$$

and is called the *ramification term* of f.

For more details on functional analysis over non-Archimedean fields, we refer the reader to books [6], [7] or [9].

3 Proof of Theorem 1.1.

Set $g := P(f)$. Without loss of generality, we may suppose $a = 0$. Otherwise, it is sufficient to consider $F = f - a$ and $G = g - a$.

At first, we consider the case that f and g share the two distinct finite values $0, b$ CM under the condition (4). By a basic unicity theorem in [7], there exists a non-zero constant $c \in \kappa$ such that

$$\frac{f}{f-b} \cdot \frac{g-b}{g} = c, \tag{9}$$

which implies that

$$f(g-b) = cg(f-b). \tag{10}$$

If $c = 1$, then $f = g$, and so we are done. Next, suppose $c \neq 1$ and a contradiction will be deduced. We rewrite (10) as

$$(g-b)\{(1-c)f + cb\} = cb(f-b).$$

Then, we have

$$f - d = \frac{cb}{1-c} \cdot \frac{f-b}{g-b},$$

where $d := \frac{cb}{c-1} \neq 0, b$. Since f and g share b CM, the zeros of $f - d$ come from poles of g, so

$$N\left(r, \frac{1}{f-d}\right) \leq m\bar{N}(r, f) + o(T(r, f)) = o(T(r, f)).$$

By using the formula (6), we obtain

$$N(r, f) \neq o(T(r, f)).$$

Thus, f and g have at least one common pole, say z_0. Letting $z \to z_0$ in (9), we immediately obtain $c = 1$. This is a contradiction. So, we derive $f = g$.

Now, we consider the general case under the assumptions of Theorem 1.1. Write

$$\varphi := \frac{f'(f-g)}{f(f-b)}. \tag{11}$$

Note that

$$\varphi = \frac{f'}{f-b} - \frac{b_{-1}}{b}\left\{\frac{f'}{f-b} - \frac{f'}{f}\right\} - \frac{f'}{f-b}\sum_{i=0}^{m} b_i \frac{f^{(i)}}{f}.$$

Then the lemma of the logarithmic derivative yields immediately

$$m(r, \varphi) = o(T(r, f)).$$

Since f and g share 0 CM and b IM with the condition (4), we easily obtain an estimate

$$N(r, \varphi) \leq (m+1)\bar{N}(r, f) + \sum_{i=-1}^{m} N(r, b_i) = o(T(r, f)). \tag{12}$$

Therefore,

$$T(r,\varphi)=o(T(r,f)).$$

Similarly, we can prove that the function

$$\psi:=\frac{f'}{f}-\frac{g'}{g} \tag{13}$$

satisfies

$$T(r,\psi)=o(T(r,f)).$$

Assume, to the contrary, that $f\not\equiv g$. Thus $\varphi\not\equiv 0$. From (11), we have

$$\varphi\frac{f-b}{f'}\equiv 1-\frac{g}{f}.$$

By taking the derivative on both sides of the above equation and substituting (13) into the resulted one, we have

$$\varphi'\frac{f-b}{f'}+\varphi\left(1-\frac{(f-b)f''}{(f')^2}\right)\equiv\psi\left(1-\varphi\frac{f-b}{f'}\right),$$

which can be rewritten as

$$(\varphi-\psi)\frac{f'}{f-b}-\varphi\frac{f''}{f'}+\varphi'+\psi\varphi\equiv 0. \tag{14}$$

We will distinguish three cases to study equation (14).

(i) $\varphi-\psi\equiv 0$. For this case, equation (14) becomes

$$-\frac{f''}{f'}+\frac{\varphi'}{\varphi}+\frac{f'}{f}-\frac{g'}{g}\equiv 0. \tag{15}$$

From (11), we have

$$\frac{\varphi'}{\varphi}=\frac{f''}{f'}+\frac{f'-g'}{f-g}-\frac{f'}{f}-\frac{f'}{f-b}.$$

Substituting this into (15), we obtain

$$\frac{f'-g'}{f-g}=\frac{f'}{f-b}+\frac{g'}{g},$$

which means that the Wronskian determinant satisfies

$$\begin{vmatrix} f-g & g(f-b) \\ f'-g' & f'g+g'(f-b) \end{vmatrix}\equiv 0.$$

Thus $f-g$ and $g(f-b)$ are linearly dependent. There exists a constant $c\in\kappa$ ($c\neq 0$) such that

$$f-g\equiv cg(f-b).$$

If z_0 is a zero of $f-b$ and $g-b$ with multiplicities p and q, respectively, then the Taylor expansions of f and g around z_0 are respectively

$$f(z)=b+\sum_{n=p}^{\infty}A_n(z-z_0)^n$$

and

$$g(z)=b+\sum_{n=q}^{\infty}B_n(z-z_0)^n.$$

By simple calculations, we find

$$p=q,\ bcA_n=A_n-B_n\ (n\geq p). \tag{16}$$

Therefore, f and g share b CM, and hence $f=g$. This is a contradiction.

(ii) $\varphi-k\psi\equiv 0$ for some integer k (>1). Then equation (14) can be rewritten as

$$\left(1-\frac{1}{k}\right)\frac{f'}{f-b}-\frac{f''}{f'}+\frac{\varphi'}{\varphi}+\frac{f'}{f}-\frac{g'}{g}\equiv 0.$$

By similar arguments as above, we can obtain

$$\frac{f'-g'}{f-g}=\frac{1}{k}\cdot\frac{f'}{f-b}+\frac{g'}{g},$$

that is, the Wronskian determinant satisfies

$$\begin{vmatrix} f-b & (\frac{f-g}{g})^k \\ & \\ f' & k\frac{(f-g)^{k-1}}{g^{k+1}}\{g(f'-g')-g'(f-g)\} \end{vmatrix}\equiv 0.$$

Hence $f-b$ and $\left(\frac{f-g}{g}\right)^k$ are linearly dependent. There exists a constant $d\in\kappa$ $(d\neq 0)$ such that

$$f=b+d\left(\frac{f-g}{g}\right)^k=b+d\left(\frac{f}{g}-1\right)^k.$$

By applying the estimate (8), we have

$$kT\left(r,\frac{f}{g}\right)=T(r,f)+O(1). \tag{17}$$

On the other hand, the poles of $\frac{g}{f}$ come only from poles of g, since f and g share 0 CM. So,

$$N\left(r,\frac{g}{f}\right)\leq m\bar{N}(r,f)+o(T(r,f))=o(T(r,f)).$$

Similarly, we also have

$$N\left(r,\frac{f}{g}\right)=o(T(r,f)).$$

By using the formula (6), we have

$$T\left(r, \frac{f}{g}\right) = \max\left\{N\left(r, \frac{f}{g}\right), N\left(r, \frac{g}{f}\right)\right\} + O(1) = o(T(r, f)).$$

This is a contradiction to (17), and so we can rule out of the case (ii), too.

(iii) $\varphi - k\psi \not\equiv 0$ for any integer $k \geq 1$. For this case, we claim

$$\bar{N}\left(r, \frac{1}{f-b}\right) = o(T(r, f)). \tag{18}$$

Let z_0 be a zero of $f - b$ with multiplicity $p \geq 1$. If $\varphi(z_0) \neq \infty$, from (14) it is easy to show

$$\varphi(z_0) - p\psi(z_0) = 0.$$

Thus we obtain

$$\bar{N}_{m+1)}\left(r, \frac{1}{f-b}\right) \leq \bar{N}(r, \varphi) + \sum_{p=1}^{m+1} \bar{N}\left(r, \frac{1}{\varphi - p\psi}\right) = o(T(r, f)). \tag{19}$$

Next, assume $p \geq m+2$. If $b_i(z_0) \neq \infty$ $(i = -1, 0, ..., m)$, $g(z_0) = b$ yields

$$b = b_{-1}(z_0) + bb_0(z_0).$$

If $b_{-1}(z) + b_0(z)b \not\equiv b$, we obtain

$$\bar{N}_{(m+2}\left(r, \frac{1}{f-b}\right) \leq \bar{N}\left(r, \frac{1}{b_{-1} + bb_0 - b}\right) + \sum_{i=-1}^{m} \bar{N}(r, b_i) = o(T(r, f)).$$

If $b_{-1}(z) + b_0(z)b \equiv b$, we have

$$g - f \equiv (b_0 - 1)(f - b) + \sum_{i=1}^{m} b_i f^{(i)},$$

which means that z_0 is a multiple zero of $f - g$, and thus a zero of φ when $b_i(z_0) \neq \infty$ $(i = -1, 0, ..., m)$. Therefore,

$$\bar{N}_{(m+2}\left(r, \frac{1}{f-b}\right) \leq \bar{N}\left(r, \frac{1}{\varphi}\right) + \sum_{i=-1}^{m} \bar{N}(r, b_i) = o(T(r, f)).$$

Hence we obtain

$$\bar{N}\left(r, \frac{1}{f-b}\right) = \bar{N}_{m+1)}\left(r, \frac{1}{f-b}\right) + \bar{N}_{(m+2}\left(r, \frac{1}{f-b}\right) = o(T(r, f)).$$

The claim (18) is proved completely. Applying the second main theorem to f and three values $0, b, \infty$, then (4) and (18) yield immediately

$$T(r, f) = \bar{N}\left(r, \frac{1}{f}\right) + o(T(r, f)).$$

Combining the above equality, the first main theorem with the fact that

$$N\left(r,\frac{1}{f}\right) \geq \bar{N}\left(r,\frac{1}{f}\right) + \frac{1}{2}N_{(2}\left(r,\frac{1}{f}\right) \geq \bar{N}\left(r,\frac{1}{f}\right)$$

derives that

$$N_{(2}\left(r,\frac{1}{f}\right) = o(T(r,f))$$

and

$$T(r,f) = N_{1)}\left(r,\frac{1}{f}\right) + o(T(r,f)). \tag{20}$$

The condition (4) and (18) imply that the function

$$\eta := \frac{f'}{f-b} - \frac{g'}{g-b} \tag{21}$$

satisfies

$$T(r,\eta) = N(r,\eta) + O(1) \leq \bar{N}\left(r,\frac{1}{f-b}\right) + o(T(r,f)) = o(T(r,f)).$$

Similarly, we can obtain the equation

$$(\varphi - \eta)\frac{f'}{f} - \varphi\frac{f''}{f'} + \varphi' + \eta\varphi \equiv 0. \tag{22}$$

We claim $\varphi - \eta \equiv 0$. Assume, to the contrary, that $\varphi - \eta \not\equiv 0$. If z_0 is a simple zero of f, then z_0 also is a simple zero of g, and so $\varphi(z_0) \neq \infty, \eta(z_0) \neq \infty$. It is easy to show $\varphi(z_0) - \eta(z_0) = 0$ from (22). Thus we obtain an estimate

$$N_{1)}\left(r,\frac{1}{f}\right) \leq \bar{N}\left(r,\frac{1}{\varphi-\eta}\right) = o(T(r,f)).$$

Combining this with (20) yields a contradiction immediately. Hence $\varphi - \eta \equiv 0$. So from (22), we obtain

$$-\frac{f''}{f'} + \frac{\varphi'}{\varphi} + \frac{f'}{f-b} - \frac{g'}{g-b} \equiv 0.$$

In an analogous way as in case (i), we can obtain

$$f - g \equiv c_0 f(g-b) \qquad (c_0 \in \kappa,\ c_0 \neq 0),$$

and similarly prove that f and g share b CM. It follows that $f = g$, a contradiction again.

Therefore from the discussions in cases (i), (ii) and (iii), we find that it must be $f = g$. The proof of Theorem 1.1 is finished completely.

4 f and $P(f)$ share two values IM.

Let f be a non-constant meromorphic function on κ satisfying the assumption (4), and let $P(f)$ be defined by (2). We further define $N_E\left(r, \frac{1}{f}\right)$ to be the valence function of common zeros of f and $P(f)$ with the same multiplicities, and $\bar{N}_E\left(r, \frac{1}{f}\right)$ the corresponding refined valence function.

Proposition 4.1. *Let f be a non-constant meromorphic function on κ satisfying the assumption (4), and let $P(f)$ be defined by (2). Assume that f and $P(f)$ share two distinct finite values a, b IM. Then we have either $P(f) = f$ or*

$$\bar{N}\left(r, \frac{1}{f-b}\right) \leq (m+1)\left\{\bar{N}\left(r, \frac{1}{f-a}\right) - \bar{N}_E\left(r, \frac{1}{f-a}\right)\right\} + o(T(r, f)).$$

Proof. Set $g = P(f)$, $a = 0$ and define φ, ψ as in the proof of Theorem 1.1. First of all, we assume

$$\bar{N}\left(r, \frac{1}{f}\right) \neq o(T(r, f)).$$

We also get

$$T(r, \varphi) = o(T(r, f)),$$

and

$$\begin{aligned} T(r, \psi) &= N(r, \psi) + O(1) = \bar{N}(r, \psi) + O(1) \\ &\leq \bar{N}\left(r, \frac{1}{f}\right) - \bar{N}_E\left(r, \frac{1}{f}\right) + O(1). \end{aligned} \tag{23}$$

Next we distinguish two cases.

(i) $\varphi - k\psi \equiv 0$ for some integer $k \geq 1$. Then by (12) and (23), we know that f and g share 0 CM, since poles of φ and ψ cannot coincide each other. Meanwhile (16) or (17) still holds, thus we get either a contradiction or $f = g$ from Theorem 1.1.

(ii) $\varphi - k\psi \not\equiv 0$ for any integer $k \geq 1$. According to the proof of Theorem 1.1, we have

$$\bar{N}_{(m+2}\left(r, \frac{1}{f-b}\right) = o(T(r, f)).$$

Now the estimate (19) still holds, and hence

$$\begin{aligned} \bar{N}_{m+1)}\left(r, \frac{1}{f-b}\right) &\leq (m+1)T(r, \psi) + o(T(r, f)) \\ &\leq (m+1)\left\{\bar{N}\left(r, \frac{1}{f}\right) - \bar{N}_E\left(r, \frac{1}{f}\right)\right\} + o(T(r, f)). \end{aligned}$$

Therefore,

$$\bar{N}\left(r, \frac{1}{f-b}\right) \leq (m+1)\left\{\bar{N}\left(r, \frac{1}{f}\right) - \bar{N}_E\left(r, \frac{1}{f}\right)\right\} + o(T(r, f)).$$

Finally, we consider the case

$$\bar{N}\left(r, \frac{1}{f}\right) = o(T(r,f)).$$

By the proof above, we can still get that either $f = g$ or

$$\bar{N}\left(r, \frac{1}{f-b}\right) = o(T(r,f)).$$

However, if the latter case holds, the second main theorem yields

$$T(r,f) \le \bar{N}(r,f) + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f-b}\right) + O(1) = o(T(r,f)).$$

This is a contradiction, and so it must be $f = g$. The proof finishes completely. ∎

5 No condition (4).

Theorem 5.1. *Let f be a non-constant meromorphic function on κ, and let $P(f)$ be defined by (2). If f and $P(f)$ share two distinct finite values a CM and b IM, then we have either $P(f) = f$ or*

$$N\left(r, \frac{1}{f-a}\right) \neq o(T(r,f)).$$

Proof. Set $g = P(f)$, $a = 0$ as in the proof of Theorem 1.1. Assume, to the contrary, that $f \not\equiv g$ and

$$N\left(r, \frac{1}{f}\right) = o(T(r,f)).$$

Since f and g share 0 CM, we also have

$$N\left(r, \frac{1}{g}\right) = o(T(r,f)).$$

Then from the formula (6), we obtain

$$T(r,f) = N(r,f) + o(T(r,f)),$$

and

$$T(r,g) = N(r,g) + o(T(r,f)). \tag{24}$$

By considering the poles of g, it is easy to show

$$N(r,g) = N(r,f) + m\bar{N}(r,f) + o(T(r,f)). \tag{25}$$

The second main theorem yields immediately

$$T(r,g) \le \bar{N}(r,g) + \bar{N}\left(r, \frac{1}{g-b}\right) + o(T(r,f)). \tag{26}$$

Since

$$\begin{aligned}\bar{N}\left(r,\frac{1}{g-b}\right) &\le N\left(r,\frac{1}{\frac{g}{f}-1}\right) \le T\left(r,\frac{g}{f}\right)+O(1)\\ &= \max\left\{N\left(r,\frac{g}{f}\right),N\left(r,\frac{f}{g}\right)\right\}+O(1)\\ &= N\left(r,\frac{g}{f}\right)+o(T(r,f))\\ &\le m\bar{N}(r,f)+o(T(r,f)),\end{aligned} \tag{27}$$

we obtain

$$T(r,g)\le (m+1)\bar{N}(r,f)+o(T(r,f))\le N(r,g)+o(T(r,f)),$$

which together with the first main theorem implies

$$T(r,g)=(m+1)\bar{N}(r,f)+o(T(r,f)). \tag{28}$$

Comparing (24), (25) and (28), we find

$$N(r,f)=\bar{N}(r,f)+o(T(r,f)),$$

and hence

$$T(r,f)=\bar{N}(r,f)+o(T(r,f)). \tag{29}$$

By using (26), (27) and (28), we also obtain

$$\bar{N}\left(r,\frac{1}{f-b}\right)=\bar{N}\left(r,\frac{1}{g-b}\right)=m\bar{N}(r,f)+o(T(r,f)). \tag{30}$$

Thus it follows that $m=1$.

Consider the function

$$\phi:=\frac{g}{f^2}.$$

Since f and g share 0 CM, $m=1$, and $N_{(2}(r,f)=o(T(r,f))$, it is obvious that

$$N\left(r,\frac{1}{\phi}\right)=o(T(r,f))$$

and

$$N(r,\phi)\le N\left(r,\frac{1}{f}\right)+o(T(r,f))=o(T(r,f)).$$

Therefore,

$$T(r,\phi)=\max\left\{N(r,\phi),N\left(r,\frac{1}{\phi}\right)\right\}+O(1)=o(T(r,f)).$$

If z_0 is a zero of $f-b$, then $\phi(z_0)=\frac{1}{b}$. If $\phi\not\equiv\frac{1}{b}$, we have

$$\bar{N}\left(r,\frac{1}{f-b}\right)\le\bar{N}\left(r,\frac{1}{\phi-\frac{1}{b}}\right)\le T(r,\phi)+O(1)=o(T(r,f)),$$

which contradicts against (29) and (30).

Therefore, it must be $\phi = \frac{1}{b}$, and so $bg \equiv f^2$. Then f has no zeros. Note that

$$b(g-b) \equiv (f-b)(f+b).$$

Then $f+b$ also has no zeros, since f and g share b IM. The formula (6) yields directly

$$T(r,f) = \max\left\{N\left(r,\frac{1}{f}\right), N\left(r,\frac{1}{f+b}\right)\right\} + O(1) = O(1),$$

which also is impossible since $T(r,f) \to \infty$. The theorem is proved completely. ■

6 Final notes.

The meromorphic function f in Theorem 1.1 is a solution of the linear differential equation

$$w^{(m)} + a_m w^{(m-1)} + \cdots + a_2 w' + a_1 w + a_0 = 0. \tag{31}$$

In [7], P. C. Hu and C. C. Yang proved that (31) has no transcendental meromorphic solutions provided that the coefficients are constants.

Take a prime number p. Here we consider the field $\kappa = \mathbb{C}_p$, completion of the algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers. Let $\bar{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in the field $\mathbb{C}_p$. A. Boutabaa ([2], [4]) studied meromorphic solutions of (31) and proved the following

Theorem 6.1. *Suppose that the equation (31) is such that $a_1(z), ..., a_m(z) \in \bar{\mathbb{Q}}(z)$, $a_0(z) \equiv 0$, and let $w(z) \in \mathcal{M}(\mathbb{C}_p)$ be a solution of (31). Then $w(z) \in \mathbb{C}_p(z)$.*

If $a_1(z), ..., a_m(z)$ are not all in $\bar{\mathbb{Q}}(z)$, A. Boutabaa ([4], [5]) shows that the Gaussian differential equation

$$z(1-z)\frac{d^2w}{dz^2} + (c-(a+b+1)z)\frac{dw}{dz} - abw = 0 \tag{32}$$

does have transcendental entire solutions on $\mathbb{C}_p$, where a, b, c are constants. We think it is interesting to further study the equation (31).

Acknowledgement. The authors owe many thanks to the referee for valuable comments and suggestions made to the present paper.

References

[1] Bernstein, C. A., Chang, D. C. and Li, B. Q., On uniqueness of entire functions in $\mathbb{C}^n$ and their partial differential polynomials, Forum Math. **8**(1996), 379-396.

[2] Boutabaa, A., On some p-adic functional equations, Lecture Notes in Pure and Applied Mathematics, Volume **192**: p-adic functional analysis, Marcel Dekker, Inc. 1997.

[3] Boutabaa, A. and Escassut, A., An improvement of the p-adic Navanlinna theory and application to meromorphic functions, Lecture Notes in Pure and Applied Mathematics, Volume **207**: p-adic functional analysis, Marcel Dekker, Inc. 1999.

[4] Boutabaa, A., A note on p-adic linear differential equations, J. Number Theory 87 (2001), 301-305.

[5] Boutabaa, A., A transcendental p-adic entire solution of a linear differential equation, manuscript.

[6] Gouvêa, F. Q., p-adic Numbers: An Introduction (2nd.ed.), Universitext, Springer, 2000.

[7] Hu, P. C. and Yang, C. C., Meromorphic Functions over Non-Archimedean Fields, Kluwer Academic, Dordrecht, 2000.

[8] Hu, P. C., Li, P. and Yang, C. C., Unicity of Meromorphic Mappings, Kluwer Academic, Dordrecht, 2003.

[9] Koblitz, N., p-adic Numbers, p-adic Analysis and Zeta-Functions, Graduate Texts in Mathematics, Volume **58**, Springer-Verlag, 1977.

[10] Li, P., Meromorphic functions sharing three values IM with its linear differential polynomials, Complex Variables Theory Appl. **41**(2000), 221-229.

[11] Li, P. and Yang, C. C., Value sharing of an entire function and its derivatives, J. Math. Soc. Japan **51**(1999), No. 4, 781-800.

[12] Yang, C. C. and Yi, H. X., Uniqueness Theory of Meromorphic Functions, Science Press & Kluwer Academic, Beijing & Dordrecht, 2003.

Department of Mathematics
Shandong University
Jinan 250100, Shandong, P. R. China
e-mail: k.l.han@mail.sdu.edu.cn (Qi Han), pchu@sdu.edu.cn (Pei-Chu Hu)