Ultrametric umbral calculus in characteristic p

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Abstract

The question to have an umbral calculus in non zero characteristic p, has been considered on some special subspaces of the space of polynomials, for instance subspaces of the space of additive polynomials with coefficients in a complete valued field of characteristic p and their closure in spaces of continuous functions. In her PhD thesis, M. Héraoua has given an umbral calculus on the so called ring of formal differential operators which has a coalgebra structure. In many respects, this umbral calculus is as in the classical umbral calculus in characteristic zero.

It turns out that the technique used by M. Héraoua can be extended in the topological case. More precisely, let $\mathbb{F}_q$ be the finite field with q elements and $\mathbb{F}_q[[T]]$ be the ring of formal power series with coefficients in $\mathbb{F}_q$. Then with the addition of formal power series and the T-adic topology, $\mathbb{F}_q[[T]]$ is a totally discontinuous compact group. Let K be a complete valued field, extension of the valued field of formal Laurent series $\mathbb{F}_q((T))$, then it is well known that the space of continuous functions $\mathcal{C}(\mathbb{F}_q[[T]], K)$ is an ultrametric Hopf algebra. The coalgebra structure of $\mathcal{C}(\mathbb{F}_q[[T]], K)$ is that of a binomial divided power coalgebra. In a previous work we have described the algebra of difference operators of the Banach coalgebra $\mathcal{C}(\mathbb{F}_q[[T]], K)$. Here, we show that the Keigher-Pritchard's divided powers of an element of the maximal ideal of the dual algebra of $\mathcal{C}(\mathbb{F}_q[[T]], K)$ can be performed. With this, and the fact that the algebra of difference operators is isomorphic to the latter dual algebra, one recovers much part of classical umbral calculus.

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1 Introduction and reminders

This paper is the sequel of [7] and gives, using the same technique as M. Héraoua in [10], an ultrametric umbral calculus on $\mathcal{C}(\mathbb{F}_q[[T]], K)$ that in many respects resembles the classical umbral calculus and differs from the umbral calculus developed by A. Kochubei in [14] that concerns the space of continuous $\mathbb{F}_q$-linear functions from $\mathbb{F}_q[[T]]$ into K. Thanks to Carlitz polynomials (cf. [2] etc.., [20] etc..), the technique is mainly founded on the divided powers structure of the dual algebra of $\mathcal{C}(\mathbb{F}_q[[T]], K)$ induced by the one defined by W.F. Keigher and F.L. Pritchard on the algebra of Hurwitz formal series [12].

We shall remind here some facts on the coalgebra structure of the space of continuous functions on $\mathbb{F}_q[[T]]$ with values in K, a complete valued field extension the valued field of formal Laurent series $\mathbb{F}_q((T))$, where $\mathbb{F}_q$ is the finite field with $q = p^m$ and of characteristic p. Naturally the absolute value on $\mathbb{F}_q((T))$ is induced by the T-adic valuation and for $b \in \mathbb{F}_q((T))$, one sets $|b| = q^{-v_T(b)}$

1.1 Carlitz - Wagner basis

The main definitions and results stated here go back to L. Carlitz (cf. [2], [3] , [4]); see also [9].

For an element a of the polynomial ring $\mathbb{F}_q[T]$, we denote by $d^\circ a$, the degree of a , with the convention $d^\circ 0 = -\infty$ and if $f \in \mathbb{F}_q[T][z]$, one sets $deg(f)$ to be the degree of f as polynomial in the indeterminate z.

The *elementary Carlitz polynomials* are defined as follows : let $j \geq 0$ be an integer, one sets $e_j(z) = \prod\limits_{d^\circ a<j} (z-a)$. One has $deg(e_j) = q^j$ and $e_0(z) = z$. It is readily seen that e_j is an additive polynomial, i.e. $e_j(z+y) = e_j(z) + e_j(y)$. Moreover, if $\alpha \in \mathbb{F}_q$, one sees that $e_j(\alpha z) = \alpha e_j(z)$.
Let us put $D_0 = 1$ and for any integer $j \geq 1 : D_j = e_j(T^j)$. Since $e_j(T^j + a) = e_j(T^j)$, for any $a \in \mathbb{F}_q[T]$, such that $d^\circ a < j$, one obtains $D_j = \prod\limits_{d^\circ b = j,\ b\ monic} b$. Furthermore, on verifies that for $j \geq 1$, $D_j = \prod\limits_{\ell=1}^{j} [\ell]^{q^{j-\ell}}$ and $D_j = [j] D_{j-1}^q$, where $[\ell] = T^{q^\ell} - T$.

On the other hand, one has $e_j(z) = e_{j-1}(z)^q - D_{j-1}^{q-1} e_{j-1}(z)$ and so, by induction, one obtains:

$e_j(z) = \sum\limits_{\ell=0}^{j} (-1)^{j-\ell} \dfrac{D_j}{D_\ell L_{j-\ell}^{q^\ell}} z^{q^\ell}$, where $L_0 = 1$ and $L_j = \prod\limits_{\ell=1}^{j} [\ell]$, for $j \geq 1$..

For instance $e_1(z) = z^q - z$ and $e_2(z) = z^{q^2} - (1 + (T^q - T)^{q-1}) z^q + (T^q - T)^{q-1} z$.

Set for $j \geq 0$, $f_j = \dfrac{e_j}{D_j}$. Let $j = \sum\limits_{\ell=0}^{s(j)} j_\ell q^\ell$ be the q-expansion of j, $s(j) = \left[\dfrac{\log j}{\log q}\right]$.

The *jth basic Carlitz polynomial* is the polynomial $h_j = \prod\limits_{\ell=0}^{s(j)} f_\ell^{j_\ell}$, with $h_0 = 1$ and $deg(h_j) = j$.
Since f_j is $\mathbb{F}_q$ - linear, one verifies by using a well known congruence of Lucas on binomial coefficients, that for $x, y \in \mathbb{F}_q[T]$, $h_j(x+y) = \sum\limits_{s+t=j} \binom{j}{s} h_s(x) h_t(y)$.

Moreover, if $S_q(j) = \sum_{\ell=0}^{s(j)} j_\ell$ is the sum of the digits of the q-expansion of j, one sees that for $\alpha \in \mathbb{F}_q$ and $x \in \mathbb{F}_q[T]$, $h_j(\alpha x) = \alpha^{S_q(j)} h_j(x) = \alpha^j h_j(x)$.

Parallel to the polynomials h_j, Carlitz has defined the polynomials $e_j^* = \prod_{\ell=0}^{s(j)} e^*_{j_\ell q^\ell}$, where $e^*_{j_\ell q^\ell} = {e_\ell}^{j_\ell}$, if $0 \le j_\ell \le q-2$ and $e^*_{(q-1)q^\ell} = {e_\ell}^{q-1} - {D_\ell}^{q-1}$, if $j_\ell = q-1$.

Furthermore, one puts $h_j^* = \dfrac{e_j^*}{d_j}$, where $d_j = \prod_{\ell=0}^{s(j)} D_\ell^{j_\ell}$. Hence $h_j^* = \prod_{\ell=0}^{s(j)} \dfrac{e^*_{j_\ell q^\ell}}{D_\ell^{j_\ell}}$.
If $0 \le j_\ell \le q-2$, for all ℓ , $0 \le \ell \le s(j)$, one has $h_j^* = h_j$.
Moreover, as for h_j, one verifies that for any $j \ge 0$, $h_j^*(x+y) = \sum_{s+t=j} \binom{j}{s} h_s^*(x) h_t(y)$.

A particular subsequence of $(h_j^*)_{j\ge 0}$ is given by $g_j = h^*_{q^{m(j)}-j-1}$, where $m(j) = s(j)+1$. One has $g_j = \prod_{\ell=0}^{s(j)} \dfrac{e^*_{(q-1-j_\ell)q^\ell}}{D_\ell^{q-1-j_\ell}}$.

Let K be a field that contains $\mathbb{F}_q[T]$, since $deg(h_j) = j$, the sequence $(h_j)_{j\ge 0}$ is a basis of the K-vector space $K[z]$.

An important property of the polynomial functions h_j (*resp.* h_j^*) is that they map $\mathbb{F}_q[T]$ into itself.

Let us put for any element a of $\mathbb{F}_q[T]$ and any polynomial $P \in K[z] : \tau_a(P)(z) = P(z+a)$.
Set for any integer $j \ge 0, m(j) = s(j)+1$, where $s(j) = \left[\dfrac{\log j}{\log q}\right]$, and let us consider the K - linear operator $A_j = (-1)^{m(j)} \sum_{d^\circ a < m(j)} g_j(a) \tau_a$. Then for $\ell \ge 0$, one has

$$A_j(h_\ell)(z) = (-1)^{m(j)} \sum_{d^\circ a < m(j)} g_j(a) h_\ell(z+a) =$$
$$\sum_{s+t=\ell} \binom{\ell}{s} \left((-1)^{m(j)} \sum_{d^\circ a < m(j)} g_j(a) h_s(a) \right) h_t(z) = \sum_{s+t=\ell} \binom{\ell}{s} \delta_{j,s} h_t(z).$$

Hence $A_j(h_\ell) = 0$, if $\ell < j$ and $A_j(h_\ell) = \binom{\ell}{j} h_{\ell-j}$, if $l \ge j$. An immediate consequence is that $A_j(P) = 0$, if $deg(P) < j$.
On the other hand, since for $a \ne 0$, $h_j(a) = 0$ if and only if $q^{d^\circ a} < q^{s(j)}$ and $q^{s(i)} = q^{d^\circ a}$, for $q^{d^\circ a} \le i < q^{d^\circ a+1}$, one has $h_j(a) \ne 0$ if and only if $j \le q^{d^\circ a+1} - 1$. Hence for a , $x \in \mathbb{F}_q[T]$ and $P \in K[z]$, one has $\tau_a P(x) = \tau_x P(a) = \sum_{j=0}^{q^{1+d^\circ a}-1} h_j(a) A_j(P)(x)$, i.e. $\tau_a = \sum_{j=0}^{q^{1+d^\circ a}-1} h_j(a) A_j$. $(\tau_0 = id)$.

Now, let K be a complete valued field, extension of the field $\mathbb{F}_q((T))$ of the formal Laurent series with coefficients in the finite field $\mathbb{F}_q$. Therefore $\mathbb{F}_q[[T]]$ is a compact sub-ring of K.

Let $\mathcal{C}(\mathbb{F}_q[[T]], K)$ be the K - Banach algebra of the continuous functions $f : \mathbb{F}_q[[T]] \longrightarrow K$. The norm on $\mathcal{C}(\mathbb{F}_q[[T]], K)$ is $\|f\| = \sup_{x\in\mathbb{F}_q[[T]]} |f(x)|$. We identify $K[x]$ with the space of the polynomial functions of $\mathbb{F}_q[[T]]$ into K.

For $a \in \mathbb{F}_q[[T]]$, one defines on $\mathcal{C}(\mathbb{F}_q[[T]], K)$ the translation operator τ_a such that $\tau_a f(x) = f(x+a),\ x \in \mathbb{F}_q[[T]]$.
For $j \geq 0$, the linear operator $A_j = (-1)^{m(j)} \sum_{d^\circ a<m(j)} g_j(a)\tau_a$ is too well defined on $\mathcal{C}(\mathbb{F}_q[[T]], K)$. Let us remind that $g_j = h^*_{q^{m(j)}-j-1}$, $m(j) = 1 + \left[\frac{\log j}{\log q}\right]$ and $g_j(\mathbb{F}_q[T]) \subset \mathbb{F}_q[T]$.
The operators τ_a are isometric and the operators A_j are bounded, with norm $\|A_j\| = 1$.
Moreover, the operators τ_a map $K[x]$ into $K[x]$; the same is true for the operators A_j.
Furthermore one can prove that for any $f \in \mathcal{C}(\mathbb{F}_q[[T]], K)$, one has $\lim_{j\to+\infty} A_j(f) = 0$ (c.f [7]) and as a consequence, one obtains a proof of the following theorem ([20], [22])

Theorem 1. *(Wagner)*
Let K be a complete valued field, extension of the field of formal Laurent series $\mathbb{F}_q((T))$.
Then the sequence of the basic Carlitz polynomials $(h_j)_{j\geq 0}$ is an orthonormal basis of the K - Banach space $\mathcal{C}(\mathbb{F}_q[[T]], K)$.
In other words, if $f \in \mathcal{C}(\mathbb{F}_q[[T]], K)$, one has $f = \sum_{j\geq 0} \alpha_j(f) h_j$, with $\alpha_j(f) \in K$, $\lim_{j\to+\infty} \alpha_j(f) = 0$ and $\|f\| = \sup_{j\geq 0} |\alpha_j(f)|$.
Moreover $\alpha_j(f) = A_j(f)(0) = (-1)^{m(j)} \sum_{d^\circ a<m(j)} g_j(a) f(a)$, where $g_j = h^*_{q^{m(j)}-j-1}$.

1.2 The Hopf algebra structure of $\mathcal{C}(\mathbb{F}_q[[T]], K)$

Notations. Let K be a complete ultrametric valued field. An ultrametric Banach space H over K is said to be a Banach coalgebra if there exist continuous linear maps $c : H \to H\widehat{\otimes}H$ = topological tensor product, called the coproduct of H, and $\sigma : H \to K$, called the counit of H, such that

(i) $\quad (c \otimes id_H) \circ c = (id_H \otimes c) \circ c$

(ii) $\quad (id_H \otimes \sigma) \circ c = id_H = (\sigma \otimes id_H) \circ c$, and $\|\sigma\| = 1$

where id_H is the identity map of H.

It follows that for $a \in H$, one has $\|a\| \leq \|c(a)\| \leq \|c\|\ \|a\|$ and c is isometric if and only if $\|c\| = 1$

Furthermore, the Banach coalgebra H is said to be a Banach bialgebra, respectively a complete Hopf algebra, if it is a unitary Banach algebra with multiplication $m : H\widehat{\otimes}H \to H$ such that c and σ are algebra homomorphisms, respectively and there exists a continuous linear map $\eta : H \to H$, called the antipode or inversion of H, such that

(iii) $\quad m \circ (id_H \otimes \eta) \circ c = k \circ \sigma = m \circ (\eta \otimes id_H) \circ c$

where k is the canonical map of K into H.

$-\bullet-$ For any Banach coalgebra H, the dual Banach space H' of H, with the convolution product $\mu * \nu = (\mu \otimes \nu) \circ c$, is a unitary normed algebra with unit σ and $\|\mu * \nu\| \le \|c\| \, \|\mu\| \, \|\nu\|$.

Definition 1. *Let us say that a continuous linear endomorphism u of the Banach H is a (left)-comodule endomorphism of H if one has $c \circ u = (id_H \otimes u) \circ c$.*

The set $End_{com}(H)$ is readily seen to be a closed Banach subalgebra of the Banach $\mathcal{L}(H)$ of the set of continuous linear endomorphisms u with the norm $\|u\| = \sup\limits_{x \neq 0} \dfrac{\|u(x)\|}{\|x\|}$.

$-\bullet-$ It is well known that if one sets for $\mu \in H' : \varphi(\mu) = (id_H \otimes \mu) \circ c$, then φ is a **continuous algebra isomorphism** of the complete algebra H' onto $End_{com}(H)$. Moreover this isomorphism is isometrical if $\|\sigma\| = 1 = \|c\|$. The reciprocal θ_1 of φ is such that $\theta_1(u) = \sigma \circ u$. (c.f [5]).

$-\bullet-$ Also, one defines the continuous coalgebra endomorphisms of H to be the continuous linear endomorphisms v of H is a continuous linear such that $(v \otimes v) \circ c = c \circ v$ and $\sigma \circ v = \sigma$.

As usual, the complete tensor product of two ultrametric Banach spaces E and F over K is the completion $E \widehat{\otimes} F$ of the algebraic tensor product $E \otimes F$ with respect to the tensor norm $\|z\| = \inf\limits_{\Sigma x_j \otimes y_j = z} \left(\max\limits_j \|x_j\| \, \|y_j\| \right)$. In the sequel all Banach spaces are ultrametric.

Let us remind that if E is a free Banach space with orthogonal (orthonormal) basis $(e_j)_{j \in I}$, then any $z \in F \widehat{\otimes} E$ can be written in the unique form $z = \sum\limits_{j \in I} y_j \otimes e_j$, with $\lim\limits_{j \in I} \|y_j\| \|e_j\| = 0$ ($\lim\limits_{j \in I} \|y_j\| = 0$).

Moreover, if F is free with orthogonal (orthonormal) basis $(f_\ell)_{\ell \in L}$; then $(f_\ell \otimes e_j)_{(\ell, j) \in L \times I}$ is an orthogonal (orthonormal) basis of $F \widehat{\otimes} E$. (cf. for example [5])

Important examples of complete ultrametric Hopf algebras are provided by the Banach algebras $\mathcal{C}(G, K)$ of continuous functions on a totally disconnected compact group G with values in K. It is well known that, if one sets for $f, g \in \mathcal{C}(G, K)$ and $x, y \in G : \Pi(f \otimes g)(x, y) = f(x)g(y)$, one obtains by linearity and completion an isometric isomorphism of Banach algebras $\Pi : \mathcal{C}(G, K) \widehat{\otimes} \mathcal{C}(G, K) \to \mathcal{C}(G \times G, K)$ (cf. [5] or [17]). On the other hand, let $\rho : \mathcal{C}(G, K) \to \mathcal{C}(G \times G, K)$ be defined by setting $\rho(f)(s, t) = f(st) = \tau_s f(t)$. Then ρ is an isometric morphism of Banach algebras.

Together with the usual multiplication of functions, the inversion operator η defined by $\eta(f)(x) = f(x^{-1})$ and the Dirac linear form $\sigma(f) = f(e)$, one obtains on $\mathcal{C}(G, K)$ a structure of complete ultrametric Hopf algebra with coproduct $c = \Pi^{-1} \circ \rho$, antipode η and counit σ (cf. [5]).

For these Banach coalgebras $\mathcal{C}(G, K)$ the dual algebra $\mathcal{C}(G, K)' = M(G, K)$ can be identified with the algebra of bounded measures of G with values in K. Here the convolution of measures $\mu \star \nu = (\mu \otimes \nu) \circ$ is the usual one such that for any continuous fonction $f : G \to K$ one has $< \mu \star \nu, f > = < \mu, < \nu, \tau_s f >>$. Furthermore, with previous reminds, one sees that the Banach algebras $M(G, K)$ and $End_{com}(\mathcal{C}(G, K))$ are isometrically isomorphic. In addition, one has the following proposition.

Proposition 2. : *The algebra $End_{com}(\mathcal{C}(G,K))$ of the comodule endomorphisms of the Hopf algebra $\mathcal{C}(G,K)$ coincides with the algebra $W(G,K)$ of the difference operators, i.e the continuous linear endomorphisms u of $\mathcal{C}(G,K)$ that commute with the translation operators τ_s : $u \circ \tau_s = \tau_s \circ u,\ \forall s \in G$. (cf.*[5])

Considering any ultrametric complete value field K and the additive group structure of the ring of formal power series $\mathbb{F}_q[[T]]$, which then becomes a compact topological group, one has on $\mathcal{C}(\mathbb{F}_q[[T]],K)$ a Hopf algebra structure with isometrical coproduct c such that $\Pi \circ c(f)(x,y) = \rho(f)(x,y) == f(x+y)$ and counity σ the linear form defined by $\sigma(f) = f(0)$.
We are concerned here by the case when K *is a complete extension of* $\mathbb{F}_q((T))$. Then thanks to the Carlitz basis on $\mathcal{C}(\mathbb{F}_q[[T]],K)$, one has the following proposition (cf. [7])

Proposition 3. *Let K be a complete valued field extension of $\mathbb{F}_q((T))$.*
Then the complete ultrametric Hopf algebra $\mathcal{C}(\mathbb{F}_q[[T]],K)$ is a binomial divided power coalgebra, having $(h_j)_{j\geq 0}$ as an associated basic binomial divided power sequence. More precisely, for any integer $j \geq 0$, one has $c(h_j) = \sum\limits_{s+t=j} \binom{j}{s} h_s \otimes h_t$.

Proof : This is an obvious consequence of the fact that if $x, y \in \mathbb{F}_q[[T]]$, one has $h_j(x+y) = \sum\limits_{s+t=j} \binom{j}{s} h_s(x)h_t(y)$, i.e. $\rho(h_j)(x,y) = \sum\limits_{s+t=j} \binom{j}{s} h_s(x)h_t(y)$.

2 Divided powers in the algebra of bounded measures

In the sequel K will be a complete valued field extension of the complete valued field $\mathbb{F}_q((T))$ of the formal Laurent power series with cœfficients in the finite field $\mathbb{F}_q$ with $q = p^m$ elements. The notations and definitions are those of the previous section.

2.1 The algebra structure of the set of bounded measures and the algebra of difference operators

$- \bullet -$ Let us remind some facts from the previous section. The continuous dual $M(\mathbb{F}_q[[T]],K)$ of the ultrametric Banach space $\mathcal{C}(\mathbb{F}_q[[T]],K)$, which can be identified with the space of the bounded measures on $\mathbb{F}_q[[T]]$ with values in K is a commutative Banach algebra for the convolution product

$\mu \star \nu = (\mu \otimes \nu) \circ c$. This product coincides with the usual one, such that for any continuous function f from $\mathbb{F}_q[[T]]$ into K : $< \mu \star \nu, f >=< \mu, < \nu, \tau_x f >>$.
Moreover, the Banach algebra $M(\mathbb{F}_q[[T]],K)$ is isometrically isomorphic to the algebra of difference operators $W(\mathbb{F}_q[[T]],K) = End_{com}(\mathcal{C}(\mathbb{F}_q[[T]],K))$. The isomorphism is given by $\varphi(\mu) = (id \otimes \mu) \circ c$.
For any non negative integer j, one defines the dual basis element h'_j of $M(\mathbb{F}_q[[T]],K)$, corresponding to the Carlitz polynomial h_j, by setting $< h'_j, h_\ell >= \delta_{j,\ell}$ (= Kronecker symbol). One has $h'_0 = \sigma$.
Since $(h_j)_{j\geq 0}$ is an orthonormal basis of $\mathcal{C}(\mathbb{F}_q[[T]],K)$, one sees (cf. for instance

[5]) that the family $(h'_j)_{j\geq 0}$ is a weak* basis of the Banach space $M(\mathbb{F}_q[[T]], K)$. In other words, any bounded measure μ on $\mathbb{F}_q[[T]]$ can be written in the unique form $\mu = \sum_{j\geq 0} \alpha_j h'_j$ in such a way that for any continuous function $f : \mathbb{F}_q[[T]] \to K$, the series $<\mu, f> = \sum_{j\geq 0} \alpha_j < h'_j, f >$ converges in K. Furthermore, one has $\alpha_j =< \mu, h_j >$ and $\|\mu\| = \sup_{j\geq 0} | < \mu, h_j > |$.

Let us notice that for the non negative integers i and j, one has $h'_i \star h'_j = \binom{i+j}{j} h'_{i+j}$.

Hence $h'^{\star 2}_j = \binom{2j}{j} h'_{2j}$ and by induction $h'^{\star n}_j = \prod_{\ell=1}^{n} \binom{\ell j}{j} \cdot h'_{nj} = \frac{(nj)!}{(j!)^n} \cdot h'_{nj}$.

It follows that for $j \geq 1$, one has $h'^{\star p}_j = 0$.

It is obvious that the operators $A_j = (-1)^{m(j)} \sum_{d^\circ a < m(j)} g_j(a)\tau_a$ commutes with the translations $\tau_b,\ b \in \mathbb{F}_q[[T]]$. Since for b and y in $\mathbb{F}_q[[T]]$, one has $f(y+b) = \tau_b(f)(y) = \sum_{j\geq 0} A_j(\tau_b f)(0)h_j(y) = \sum_{j\geq 0} A_j(f)(b)h_j(y)$, one sees that $c(f) = \sum_{j\geq 0} A_j(f) \otimes h_j$, therefore $\varphi(h'_j)(f) = (id \otimes h'_j) \circ c(f) = \sum_{\ell\geq 0} A_\ell(f) < h'_j, h_\ell >= A_j(f)$, that is $\varphi(h'_j) = A_j$.

On the one hand, for the integers $j, \ell \geq 0$, one has $A_j(h_\ell) = (id \otimes h'_j) \circ c(h_\ell) = \sum_{i+k=\ell} < h'_j, h_k > \binom{\ell}{i} h_i = \binom{\ell}{j} h_{\ell-j}$, for $\ell \geq j$ and $\varphi(h'_j)(h_\ell) = 0$, for $\ell < j$.

On the other hand one sees that $A_i \circ A_j = \binom{i+j}{j} A_{i+j}$, etc......

$-\bullet-$ Let u be an element of $W(\mathbb{F}_q[[T]], K)$, one has $u = \varphi(\sigma \circ u) = (id \otimes (\sigma \circ u)) \circ c$. Hence for any continuous function f from $\mathbb{F}_q[[T]]$ to K, one has $u(f) = (id \otimes (\sigma \circ u)) \circ c(f) = \sum_{j\geq 0} A_j(f) \otimes (\sigma \circ u)(h_j) = \sum_{j\geq 0} u(h_j)(0)A_j(f)$. It follows that the family $(A_j)_{j\geq 0}$ is a topological basis of $W(\mathbb{F}_q[[T]], K)$ on which one considers the strong topology (called also the topology of pointwise uniform convergence), that is the topology induced by the semi-norms $\|u(f)\|, f \in \mathcal{C}(\mathbb{F}_q[[T]], K)$. More precisely any difference operator u can be written in a unique form as a pointwise uniform convergent series $u = \sum_{j\geq 0} u(h_j)(0)A_j$. Moreover, one has, $\|u\| = \sup_{j\geq 0} |u(h_j)(0)|$.

For complements on this subsection see [7].

2.2 The sequence of divided powers of some bounded measures

Let μ be a bounded measure expanded in the weak* topology as the unique convergent series $\mu = \sum_{j\geq 0} < \mu, h_j > h'_j$, one has $\mu^{\star p} = \sum_{j\geq 0} < \mu, h_j >^p h'^{\star p}_j =< \mu, h_0 >^p$.

One then deduces that $M(\mathbb{F}_q[[T]], K)$ is a local algebra with maximal ideal $\mathcal{M}_q = M_0(\mathbb{F}_q[[T]], K) = (K.h_0)^\perp$.

For $\mu = \sum_{j\geq 0} a_j h'_j$ and $\nu = \sum_{j\geq 0} b_j h'_j$ two elements of $M(\mathbb{F}_q[[T]], K)$, one has $\mu \star \nu = \sum_{j\geq 0} c_j h'_j$, with $c_j = \sum_{i+k=j} \binom{j}{i} a_i b_k$. Hence the algebra $M(\mathbb{F}_q[[T]], K)$ is isomorphic to a subalgebra of the algebra of Hurwitz formal series.

Therefore, one can define, as done by W. F. Keigher et F. L. Pritchard [12], the divided powers of an element of $\mathcal{M}_q$ (see also [10]).

For that, let ∂ be the continuous endomorphism of the Banach space $M(\mathbb{F}_q[[T]], K)$ which associates to $\mu = \sum_{j\geq 0} < \mu, h_j > h'_j$ the measure $\partial(\mu) = \sum_{j\geq 1} < \mu, h_j > h'_{j-1}$. One verifies easily that ∂ is a derivation of the algebra $M(\mathbb{F}_q[[T]], K)$, that is for the measures μ and ν, one has $\partial(\mu \star \nu) = \partial(\mu) \star \nu + \mu \star \partial(\nu)$. Moreover $\|\partial(\mu)\| \leq \|\mu\|$ and ∂ is weak*-continuous on any bounded subset of $M(\mathbb{F}_q[[T]], K)$.

In the opposite, one defines the operator of integration ι by setting $\iota(\mu) = \sum_{j\geq 0} < \mu, h_j > h'_{j+1}$.
Then ι is an isometrical linear endomorphim of $M(\mathbb{F}_q[[T]], K)$ such that $\partial \circ \iota = id$. Moreover, for any bounded measure μ, one has $\iota \circ \partial(\mu) = \mu - < \mu, h_0 > \sigma$ and for μ and ν in the maximal ideal $\mathcal{M}_q$, one has $\partial(\mu) = \partial(\nu)$ if and only if $\mu = \nu$.

Definition 2. *To any measure $\mu \in \mathcal{M}_q$ is associated the sequence of measures $(\gamma_n(\mu))_{n\geq 0}$, called the sequence of divided powers of μ, defined recursively by setting :* $\gamma_0(\mu) = \sigma,\ \gamma_1(\mu) = \mu$, *and for* $n \geq 1,\ \gamma_n(\mu) = \iota(\gamma_{n-1}(\mu) \star \partial(\mu))$.

For any integer $n \geq 1$ and any measure $\mu \in \mathcal{M}_q$, one has $\gamma_n(\mu) \in \mathcal{M}_q$.
Let us notice that the above definition of sequence of divided powers can be performed for any bounded measure.

Lemma 4. *Let ψ be an element of $M(\mathbb{F}_q[[T]], K)$ and $\mu,\ \nu \in \mathcal{M}_q$.*
Then for the non negative integers m and n, one has :

- -(i)- $\gamma_n(\mu + \nu) = \sum_{i+j=n} \gamma_i(\mu) \star \gamma_j(\nu)$.
- - (ii) - $\gamma_n(\psi \star \mu) = \psi^{\star n} \star \gamma_n(\mu)$.
- - (iii) - $\gamma_m(\mu) \star \gamma_n(\mu) = \binom{m+n}{n} \gamma_{m+n}(\mu)$.
- - (iv) - $\gamma_n(\gamma_m(\mu)) = \frac{(mn)!}{(m!)^n n!} \gamma_{mn}(\mu)$, *for* $m \geq 1$.
- - (v) - $n!\gamma_n(\mu) = \mu^{\star n}$.

Proof : According to the definition, one has : $\partial(\gamma_n(\mu)) = \gamma_{n-1}(\mu) \star \partial(\mu)$.

The relation (v) follows readily from (iii).

The other formulae are obtained by induction .

For instance, to prove (ii), let us suppose that for the integer $n \geq 1$, one has $\gamma_{n-1}(\psi\star\mu) == \psi^{\star n-1}\star\gamma_{n-1}(\mu)$. Then, one sees that $\partial(\gamma_n(\psi\star\mu)) = \gamma_{n-1}(\psi\star\mu)\partial(\psi\star\mu)$ $= \psi^{\star n-1}\star\gamma_{n-1}(\mu)\star(\partial(\psi) \star \mu + \psi \star \partial(\mu)) = \psi^{\star n-1}\star\partial(\psi)\star\gamma_{n-1}(\mu)\star\mu + \psi^{\star n}\star\gamma_{n-1}(\mu)\star \partial(\mu)$. But, one shows that $\gamma_{n-1}(\mu) \star \mu = n\gamma_n(\mu)$. This remark is also the beginning of the the proof of (iii). Hence, $\partial(\gamma_n(\psi\star\mu)) = n\psi^{\star n-1}\star\partial(\psi)\star\gamma_n(\mu) + \psi^{\star n}\star\partial(\gamma_n(\mu)) = \partial(\psi^{\star n} \star \gamma_n(\mu))$. And one has proved that $\gamma_n(\psi \star \mu) = \psi^{\star n} \star \gamma_n(\mu)$.

Lemma 5. *Let μ be an element of $\mathcal{M}_q$. Then, for any integer $n \geq 0$, one has* $\|\gamma_n(\mu)\| \leq \|\mu\|^n$.

Proof : Indeed, $\|\gamma_n(\mu)\| = \|\iota(\gamma_{n-1}(\mu) \star \partial(\mu))\| = \|\gamma_{n-1} \star \partial(\mu)\| \leq \|\gamma_{n-1}(\mu)\|\|\mu\|$. Then one obtains by induction that $\|\gamma_n(\mu)\| \leq \|\mu\|^n$.

Lemma 6. *Put $\delta = h'_1$, one has for any integer $n \geq 0$: $\gamma_n(\delta) = h'_n$.*

Proof : By definition $\partial(h'_n) = h'_{n-1}, \gamma_0(\delta) = h'_0 = \sigma$ and $\gamma_1(\delta) = \delta$. Hence, if $n = 2$, one has $\partial(\gamma_2(\delta)) = \gamma_1(\delta) \star \partial(\delta) = h'_1 = \partial(h'_2)$, then $\gamma_2(\delta) = h'_2$.

Assume that, $\gamma_{n-1}(\delta) = h'_{n-1}$, then $\partial(\gamma_n(\delta)) = \gamma_{n-1}(\delta) \star \partial(\delta) = \gamma_{n-1}(\delta) = h'_{n-1} = \partial(h'_n)$. It follows that $\gamma_n(\delta) = h'_n$.

2.3 Logarithm and exponential maps

2.3.1 Truncated logarithm and exponential maps

Let μ be an element of $\mathcal{M}_q$, with the following fact in mind, i.e $\mu^{\star p} = p!\gamma_p(\mu) = 0$, on can define logarithm and exponential mappings from $\sigma + \mathcal{M}_q$ (resp. $\mathcal{M}_q$) into $M(\mathbb{F}_q[[T]], K)$ by setting :
$\log(\sigma + \mu) = \sum_{j=1}^{p-1} \frac{(-1)^{j-1}}{j} \mu^{\star j} = \sum_{j=1}^{p-1} (-1)^{j-1}(j-1)!\gamma_j(\mu)$ and $\exp(\mu) = \sum_{j=0}^{p-1} \frac{1}{j!} \mu^{\star j} = \sum_{j=0}^{p-1} \gamma_j(\mu)$.

We have unfortunately stated an *inexact fact* in [7], Corollary 3-2-5, by saying that the corresponding maps defined for specific subspaces of $W(\mathbb{F}_q[[T]], K)$ were group isomorphisms. Indeed it is readily seen, within $M(\mathbb{F}_q[[T]], K)$, that the above exp is not a group homomorphism. The cases $p = 2$ and $p = 3$ are readily examined. But more generally, $\exp(\mu) \star \exp(\nu)$ contains terms as $\mu^{\star(p-1)} \star \nu^{\star(p-1)}$ which is no longer a term of $\exp(\mu + \nu)$.

What is only true is that the corresponding maps are bijective. We state here an **erratum** to this Corollary as a proposition within $M(\mathbb{F}_q[[T]], K)$.

Proposition 7. *Put for $\mu \in \mathcal{M}_q$, $\log(\sigma + \mu) = \sum_{j=1}^{p-1} \frac{(-1)^{j-1}}{j} \mu^{\star j}$ and $\exp(\mu) = \sum_{j=0}^{p-1} \frac{1}{j!} \mu^{\star j}$. Then $\exp : \mathcal{M}_q \longrightarrow \sigma + \mathcal{M}_q$ is a bijection of the set $\mathcal{M}_q$ onto $\sigma + \mathcal{M}_q$, with reciprocal map $\log : \sigma + \mathcal{M}_q \longrightarrow \mathcal{M}_q$.*

Proof : Let $\mu \in \mathcal{M}_q$, then $\log \circ \exp(\mu) = \log(1 + \omega(\mu)) = \sum_{\ell=1}^{p-1} \frac{(-1)^{\ell-1}}{\ell} \omega(\mu)^{\star \ell}$, where $\omega(\mu) = \sum_{j=1}^{p-1} \frac{1}{j!} \mu^{\star j}$. However, one has $\omega(\mu)^{\star \ell} = \sum_{n \geq \ell} \sum^{*}_{j_1 + \cdots j_k + \cdots j_\ell = n} \frac{1}{j_1!} \cdots \frac{1}{j_\ell!} \mu^{\star n}$.

The notation $\sum^{*}_{j_1 + \cdots j_k + \cdots j_\ell = n}$ means that in the sum the j_k are different from 0. Since

$\mu^{\star n} = 0$ for $n \geq p-1$, one has $\omega(\mu)^\ell = \sum_{n=\ell}^{p-1} \sum^{*}_{j_1+\cdots j_k+\cdots j_\ell=n} \frac{1}{j_1!}\cdots\frac{1}{j_\ell!}\mu^{\star n}$.

Hence $\log\circ\exp(\mu) = \sum_{\ell=1}^{p-1}\frac{(-1)^{j\ell-1}}{\ell}\sum_{n=\ell}^{p-1}\sum^{*}_{j_1+\cdots j_k+\cdots j_\ell=n}\frac{1}{j_1!}\cdots\frac{1}{j_\ell!}\mu^{\star n} =$

$= \sum_{n=1}^{p-1}\left(\sum_{\ell=1}^{n}\frac{(-1)^{\ell-1}}{\ell}\sum^{*}_{j_1+\cdots j_k+\cdots j_\ell=n}\frac{1}{j_1!}\cdots\frac{1}{j_\ell!}\right)\mu^{\star n}$.

One deduces from the fact that the formal power series with rational cœfficients $\log(1+X) = \sum_{j\geq 1}\frac{(-1)^{j-1}}{j}X^j$ and $\exp(X) = \sum_{j\geq 0}\frac{1}{j!}X^j$ are such that $\log(\exp X) = X$ that for $n \geq 2$, one has $\sum_{\ell=1}^{n}\frac{(-1)^{\ell-1}}{\ell}\sum^{*}_{j_1+\cdots j_k+\cdots j_\ell=n}\frac{1}{j_1!}\cdots\frac{1}{j_\ell!} = 0$.

For $n \leq p-1$ in the above sum the numbers lie in the ring of p-adic numbers therefore by reducing modulo p, we have the same identities. It follows that $\log\circ\exp(\mu) = \sum^{*}_{j_1+\cdots j_k+\cdots j_\ell=1}\frac{1}{j_1!}\cdots\frac{1}{j_\ell!}\mu + \sum_{\ell=2}^{n}\left(\frac{(-1)^{\ell-1}}{\ell}\sum^{*}_{j_1+\cdots j_k+\cdots j_\ell=n}\frac{1}{j_1!}\cdots\frac{1}{j_\ell!}\right)\mu^{\star n} =$ $\mu + 0 = \mu$.

In the same way, one proves that for $\mu \in \mathcal{M}_q$, one has $\exp\circ\log(\sigma+\mu) = \sigma+\mu$. ∎

N. B : $-\dagger-$ *Let endow $\mathcal{M}_q$ with the group law $\mu \odot \nu = \log(\exp(\mu) \star \exp(\nu))$ and $\mathcal{M}_q$ with the convolution, then the maps* exp *and* log *are reciprocal group isomorphisms.*
$-\dagger\dagger-$ *By the same way, with the group law $(\sigma+\mu)\oplus(\sigma+\nu) = \exp(\log(\sigma+\mu)+\log(\sigma+\nu))$ on $\sigma+\mathcal{M}_q$ and the addition on $\mathcal{M}_q$, one sees that the maps* log *and* exp *are again reciprocal group isomorphisms.*

2.3.2 Non truncated logarithm and exponential maps

One can define logarithm and exponential functions which in many respects correspond to the classical ones and are the "true" logarithm and exponential functions in our context here. See [8], [18] and [19].

Let r be a real number > 0 or $+\infty$. Let us set $\mathcal{M}_q^{r+} = \{\mu \in \mathcal{M}_q \ / \ \|\mu\| \leq r\}$ and $\mathcal{M}_q^{r-} = \{\mu \in \mathcal{M}_q \ / \ \|\mu\| < r\}$. One sees easily that $\mathcal{M}_q^{r\pm}$ is a additive subgroup of $\mathcal{M}_q$.

Put $\mathbb{A}_q = M(\mathbb{F}_q[[T]], K))$. A function $f : \mathcal{M}_q^{r-} \longrightarrow \mathbb{A}_q$ is said to be analytic if there exists a sequence $(a_n)_{n\geq 0} \subset \mathbb{A}_q$ such that the series $\sum_{n\geq 0} a_n\gamma_n$ converges uniformly to f on any $\mathcal{M}_q^{s+}$, $s < r$. One then writes $f = \sum_{n\geq 0} a_n\gamma_n$ and puts $An(\mathcal{M}_q^{r-})$ = the set of analytic functions on $\mathcal{M}_q^{r-}$. With the usual operations on sets of functions, $An(\mathcal{M}_q^{r-})$ becomes an algebra over the ring $\mathbb{A}_q$. One defines on this algebra a $\mathbb{A}_q$-derivation by setting $D\gamma_0 = 0$ and $D(\gamma_n) = \gamma_{n-1}, n \geq 1$. Let f and g be two analytic functions defined on possibly different subsets, if one can do the composition $f\circ g$ of this two functions, then one has the chain rule $D(f\circ g) = (D(f)\circ g)\star D(g)$.
For more informations on the on the algebra of analytic functions in this context, see [8] and [18].

$-\bullet-$ Let us put for $\mu \in \mathcal{M}_q$: $Log(\sigma+\mu) = \sum_{j\geq 1}(-1)^{j-1}(j-1)!\gamma_j(\mu) = \sum_{j=1}^{p}(-1)^{j-1}(j-1)!\gamma_j(\mu)$. This function is a polynomial function and then is an analytic function(first studied in [19]).

On sees that $D(Log(\sigma+\mu)) = \sum_{j=1}^{p}(-1)^{j-1}(j-1)!\gamma_{j-1}(\mu) = \sum_{j=0}^{p-1}(-\mu)^{\star j} = (\sigma+\mu)^{\star -1}$.

We know that the convolution product induces on $\sigma+\mathcal{M}_q$ a structure of commutative multiplicative group. The map $Log : \sigma+\mathcal{M}_q \longrightarrow M(\mathbb{F}_q[[T]],K)$ is a group homomorphism of the multiplicative group $\sigma+\mathcal{M}_q$ into the additive group $M(\mathbb{F}_q[[T]],K)$. This can be proved by fixing ν and showing that the derivative of the function $f(\mu) = Log((\sigma+\mu)(\sigma+\mu)) - Log(\mu) - Log(\nu)$ is zero.

$-\bullet-$ For the "exponential", if μ is an element of $\mathcal{M}_q$ there is no reason that the series $Exp(\mu) = \sum_{j\geq 0}\gamma_j(\mu)$ converges. But if $\|\mu\| \leq 1$, we shall see later that this series is weak*-convergent.

Thanks to the inequalities $\|\gamma_j(\mu)\| \leq \|\mu\|^j$, if $\|\mu\| < 1$, one sees that the series $Exp(\mu) = \sum_{j\geq 0}\gamma_j(\mu)$ is norm convergent, with $\|Exp(\mu)-\sigma\| \leq \|\mu\|$. Furthermore, one has $Exp = \sum_{j\geq 0}\gamma_j \in An(\mathcal{M}_q^{1-})$.

With the property (i) of Lemma 4, if μ and ν in $\mathcal{M}_q^{1-}$, one proves that $Exp(\mu+\nu) = Exp(\mu)\star Exp(\nu)$.

Proposition 8. *Let r be a real number such that $0 < r \leq 1$*
The mapping Exp is an isomorphism of the additive subgroup $\mathcal{M}_q^{r-}$ of $\mathcal{M}_q^{1-}$ onto the multiplicative subgroup $\sigma+\mathcal{M}_q^{r-}$ of $\sigma+\mathcal{M}_q^{1-}$. The reciprocal isomorphism is then restriction of the polynomial function Log.

Proof : See [8] and [18] for a full proof. For instance, one can calculate the derivatives of the analytic functions $Exp\circ Log$ and $Log\circ Exp$ and finds that $Exp\circ Log(\mu) = \sigma+\mu$ and $Log\circ Exp(\mu) = \mu$.

N.B : Let us notice that, for $0 < r < 1$, one obtains the same isomorphisms between the groups $\mathcal{M}_q^{r+}$ and $\sigma+\mathcal{M}_q^{r+}$.

2.4 The operations of substitution in the algebra of bounded measures

In the sequel we shall put for $\mu \in \mathcal{M}_q : \gamma_n(\mu) = \mu^{[n]}$. In particular $\gamma_n(\delta) = \delta^{[n]}$ and then for any measure $\mu \in M(\mathbb{F}_q[[T]],K)$, we write $\mu = \sum_{n\geq 0} < \mu, h_n > \delta^{[n]}$.

Let us consider $\mu = \sum_{j\geq 1}\alpha_j\delta^{[j]} \in \mathcal{M}_q^{1+}$ and for any integer $n \geq 1$, let us put $\mu^{[n]} = \sum_{j\geq 1}\alpha_j(n)\delta^{[j]}$.

One has $\partial(\mu^{[n]}) = \sum_{j\geq 1}\alpha_j(n)\delta^{[j-1]} = \mu^{[n-1]}\star\partial(\mu) = \sum_{j\geq 1}\alpha_j(n)\delta^{[j]}\star\sum_{j\geq 1}\alpha_j\delta^{[j-1]} =$

$$= \sum_{j\geq 1}\left(\sum_{k+\ell=j}\binom{k+\ell-1}{k}\alpha_k(n-1)\alpha_\ell\right)\delta^{[j-1]}.$$

Hence $\alpha_j(n) = \sum_{k+\ell=j} \binom{k+\ell-1}{k} \alpha_k(n-1)\alpha_\ell = \sum_{k=0}^{j-1} \binom{j-1}{k} \alpha_k(n-1)\alpha_{j-k}$.
For $n = 2$, one has $\alpha_1(2) = 0$, $\alpha_2(2) = \alpha_1^2$. Hence, $\mu^{[2]} = \sum_{j\geq 2} \alpha_j(2)\delta^{[j]}$, with $\alpha_2(2) = \alpha_1^2$.
One sees recursively that $\alpha_j(n) = 0, \forall j, 0 \leq j \leq n-1$, $\alpha_n(n) = \alpha_{n-1}(n-1)\alpha_1 = \alpha_1^n$.
In other words, one has $\mu^{[n]} = \sum_{j\geq n} \alpha_j(n)\delta^{[j]}$, with $\alpha_n(n) = \alpha_1^n$ and

$$\alpha_j(n) = \sum_{k=0}^{j-1} \binom{j-1}{k} \alpha_k(n-1)\alpha_{j-k} = \sum_{k=n-1}^{j-1} \binom{j-1}{k} \alpha_k(n-1)\alpha_{j-k}, \forall j \geq n.$$

For any continuous function $f : \mathbb{F}_q[[T]] \to K$ with Wagner-Carlitz expansion $f = \sum_{\ell\geq 0} a_\ell h_\ell$, one has $< \mu^{[n]}, f >= \sum_{j\geq n} a_j\alpha_j(n)$.

Assume $\|\mu\| = \sup_{j\geq 1} |\alpha_j| \leq 1$; by induction, one sees that $|\alpha_j(n)| \leq \max_{0\leq k\leq j-1} |a_k(n-1)||\alpha_{j-k}| \leq 1$.
Hence, for any continuous function $f = \sum_{\ell\geq 0} a_\ell h_\ell$, one obtains $| < \mu^{[n]}, f > | \leq \sup_{j\geq n} |a_j||\alpha_j(n)| \leq \sup_{j\geq n} |a_j|$ et $\lim_{n\to+\infty} | < \mu^{[n]}, f > | = 0$.

For $\psi = \sum_{n\geq 0} \beta_n\delta^{[n]} \in M(\mathbb{F}_q[[T]], K)$ and $\mu \in \mathcal{M}_q^{1+}$ = the subset of the elements μ of the maximal ideal $\mathcal{M}_q$ with norm ≤ 1. Since $|\beta_n < \mu^{[n]}, f > | \leq \|\psi\|| < \mu^{[n]}, f > |$, the series $\sum_{n\geq 0} \beta_n < \mu^{[n]}, f >$ converges in K. By the way, one defines a bounded measure noticed by $\psi \diamond \mu$ such that $< \psi \diamond \mu, f >= \sum_{n\geq 0} \beta_n < \mu^{[n]}, f >$.

Definition 3. *Let μ be an element of the maximal ideal $\mathcal{M}_q$ with norm ≤ 1 and let $\psi = \sum_{n\geq 0} \beta_n\delta^{[n]}$ be any element of $M(\mathbb{F}_q[[T]], K)$. The weak* convergent series $\psi \diamond \mu = \sum_{n\geq 0} \beta_n\mu^{[n]}$ is said to be obtained by substitution of μ into ψ. Moreover, one has $\psi \diamond \mu = \beta_0\sigma + \sum_{j\geq 1} \left(\sum_{n=1}^{j} \alpha_j(n)\beta_n \right) \delta^{[j]}$ and $\|\psi \diamond \mu\| \leq \|\psi\|$.*

Example : Let $\psi_1 = \sum_{n\geq 0} \delta^{[n]}$. By substitution of $\mu \in \mathcal{M}_q^{1+}$ into ψ_1, one obtains an extension to $\mathcal{M}_q^{1+}$ of the exponential map by setting $Exp(\mu) = \psi_1 \diamond \mu = \sum_{n\geq 0} \mu^{[n]}$.
Furthermore this mapping is an isomorphism of the additive group $\mathcal{M}_q^{1+}$ onto the multiplicative group $\sigma + \mathcal{M}_q^{1+}$.

Let notice that $\|Exp(\mu) - Exp(\nu)\| \leq \|\mu - \nu\|$, for μ, $\nu \in \mathcal{M}_q^{1+}$.

Remark 1 : *Let $\psi, \psi_1, \in M(\mathbb{F}_q[[T]], K)$ and $\alpha \in K$ and let μ and ν be two elements of $\mathcal{M}_q^{1+}$.*
Then, one has :

-(1)- $(\psi + \alpha\psi_1) \diamond \mu = \psi \diamond \mu + \alpha\psi_1 \diamond \mu$
-(2)- $(\psi \star \psi_1) \diamond \mu = (\psi \diamond \mu) \star (\psi_1 \diamond \mu)$
-(3)- $\delta \diamond \mu = \mu = \mu \diamond \delta$

-(4)- $(\psi \diamond \mu) \diamond \nu = \psi \diamond (\mu \diamond \nu)$
-(5)- $\partial(\psi \diamond \mu) = (\partial(\psi) \diamond \mu) \star \partial(\mu)$.
-(6)- $(\mu \diamond \nu)^{[n]} = \mu^{[n]} \diamond \nu$.

Proof : Let us remind that the derivation ∂ of $M(\mathbb{F}_q[[T]], K)$ is norm continuous as well weak*-continuous on any bounded subset of $M(\mathbb{F}_q[[T]], K)$. Then, since $\psi \diamond \mu = \sum_{n\geq 0} \beta_n \mu^{[n]}$ and $\|b_n \mu^{[n]}\| \leq \|\psi\|$, one has $\partial(\psi \diamond \mu) = \sum_{n\geq 0} \beta_n \partial(\mu^{[n]}) = \sum_{n\geq 0} \beta_n \mu^{[n-1]} \star \partial(\mu) = (\partial(\psi)) \diamond \mu) \star \partial(\mu)$. Hence (5) is proved. One deduces by induction (6) from (5). The relation (4) follows from (6). The other relations are easy to prove.

Proposition 9. *Let μ be an element of $\mathcal{M}_q^{1+}$.*
The mapping $w : \psi \longrightarrow w(\psi) = \psi \diamond \mu$ is a continuous endomorphism of the unitary Banach algebra $M(\mathbb{F}_q[[T]], K)$, which is weak*-continuous and has norm 1.
With the above notations, one has $< \psi \diamond \mu, f >=< \psi, g >$, where $g = \sum_{n\geq 0} b_n h_n$, with $b_0 = a_0$ and $b_n = \sum_{\ell\geq n} \alpha_\ell(n) a_\ell$, for $n \geq 1$.

Proof : From (1) and (2) of the above Remark 1 and from the fact that $\sigma \diamond \mu = \sigma$, one deduces that the map $w : \psi \longrightarrow w(\psi) = \psi \diamond \mu$ is an endomorphism of the unitary algebra $M(\mathbb{F}_q[[T]], K)$. As already noticed, $\|\psi \diamond \mu\| \leq \|\psi\|$, hence $\|w\| \leq 1$. Since $w(\sigma) = \sigma$, one obtains that $\|w\| = 1$.
By definition $\mu^{[0]} = \gamma_0(\mu) = \sigma$, hence $\alpha_0(0) = 1$. Let us remind that for $n \geq 1$ and with the above notations, one has $\mu^{[n]} = \sum_{j\geq 0} \alpha_j(n) \delta^{[j]}$, $\alpha_j(n) = 0,\ 0 \leq j \leq n-1$. Let $\psi = \sum_{n\geq 0} \beta_n \delta^{[n]}$ be a bounded measure. For any continuous function $f = \sum_{n\geq 0} a_n h_n$, one has $< \psi \diamond \mu, f >= \sum_{n\geq 0} \beta_n < \mu^{[n]}, f >= \sum_{n\geq 0} \beta_n b_n$, where $b_n =< \mu^{[n]}, f >= \sum_{\ell\geq 0} \alpha_\ell(n) a_\ell = \sum_{\ell\geq n} \alpha_\ell(n) a_\ell$. Since $|\alpha_\ell(n)| \leq 1$, one has $|b_n| \leq \sum_{\ell\geq n} |a_\ell|$ and $\lim_{n\to+\infty} b_n = 0$. Then one defines a continuous function g by setting $g = \sum_{n\geq 0} b_n h_n$, and one has $< w(\psi), f >=< \psi \diamond \mu, f >= \sum_{n\geq 0} \beta_n < \mu^{[n]}, f >=< \psi, g >$. From this identity, one deduces that the operator w is weak*-continuous.

Corollary 10. *Let μ be an element of $\mathcal{M}_q^{1+}$*
The linear operator ${}^t w$, transpose of the endomorphism $w : \psi \longrightarrow w(\psi) = \psi \diamond \mu$ of the unitary algebra $M(\mathbb{F}_q[[T]], K)$, induces a continuous coalgebra endomorphism of the Banach coalgebra $\mathcal{C}(\mathbb{F}_q[[T]], K)$.
In other words, one has $c \circ {}^t w = ({}^t w \otimes {}^t w) \circ c$ and $\sigma \circ {}^t w = \sigma$.

Proof : One has a natural isometrical identification of $\mathcal{C}(\mathbb{F}_q[[T]], K)$ as a subspace of its strong bidual $\mathcal{C}(\mathbb{F}_q[[T]], K)'' = M(\mathbb{F}_q[[T]], K)'$. Thus, $< w(\psi), f >=< \psi, {}^t w(f) >=< \psi, g >\Longrightarrow {}^t w(f) = g = \sum_{n\geq 0} (\sum_{\ell\geq n} \alpha_\ell(n) a_\ell) h_n \in \mathcal{C}(\mathbb{F}_q[[T]], K)$.
Let ψ and ψ_1 be two bounded measures. For any continuous function f, one has $< \psi \otimes \psi_1, c \circ {}^t w(f) >= (\psi \otimes \psi_1) \circ c({}^t w(f)) =< \psi \star \psi_1, {}^t w(f) >=< w(\psi \star \psi_1), f >=$

$$=< w(\psi) \star w(\psi_1), f >= (w(\psi) \otimes w(\psi_1)) \circ c(f) = (w(\psi) \otimes w(\psi_1)) \left(\sum_{i \geq 1} f_i^1 \otimes f_i^2 \right) =$$
$$= \sum_{i \geq 1} w(\psi)(f_i^1) \otimes w(\psi_1)(f_i^2) = \sum_{i \geq 1} < \psi, {}^t w(f_i^1) >< \psi_1, {}^t w(f_i^2) >=$$
$< \psi \otimes \psi_1, ({}^t w \otimes {}^t w) \circ c(f) >$.
It follows that $c \circ {}^t w = ({}^t w \otimes {}^t w) \circ c$. ∎

Remark 2 :
More generally, one can prove that for any ultrametric Banach coalgebra H which is a pseudo-reflexive normed space (i.e the canonical application of H into its bidual is an isometry), the operation of transposition of linear operators induces an isomorphism between the monoid of continuous coalgebra endomorphisms of H and that of the continuous algebra endomorphisms of the dual algebra H' of H.

$- \bullet -$ One deduces from Remark 1-(6) that the endomorphism of the algebra $M(\mathbb{F}_q[[T]], K)$ defined by $w(\psi) = \psi \diamond \mu$ (for $\mu \in \mathcal{M}_q^{1+}$ commutes with the divided powers, that is for $\nu \in \mathcal{M}_q^{1+}$, one has $w(\nu^{[n]}) = w(\nu)^{[n]}$.
$- \bullet -$ Let w be an algebra endomorphism of $M(\mathbb{F}_q[[T]], K)$, since for any element ν of the maximal ideal $\mathcal{M}_q$, one has $\nu^{\star p} = 0$, one sees that $0 = w(\nu^{\star p}) = w(\nu)^{\star p}$ and $w(\nu)$ belongs to $\mathcal{M}_q$. In particular $w(\delta) \in \mathcal{M}_q$. Moreover, if w is continuous with norm ≤ 1, weak*-continuous and commutes with the divided powers, then setting $w(\delta) = \mu$, one has $\|\mu\| \leq 1$. And for any bounded measure $\psi = \sum_{n \geq 0} \beta_n \delta^{[n]}$, one has $w(\psi) = \sum_{n \geq 0} \beta_n w(\delta^{[n]}) = \sum_{n \geq 0} \beta_n w(\delta)^{[n]} = \psi \diamond \mu$. The following proposition is a topological counterpart of the one given in [10].

Proposition 11. *Let w be a continuous algebra endomorphism of the unitary Banach algebra $M(\mathbb{F}_q[[T]], K)$.*
-(*i*)- *Assume that there exists a bounded measure ν such $\partial \circ w = w \circ m_\nu \circ \partial$, where m_ν is the multiplication operator defined defined by $m_\nu(\mu) = \mu \star \nu$. Then w commutes with divided powers.*
-(*ii*)- *Conversely, assume that $Im(\partial \circ w) \subseteq Im(w)$ and w is weak*- continuous and commutes with divided powers, then for any bounded measure ν such that $\partial \circ w(\delta) = w(\nu)$, one has $\partial \circ w = w \circ m_\nu \circ \partial$.*

Proof : $-(i)-$ Let w be a continuous algebra endomorphism of the unitary algebra $(M(\mathbb{F}_q[[T]], K)$, such that there exists a bounded measure ν satisfying $\partial \circ w = w \circ m_\nu \circ \partial$.
Then, one has $\partial \circ w(\mu^{[n]}) = w \circ m_\nu \circ \partial(\mu^{[n]}) = w \circ m_\nu(\mu^{[n-1]} \star \partial(\mu)) = w(\nu \star \mu^{[n-1]} \star \partial(\mu)) = w(\mu^{[n-1]} \star \nu \star \partial(\mu)) = w(\mu^{[n-1]}) \star w(\nu \star \partial(\mu)) = w(\mu^{[n-1]}) \star (w \circ m_\nu \circ \partial)(\mu) = w(\mu^{[n-1]}) \star \partial(w(\mu))$.
Let us suppose, by induction hypothesis, that $w(\mu^{[n-1]}) = w(\mu)^{[n-1]}$, then one obtains $w(\mu^{[n]}) = \iota \circ \partial(w(\mu^{[n]})) = \iota(w(\mu)^{[n-1]} \star \partial(w(\mu))) = w(\mu)^{[n]}$.

$-(ii)-$ Assume $Im(\partial \circ w) \subseteq Im(w)$, then there exists a bounded measure ν such that $\partial \circ w(\delta) = w(\nu)$. If w commutes with the divided powers, for any integer $n \geq 0$, one has $w(\delta^{[n]}) = w(\delta)^{[n]}$ and $\partial \circ w(\delta^{[n]}) = \partial(w(\delta)^{[n]}) = w(\delta)^{[n-1]} \star \partial(w(\delta)) = w(\delta^{[n-1]}) \star \partial(w(\delta)) = w(\delta^{[n-1]}) \star w(\nu) = w(\nu \star \delta^{[n-1]}) = w \circ m_\nu \circ \partial(\delta^{[n]})$, because $\partial(\delta) = \sigma$.
If in addition, w is weak*-continuous, since the sequence $(\delta^{[n]})_{n \geq 0}$ converges weakly

to zero, the sequence $(w(\delta^{[n]}))_{n\geq 0}$ converges also weakly to zero. Hence for any bounded measure
$\mu = \sum_{n\geq 0} < \mu, h_n > \delta^{[n]}$ and any continuous fonction $f : \mathbb{F}_q[[T]] \to K$, one has $< w(\mu), f >=< \mu, {}^t w(f) >=< \sum_{n\geq 0} < \mu, h_n > \delta^{[n]}, {}^t w(f) >= \sum_{n\geq 0} < \mu, h_n > \delta^{[n]}, {}^t w(f) >= \sum_{n\geq 0} < \mu, h_n >< \delta^{[n]}, {}^t w(f) >= \sum_{n\geq 0} << \mu, h_n > w(\delta^{[n]}), f >= \sum_{n\geq 0} << \mu, h_n > w(\delta)^{[n]}, f > =< \sum_{n\geq 0} < \mu, h_n > w(\delta)^{[n]}, f >$.
Therefore one has the weak convergent sum $w(\mu) = \sum_{n\geq 0} < \mu, h_n > w(\delta)^{[n]}$.

Since $\partial \circ w(\delta^{[n]}) = w \circ m_\mu \circ \partial(\delta^{[n]})$, $\forall n \geq 0$, one obtains by linearity and weak* convergence that $\partial \circ w(\mu) = w \circ m_\nu \circ \partial(\mu)$, for any bounded measure μ, i.e. $\partial \circ w = w \circ m_\nu \circ \partial$.

Lemma 12. *Let* $\mu = \sum_{j\geq 1} \alpha_j \delta^{[j]}$ *be an element of* $\mathcal{M}_q^{1+}$*.*
Then, there exists $\nu \in \mathcal{M}_q^{1+}$ *such that* $\nu \diamond \mu = \delta = \mu \diamond \nu$ *if and only if* $\|\mu\| = |\alpha_1| = 1$*. In this case,* μ *is said to be reversible with reverse* $\nu = \mu^{\diamond -1}$*.*

Proof : Let $\nu = \sum_{j\geq 1} \beta_j \delta^{[j]}$ be a measure such that $\|\nu\| \leq 1$. One has $\nu \diamond \mu = \sum_{j\geq 1} \left(\sum_{n=1}^{j} \alpha_j(n) \beta_n \right) \delta^{[j]}$ and $\nu \diamond \mu = \delta \iff \alpha_1 \beta_1 = 1$ and for $j \geq 2 :$ $\sum_{n=1}^{j-1} \alpha_j(n) \beta_n + \alpha_1^j \beta^j = 0$. Since $|\alpha_1| \leq 1$ and $|\beta_1| \leq 1$, with $\alpha_1 \beta_1 = 1$, one necessary obtains $|\alpha_1| = 1 = |\beta_1|$. Hence $\|\mu\| = |\alpha_1| = 1$.
Reciprocally, assume that $\|\mu\| = |\alpha_1| = 1$, then $|\alpha_j| \leq 1, \forall j \geq 1$ and $|\alpha_j(n)| \leq 1$ for any integer $n \geq 1$ and any integer $j \geq n$.
Take $\beta_1 = \alpha_1^{-1}$, with the relations $\beta_j = -\alpha_1^{-j} \sum_{n=1}^{j-1} \alpha_j(n) \beta_n$, we shall determine recursively the β_j. Indeed $\beta_2 = -\alpha_1^{-2} \alpha_2$ is such that $|\beta_2| \leq 1$. Assume by induction hypothesis that $|\beta_n| \leq 1, \forall n, 1 \leq n \leq j-1$, then one sees that $|\beta_j| \leq |-\alpha_1^{-j}| \sup_{1\leq n\leq j-1} |\alpha_j(n)||\beta_n| \leq 1$. Setting $\nu = \sum_{j\geq 1} \beta_j \delta^{[j]}$, one obtains $\nu \diamond \mu = \delta$.
Assume that we have found $\nu = \sum_{j\geq 1} \beta_j \delta^{[j]}$ such that $\nu \diamond \mu = \delta$, then $|\beta_1| = 1$. Hence, there exists ν_1 such that $\nu_1 \diamond \nu = \delta$. Therefore $\mu = \delta \diamond \mu = (\nu_1 \diamond \nu) \diamond \mu = \nu_1 \diamond (\nu \diamond \mu) = \nu_1 \diamond \delta = \nu_1$.

Examples : The measures $\mu_0 = \exp(\delta) - \sigma = \sum_{j=1}^{p-1} \frac{1}{j!} \delta^j = \sum_{j=1}^{p-1} \delta^{[j]}$ and $\mu_1 = Exp(\delta) - \sigma = \sum_{j\geq 1} \delta^{[j]}$ belongs to $\mathcal{M}_q^{1+}$ and are reversible with reverse respectively $\log(\delta)$ and $Log(\delta)$.

Proposition 13. *Let* $\mu = \sum_{j\geq 1} \alpha_j \delta^{[j]} \in \mathcal{M}_q^{1+}$ *such that* $\|\mu\| = |\alpha_1\| = 1$*.*
The map w *which associates to any bounded measure* ψ *the bounded measure* $w(\psi) = \psi \diamond \mu$ *is an isometrical automorphism of the unitary Banach algebra* $M(\mathbb{F}_q[[T]], K)$ *and is weak***-continuous .*

Proof : It remains to show that the endomorphism w of the algebra is an isometrical bijection.
We know that if μ is a measure such that $\|\mu\| \leq 1$, then the corresponding algebra endomorphism w obtained by substitution of μ has norm ≤ 1.
Since $\|\mu\| = 1 = |\alpha_1|$, the measure μ is reversible of reverse $\mu^{\diamond -1}$. By definition $w(\psi) = \psi \diamond \mu$. Put $\omega(\psi) = \psi \diamond \mu^{\diamond -1}$, which also defines an algebra endomorphism of $M(\mathbb{F}_q[[T]], K)$. One verifies that $w \circ \omega(\psi) = w(\psi \diamond \mu^{\diamond -1}) = (\psi \diamond \mu^{\diamond -1}) \diamond \mu = \psi \diamond (\mu^{\diamond -1} \diamond \mu) = \psi \diamond \delta = \psi$. And also $\omega \circ w(\psi) = \psi$.
Since $\|w(\psi)\| = \|\psi \diamond \mu\| \leq \|\psi\|$, one has $\|\psi\| = \|\omega(w(\psi))\| \leq \|w(\psi)\| \leq \|\psi\| \Longrightarrow \|w(\psi)\| = \|\psi\|$.
Which finishes the proof of the proposition. ∎

3 Ultrametric umbral calculus in characteristic p

With the above propositions we are able to recover some important statements of classical umbral calculus.

3.1 Orthonormal basis of binomial divided power sequences of polynomials

A sequence of polynomial functions $(Q_n)_{n\geq 0}$ is said to a sequence of binomial divided power sequence if for any integer $n \geq 0$ the polynomial Q_n is of degree n with $Q_0 = 1$ and $Q_n(x+y) = \sum\limits_{i+j=n} \binom{n}{i} Q_i(x)Q_j(y),\ \forall n \geq 0$. This condition can be expressed with the coproduct, i.e $c(Q_n) = \sum\limits_{i+j=n} \binom{n}{i} Q_i \otimes Q_j$.

Theorem 14. *Let* $\mu = \sum\limits_{j\geq 1} \alpha_j \delta^{[j]} \in \mathcal{M}_q^{1+}$ *such that* $\|\mu\| = |\alpha_1\| = 1$.
There exists a unique sequence of polynomial functions $(Q_n)_{n\geq 0}$ *in* $\mathcal{C}(\mathbb{F}_q[[T]], K)$ *such that* $deg(Q_n) = n$ *and* $< \mu^{[n]}, Q_k >= \delta_{n,k}$.
Moreover $(Q_n)_{n\geq 0}$ *is an orthonormal basis of* $\mathcal{C}(\mathbb{F}_q[[T]], K)$.

Proof : -(i)- Let us remind that for $\mu = \sum\limits_{j\geq 1} \alpha_j \delta^{[j]}$, one has $\mu^{[n]} = \sum\limits_{j\geq n} \alpha_j(n) \delta^{[j]}$, with $\alpha_n(n) = \alpha_1^n$. Moreover $\|\mu^{[n]}\| = 1$.
If Q is a polynomial of degree $deg(Q) \leq n-1$, one has $< \mu^{[n]}, Q >= 0$.
Since $\mu^{[0]} = \sigma$, for Q_0 a polynomial of degree 0, one has $< \sigma, Q_0 >= 1 \Longrightarrow Q_0 = 1$.
Let $Q_1 = a_0 h_0 + a_1 h_1$ be such that $< \sigma, Q_1 >= \alpha_0 = 0$ and $< \mu^{[1]}, Q_1 >= 1$.
By hypothesis $1 = \sum\limits_{j\geq 1} \alpha_j < \delta^{[j]}, Q_1 >= a_1 \alpha_1 \Longrightarrow a_1 = \alpha_1^{-1}\ Q_1 = \alpha_1^{-1} h_1$.

Let us consider a polynomial $Q_n = \sum\limits_{\ell=0}^{n} a_{n,\ell} h_\ell$ such that $< \mu^{[j]}, Q_n >= \delta_{j,n}$.

Hence, for $0 \leq j \leq n$, one has $\sum\limits_{k\geq j} \sum\limits_{\ell=0}^{n} \alpha_k(j) a_{n,\ell} < \delta^{[k]}, h_\ell >= \sum\limits_{\ell=j}^{n} \alpha_\ell(j) a_{n,\ell} = \delta_{j,n}$.

For $j = n$, one has $1 =< \mu^{[n]}, \sum\limits_{\ell=0}^{n-1} a_{n,\ell} h_\ell > + \alpha_n(n) a_{n,n} = 0 + \alpha_n(n) a_{n,n}$

$\Longrightarrow a_{n,n} = \alpha_n(n)^{-1} = \alpha_1^{-n}$.
For $j = n-1$, one has $0 = \sum_{\ell=n-1}^{n} \alpha_\ell(n-1)a_{n,\ell} \Longrightarrow$
$a_{n,n-1} = -\alpha_{n-1}(n-1)^{-1}a_{n,n}\alpha_n(n-1) = -\alpha_1^{-n+1}a_{n,n}\alpha_n(n-1) = -\alpha_1^{-2n+1}\alpha_n(n-1)$.
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$
$0 =< \mu^{[j]}, Q_n >= \sum_{\ell=j}^{n} \alpha_\ell(j)a_{n,\ell} \Longrightarrow a_{n,j} = -\alpha_j(j)^{-1} \sum_{\ell=j+1}^{n} \alpha_\ell(j)a_{n,\ell} = -\alpha_1^{-j} \sum_{\ell=j+1}^{n} \alpha_\ell(j)a_{n,\ell}$.
By the way, one determines the cœfficients of the polynomial Q_n. For $j = 1$, one has $\alpha_\ell(1) = \alpha_\ell$ and $0 =< \mu, Q_n >= \sum_{\ell=1}^{n}\alpha_\ell a_{n,\ell} = \alpha_1 a_{n,1} + \sum_{\ell=2}^{n}\alpha_\ell a_{n,\ell} \Longrightarrow a_{n,1} = -\alpha_1^{-1}\sum_{\ell=2}^{n}\alpha_\ell a_{n,\ell}$.
Since $0 =< \sigma, Q_n >= a_{n,0}$, one sees that Q_n is unique such that $< \mu^{[k]}, Q_n >= \delta_{k,n}, \forall k \geq 0$.

- (ii) - We have just proved that $Q_n = \sum_{\ell=0}^{n} a_{n,\ell}h_\ell$, with $a_{n,n} = -\alpha_1^{-n}$, $a_{n,n-1} = -\alpha_1^{-2n+1}\alpha_n(n-1)$. Therefore $|a_{n,n}| = |\alpha_1|^{-n} = 1$ and $|a_{n,n-1}| = |\alpha_1|^{-2n+1}|\alpha_n(n-1)| \leq 1$.
Assuming that $|a_{n,\ell}| \leq 1$, for $j+1 \leq \ell \leq n$, one obtains
$|a_{n,j}| \leq |\alpha_1|^{-1} \sup_{j+1\leq\ell\leq n} |\alpha_\ell(j)|a_{n,\ell}| \leq 1$.
Since for $n \geq 1$, one has $a_{n,0} = 0$, then one sees that $|a_{n,\ell}| \leq 1, \forall 0 \leq \ell \leq n$, with $|a_{n,n}| = 1$.
Therefore, since $Q_0 = 1$, one has $\|Q_n\| = \sup_{0\leq\ell\leq n} |a_{n,\ell}| = 1, \forall n \geq 0$.
Since any polynomial Q_n is of degree n, one verifies that $(Q_n)_{n\geq 0}$ is a basis of the vector space of polynomials $K[x]$. Let Q be an element of $K[x]$ of degree $deg(Q) = m$, one has $Q = \sum_{n=0}^{m} b_nQ_n$. On one hand one has a $\|Q\| \leq \max_{0\leq n\leq m} |b_n|\|Q_n\| = \max_{0\leq n\leq m} |b_n|$.
On the other hand $|b_n| = |< \mu^{[n]}, Q >| \leq \|\mu^{[n]}\|\|Q\| \leq \|Q\|$. Hence $\max_{0\leq n\leq m} |b_n| = \|Q\|$.
It follows that $(Q_n)_{n\geq 0}$ is an orthonormal family. Since this family is a basis of the space of polynomial functions that is a dense subspace of the space of continuous functions $\mathcal{C}(\mathbb{F}_q[[T]], K)$, one sees that it is an orthonormal basis of $\mathcal{C}(\mathbb{F}_q[[T]], K)$. ■

$- \bullet -$ With the notations of the above theorem, one has that the sequence $(\mu^{[n]})_{n\geq 0}$ is the dual basis of the orthonormal basis $(Q_n)_{n\geq 0}$. Let $\psi = \sum_{n\geq 0} \beta_n\delta^{[n]}$ be the weak*-expansion of the bounded measure ψ and $w(\psi) = \psi \diamond \mu = \sum_{n\geq 0} \beta_n\mu^{[n]}$.
If $f = \sum_{k\geq 0} c_kQ_k$ is the expansion of the continuous function $f : \mathbb{F}_q[[T]] \longrightarrow K)$ in the orthonormal basis $(Q_k)_{k\geq 0}$; since $< \mu^{[n]}, Q_k >= \delta_{n,k}$, one has $< w(\psi), f >= \sum_{n\geq 0} \beta_nc_n$. But $\lim_{k\to+\infty} c_k = 0$, hence the following continuous $g = \sum_{k\geq 0} c_kh_k$ is such that $< \psi, g >= \sum_{n\geq 0} \beta_nc_n =< w(\psi), f >$. ■

Corollary 15. *Let* $\mu = \sum_{j\geq 1} \alpha_j \delta^{[j]} \in \mathcal{M}_q^{1+}$ *be such that* $\|\mu\| = |\alpha_1\| = 1$.
The transpose ${}^t w$ *of the algebra automorphism* w *associated to* μ *induces on* $\mathcal{C}(\mathbb{F}_q[[T]], K)$ *a continuous linear endomorphism* W *which is an isometrical coalgebra automorphism.*
One has $W(Q_n) = h_n,\ \forall n \geq 0$, *and* $c(Q_n) = \sum_{i+j=n} \binom{n}{i} Q_i \otimes Q_j$.
Setting $\nu = \sum_{n\geq 1} \eta_n \delta^{[n]}$ *the reverse* $\mu^{\diamond -1}$ *of* μ, *and* $\nu^{[n]} = \sum_{j\geq n} \eta_j(n)\delta^{[j]}$, *one has* $Q_n = \sum_{j=1}^{n} \eta_n(j) h_j$.

Proof : The transpose ${}^t w$ of the isometrical bijective linear operator w is an isometrical linear operator of the Banach dual space of $M(\mathbb{F}_q[[T]], K)$, this dual is the strong bidual of $\mathcal{C}(\mathbb{F}_q[[T]], K)$. Since $\mathcal{C}(\mathbb{F}_q[[T]], K)$ is pseudo-reflexive, that is the canonical map of $\mathcal{C}(\mathbb{F}_q[[T]], K)$ into it bidual is isometrical, one sees that the weak*-dual of $M(\mathbb{F}_q[[T]], K)$ is equal to $\mathcal{C}(\mathbb{F}_q[[T]], K)$. Since w is weak*-continuous its transpose ${}^t w$ induces by restriction, a continuous linear W of $\mathcal{C}(\mathbb{F}_q[[T]], K)$.
Then, as already seen, W is a Banach coalgebra endomorphism of $\mathcal{C}(\mathbb{F}_q[[T]], K)$.
Since $< \delta^{[n]}, h_k >= \delta_{n,k} =< \mu^{[n]}, Q_k >=< w(\delta^{[n]}), Q_k >=< \delta^{[n]}, {}^t w(Q_k) >$, one has $< \delta^{[n]}, {}^t w(Q_k) - h_k >= 0,\ \forall n \geq 0$. Hence $W(Q_k) = {}^t w(Q_k) = h_k$.
Let $U = W^{\circ -1}$ be the reciprocal of W.
Obviously U is a coalgebra endomorphism and one has $U(h_n) = Q_n,\ \forall n \geq 0$.
Therefore $c(Q_n) = c \circ U(h_n) = (U \otimes U) \circ c(h_n) = (U \otimes U)\left(\sum_{i+j=n} \binom{n}{i} h_i \otimes h_j\right) =$
$= \sum_{i+j=n} \binom{n}{i} U(h_i) \otimes U(h_j) = \sum_{i+j=n} \binom{n}{i} Q_i \otimes Q_j$. This means that the sequence $(Q_n)_{n\geq 0}$ is a binomial divided power sequence.

N.B : Keep the notations of Corollary 15.
Since for any bounded measure ψ, one has ${}^t W(\psi) = \psi \diamond \mu$, one sees that ${}^t U(\psi) = {}^t W^{\circ -1}(\psi) = \psi \diamond \nu$. Hence $< \delta^{[j]}, Q_n >=< \delta^{[j]}, U(h_n) >=< {}^t U(\delta^{[j]}), h_n >=< {}^t U(\delta)^{[j]}, h_n >=< \nu^{[j]}, h_n >$.
Furthermore $Q_n = \sum_{j=1}^{n} < \delta^{[j]}, Q_n > h_j = \sum_{j=1}^{n} < \nu^{[j]}, h_n > h_j = \sum_{j=1}^{n} \gamma_n(j) h_j$.
What we have proved is that given a bounded measure $\mu = \sum_{n\geq 1} \alpha_n \delta^{[n]}$ such that $\|\mu\| = |\alpha_1| = 1$, there exists a unique binomial divided power sequence $(Q_n)_{n\geq 0}$ such that $< \mu^{[n]}, Q_k >= \delta_{n,k},\ \forall n, k \geq 0$. Moreover this sequence is an orthonormal basis of $\mathcal{C}(\mathbb{F}_q[[T]], K)$.

3.2 The algebra of difference operators

Let us remind that $W(\mathbb{F}_q[[T]], K)$ is the algebra of difference operators for the compact additive group $\mathbb{F}_q[[T]]$, that is the same as the comodule endomorphisms of the Banach coalgebra $\mathcal{C}(\mathbb{F}_q[[T]], K)$, in other words, these operators are the continuous linear endomorphism u of $\mathcal{C}(\mathbb{F}_q[[T]], K)$ such that $c \circ u = (id \otimes u) \circ c$.

Also, considering on $W(\mathbb{F}_q[[T]],K)$ the strong topology, one has a topological basis $(A_j)_{j\geq 0}$ of $W(\mathbb{F}_q[[T]],K)$ where A_j has been defined previously as follows : $A_j = (-1)^{m(j)} \sum_{d^\circ a<m(j)} g_j(a)\tau_a$, furthermore $A_j = \varphi(h'_j) = (id\otimes h'_j)\circ c$. Then if u is an element of $W(\mathbb{F}_q[[T]],K)$, one has the strong sum $u = \sum_{j\geq 0} a_j A_j$, with $a_j = u(h_j)(0)$ and $\|u\| = \sup_{j\geq 0} |a_j|$.

Let d be the derivation of $W(\mathbb{F}_q[[T]],K)$ analogous to ∂, hence such that $d(u) = \sum_{j\geq 1} a_j A_{j-1}$ and $\mathcal{I}$ the operator of integration $\mathcal{I}(u) = \sum_{j\geq 0} a_j A_{j+1}$. Let $\mathcal{W}_q = W_0(\mathbb{F}_q[[T]],K)$ the space of the difference operator v such that $v(h_0)(0) = 0$. As for bounded measures, on defines the divided powers of an element v of $\mathcal{W}_q$ = by setting $v^{[0]} = id$ and inductively for the integers $n\geq 1$, $v^{[n]} = \mathcal{I}(v^{[n-1]}\circ d(v))$.

Let φ be the isometric isomorphism of Banach algebra of $M(\mathbb{F}_q[[T]],K)$ onto $W(\mathbb{F}_q[[T]],K)$ such that $\varphi(\mu) = (id\otimes\mu)\circ c$. One has $d\circ\varphi = \varphi\circ\partial$ et $\mathcal{I}\circ\varphi = \varphi\circ\iota$. From these relations one deduces that for any integer $n\geq 0$, one has $\varphi(\mu^{[n]}) = \varphi(\mu)^{[n]}$. If θ_1 is the reciprocal isomorphism of φ, one also has $\theta_1(v^{[n]}) = (\theta_1(v))^{[n]}$.

We have set $h'_1 = \delta$ and have seen that $\delta^{[n]} = h'_n$. Put $A_1 = A$, it is readily seen that $A^{[n]} = \varphi(\delta^{[n]}) = A_n$. Therefore any difference operator u can be written in the strong sum $u = \sum_{j\geq 0} a_j A^{[j]}$. Let $v = \sum_{j\geq 1} \alpha_j A^{[j]} \in \mathcal{W}_q$ ($\alpha_j = v(h_j)(0)$), then as for bounded measures, one has $v^{[n]} = \sum_{j\geq n} \alpha_j(n) A^{[n]}$, with $\alpha_j(n) = \sum_{k=0}^{j-1} \binom{j-1}{k} \alpha_k(n-1)\alpha_{j-k}$, if $j\geq n$ and $\alpha_j(n) = 0$, for $j<n$.

Lemma 16. *Let $v = \sum_{j\geq 1} \alpha_j A^{[j]} \in \mathcal{W}_q$ and let the divided powers of v be $v^{[n]} = \sum_{j\geq n} \alpha_j(n) A^{[n]}$.*

For any continuous function $f = \sum_{\ell\geq 0} a_\ell h_\ell$, one has $v^{[n]}(f) = \sum_{\ell\geq 0}\left(\sum_{j\geq n}\binom{j+\ell}{j}\alpha_j(n)a_{j+\ell}\right)h_\ell$.
In particular, for any polynomial function P of degree $\leq n-1$, one has $v^{[n]}(P) = 0$ and if Q is a polynomial function of degree n and leading cœfficient β_n, one has $v^{[n]}(Q) = \alpha_1^n\beta_n h_0$.

Proof: Let $f = \sum_{\ell\geq 0} a_\ell h_\ell$ be a continuous function of $\mathbb{F}_q[[T]]$ into K.

For any integer $j\geq 0$, one has $A^{[j]}(f) = \sum_{\ell\geq 0}\binom{j+\ell}{j} a_{j+\ell}h_\ell$. Hence, $v^{[n]}(f) =$

$$= \sum_{j\geq n}\alpha_j(n)A^{[j]}(f) = \sum_{j\geq n}\alpha_j(n)\sum_{\ell\geq 0}\binom{j+\ell}{j}a_{j+\ell}h_\ell = \sum_{\ell\geq 0}\left(\sum_{j\geq n}\binom{j+\ell}{j}\alpha_j(n)a_{j+\ell}\right)h_\ell.$$

For integers such $\ell < n \leq j$, one has $A^{[j]}(h_\ell) = 0$. Hence, in one hand, for $\ell<n$, one sees that $v^{[n]}(h_\ell) = \sum_{j\geq n}\alpha_j(n)A^{[j]}(h_\ell) = 0$.

On the other hand $v^{[n]}(h_n) = \alpha_n(n)A^{[n]}(h_n) + \sum_{j\geq n+1}\alpha_j(n)A^{[j]}(h_n) = \alpha_n(n)h_0 + 0 = \alpha_1^n h_0$. Then if $P = \sum_{k=0}^{m}\beta_k h_k$, with $m<n$, one obtains $v^{[n]}(P) = 0$.

And for $Q = \sum_{k=0}^{n-1} \beta_k h_k + \beta_n h_n = P + \beta_n h_n$, one has $v^{[n]}(Q) = 0 + \beta_n v^{[n]}(h_n) = \alpha_1^n \beta_n h_0$. ■

$- \bullet \bullet -$ When $\|v\| \leq 1$, one can substitute v in any difference operator $u = \sum_{n\geq 0} \beta_n A^{[n]}$, by setting $u \diamond v = \sum_{n\geq 0} \beta_n v^{[n]}$. One has $u \diamond v = \beta_0 id + \sum_{j\geq 1} \left(\sum_{n=1}^{j} \alpha_j(n)\beta_n \right) A^{[j]}$ and $\|u \diamond v\| \leq \|u\|$.

Assume that $v = \sum_{j\geq 1} \alpha_j A^{[j]}$ is such that $\|v\| = |\alpha_1| = 1$. One has $\|v^{[n]}\| = |\alpha_n(n)| = |\alpha_1|^n = 1$.

Put $\mu = \theta_1(v)$. One has $\mu = \sum_{j\geq 1} \alpha_j \delta^{[j]}$ et $\|\mu\| = |\alpha_1| = 1$. We know that there exists a sequence of polynomial functions $(Q_n)_{n\geq 0}$ such that $deg(Q_n) = n$ and $< \mu^{[k]}, Q_n >= \delta_{k,n}$. Moreover $(Q_n)_{n\geq 0}$ is a sequence of binomial divided power and is an orthonormal basis of $\mathcal{C}(\mathbb{F}_q[[T]], K)$.
Let us notice that since the sequence $(Q_n)_{n\geq 0}$ is a binomial divided power sequence, one has $Q_n(0) = 0, \ \forall n \geq 1$.
For any integer $n \geq 0$, one has $v^{[n]} = (id \otimes \mu^{[n]}) \circ c$. Hence $v^{[n]}(Q_m) = (id \otimes \mu^{[n]}) \circ c(Q_m) = \sum_{i+j=m} \binom{m}{i} Q_i < \mu^{[n]}, Q_j >= \sum_{i+j=m} \binom{m}{i} Q_i \delta_{n,j}$.
Then $v^{[n]}(Q_m) = 0$, if $m < n$ and $v^{[n]}(Q_m) = \binom{m}{n} Q_{m-n}$, for $0 \leq n \leq m$. Furthermore if Q is a polynomial with cœfficients in K and degree $deg(Q) = m$, one has $Q = \sum_{n=0}^{m} v^{[n]}(Q)(0) Q_n$.

Let us notice that setting $Q_n = \sum_{\ell=0}^{n} \beta_\ell(n) h_\ell$, one has $\beta_0(n) = 0$, if $n \geq 1$ and, since $h_0 = v^{[n]}(Q_n) = \alpha_1^n \beta_n(n)$, one obtains $\beta_n(n) = \alpha_1^{-n}$.
Now, if $(Q_n^1)_{n\geq 0}$ is an other sequence of polynomials such that $deg(Q_n^1) = n$, $Q_0^1 = 1$, $Q_n^1(0) = 0$, for $n \geq 1$ and satisfying $v^{[n]}(Q_m^1) = 0$, if $m < n$ and $v^{[n]}(Q_m^1) = \binom{m}{n} Q_{m-n}^1$, for $0 \leq n \leq m$, then $Q_n^1 = Q_n$, $\forall n \geq 0$. Indeed for any integer $m \geq 1$, one sees that $Q_m^1 = \sum_{n=1}^{m} \binom{m}{n} Q_{m-n}^1(0) Q_n = Q_m$.

Theorem 17. *Let $v = \sum_{j\geq 1} \alpha_j A^{[j]}$ be a difference operator such that $\|v\| = |\alpha_1| = 1$. There exists a unique binomial divided power sequence $(Q_n)_{n\geq 0}$ such that $v^{[n]}(Q_m) = 0$, if $m < n$, $v^{[n]}(Q_m) = \binom{m}{n} Q_{m-n}$, for $m \geq n$. This sequence is an orthonormal basis of the Banach space $\mathcal{C}(\mathbb{F}_q[[T]], K)$.*
More precisely, any continuous function $f : \mathbb{F}_q[[T]] \longrightarrow K$ can be expanded as a unique uniformly convergent series $f = \sum_{n\geq 0} v^{[n]}(f)(0) Q_n$ and $\|f\| = \sup_{n\geq 0} |v^{[n]}(f)(0)|$.

Proof : The existence and unicity of the sequence of polynomials $(Q_n)_{n\geq 0}$ satisfying the properties in the theorem have been verified just above.

Let $f = \sum_{n\geq 0} b_n Q_n$ be the expansion in the orthonormal basis $(Q_n)_{n\geq 0}$ of the continuous function $f : \mathbb{F}_q[[T]] \longrightarrow K$. Then for any integer $k \geq 0$, one has $v^{[k]}(f) = \sum_{n\geq k} b_n \binom{n}{k} Q_{n-k} \Longrightarrow v^{[k]}(f)(0) = b_k$.

Remark 3: -(i)- *Let as above K be a complete valued field extension of the field $\mathbb{F}_q((T))$ of Laurent formal series.*
Let $(Q_n)_{n\geq 0}$ be a sequence of polynomials with cœfficients in K and which is a binomial divided power sequence. Then for any integer $\ell \geq 0$, the polynomial Q_{p^ℓ} is additive, that is $Q_{p^\ell}(x+y) = Q_{p^\ell}(x) + Q_{p^\ell}(y)$.

-(ii)- *Let $x \in \mathbb{F}_q[[T]]$, for any continuous function $f : \mathbb{F}_q[[T]] \longrightarrow K$, with the notations of Theorem 17, one has $\tau_x(f) = \sum_{n\geq 0} v^{[n]}(\tau_x f)(0) Q_n$.*
Furthermore, if u is a difference operator of $\mathcal{C}(\mathbb{F}_q[[T]], K)$, one has $\tau_x \circ u(f) = u \circ \tau_x(f) = \sum_{n\geq 0} v^{[n]}(\tau_x f)(0) u(Q_n)$. Therefore, for $x,\ y \in \mathbb{F}_q[[T]]$, one has $u(f)(x+y) = \sum_{n\geq 0} v^{[n]}(f)(x) u(Q_n)(y)$. It follows that $u(f)(x) = \sum_{n\geq 0} u(Q_n)(0) v^{[n]}(f)(x)$, the series of functions being uniformly convergent. This last formula can also be obtained by the substitution in the bounded measures. ■

$-\dagger-$ Again, let $(Q_n)_{n\geq 0}$ be a sequence of polynomial functions which is an orthonormal basis such that $Q_0 = h_0, deg(Q_n) = n$ and is a binomial divided power sequence, i.e $c(Q_n) = \sum_{i+j=n} \binom{n}{i} Q_i \otimes Q_j$.
One defines a continuous linear endomorphism v of $\mathcal{C}(\mathbb{F}_q[[T]], K)$ by setting $v(Q_0) = 0$ and for any integer $n \geq 1 : v(Q_n) = nQ_{n-1}$. Hence if $f = \sum_{n\geq 0} a_n Q_n$ is a continuous function, one has $v(f) = \sum_{n\geq 1} n a_n Q_{n-1}$. Furthermore $\|v\| = 1$. Notice that if the integer n is a *multiple* of p, then $v(Q_n) = 0$.
Reducing modulo p the following identities between integers numbers $n\binom{n-1}{i} = \dfrac{n!}{i!(n-i-1)!} = (n-i)\dfrac{n!}{i!(n-i)!} = (n-i)\binom{n}{i}$, one obtains $n\binom{n-1}{i} \equiv (n-i)\binom{n}{i}$ (mod. p).

From $c(Q_n) = \sum_{i=0}^{n} \binom{n}{i} Q_i \otimes Q_{n-i}$, setting $Q_{-1} = 0$, one sees that

$$c \circ v(Q_n) = nc(Q_{n-1}) = \sum_{i=0}^{n-1} n\binom{n-1}{i} Q_i \otimes Q_{n-1-i} = \sum_{i=0}^{n} (n-i)\binom{n}{i} Q_i \otimes Q_{n-i-1} =$$
$$= \sum_{i=0}^{n} \binom{n}{i} Q_i \otimes (n-i) Q_{n-i-1} = \sum_{i=0}^{n} \binom{n}{i} Q_i \otimes v(Q_{n-i}) = (id \otimes v)\left(\sum_{i=0}^{n} \binom{n}{i} Q_i \otimes Q_{n-i}\right) =$$
$$= (id \otimes v) \circ c(Q_n) \Longrightarrow c \circ v = (id \otimes v) \circ c.$$

In other words, v is a difference operator of the Banach coalgebra $\mathcal{C}(\mathbb{F}_q[[T]], K)$ such that $v(h_0) = v(Q_0) = 0$. Hence $v = \sum_{j\geq 1} \alpha_j A^{[j]}$. But $Q_1 = \alpha_{1,1} h_1$ et $1 = \|Q_1\| =$

$|\alpha_{1,1}|$, therefore $h_0 = Q_0 = v(Q_1) = \alpha_{1,1}v(h_1) = \alpha_{1,1} \cdot \alpha_1 h_0 \Longrightarrow \alpha_1 = \alpha_{1,1}^{-1}$. As a consequence $\|v\| = 1 = |\alpha_1|$.

Let $\mu = \sigma \circ v = \theta_1(v)$ be the linear form corresponding to v. One has $\mu = \sum_{j\geq 1} \alpha_j \delta^{[j]}$ and $\|\mu\| = 1 = |\alpha_1|$. We have shown that in this case, there exists a unique orthonormal basis $(Q_n^1)_{n\geq 0}$ of $\mathcal{C}(\mathbb{F}_q[[T]], K)$ such that $< \mu^{[n]}, Q_k^1 >= \delta_{n,k}$. This basis is a binomial divided power sequences of polynomial functions.

Furthermore $Q_0^1 = h_0$ and $v^{[n]}(Q_k^1) = \binom{k}{n} Q_{k-n}^1$, if $n \leq k$, and $v^{[n]}(Q_k^1) = 0$, for $k < n$.

It is readily seen as above that from $v(Q_1^1) = h_0 = v(Q_1)$ one deduces $Q_1^1 = Q_1$.

For any integer $n \geq 1$, one has $\tau_x(Q_n)(y) = \sum_{k=0}^{n} v^{[k]}(\tau_x(Q_n))(0)Q_k^1 = \sum_{k=0}^{n} \tau_x \circ v^{[k]}(Q_n)(0)Q_k^1$.

Hence $Q_n(x+y) = \sum_{k=0}^{n} v^{[k]}(Q_n)(x)Q_k^1(y)$. Since $c(Q_n) = \sum_{k=0}^{n} \binom{n}{k} Q_{n-k} \otimes Q_k$, one obtains the identities $\sum_{k=0}^{n} v^{[k]}(Q_n)(x)Q_k^1(y) = Q_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} Q_{n-k}(x)Q_k(y)$.

One deduces by induction from $v(Q_n) = nQ_{n-1}$, that for any integer $j \geq 1$, one has $v^j(Q_n) = n(n-1)\cdots(n-j+1)Q_{n-j}$.

But $j!v^{[j]} = v^j$, hence $j!v^{[j]}(Q_n) = n(n-1)\cdots(n-j+1)Q_{n-j}$. Therefore for j an integer such that

$0 \leq j \leq p-1$, one has $v^{[j]}(Q_n) = \dfrac{n(n-1)\cdots(n-j+1)}{j!}Q_{n-j} = \binom{n}{j} Q_{n-j}$, with the convention $Q_{n-j} = 0$, if $n < j$.

$-\bullet-$ Assume that $n \leq p-1$, for $0 \leq j \leq n$, one has $v^{[j]}Q_n = \binom{n}{j} Q_{n-j}$. Hence $Q_n = \sum_{j=0}^{n} v^{[j]}(Q_n)(0)Q_j^1 = \sum_{j=0}^{n} \binom{n}{j} Q_{n-j}(0)Q_j^1 = \sum_{j=0}^{n-1} \binom{n}{j} Q_{n-j}(0)Q_j^1 + Q_0(0)Q_n^1 = Q_n^1$.

However, one has $Q_p = \sum_{j=0}^{p} v^{[j]}(Q_p)(0)Q_j^1 = \sum_{j=0}^{p-1} \binom{p}{j} Q_{p-j}(0)Q_j^1 + v^{[p]}(Q_p)(0)Q_p^1 = v^{[p]}(Q_p)(0)Q_p^1$ and $Q_{p+1} = \sum_{j=0}^{p+1} v^{[j]}(Q_{p+1})(0)Q_j^1 = \sum_{j=0}^{p-1} v^{[j]}(Q_{p+1})(0)Q_j^1 + v^{[p]}(Q_{p+1})(0)Q_j^1 + v^{[p+1]}(Q_{p+1})(0)Q_j^1 = \sum_{j=0}^{p-1} \binom{p+1}{j} Q_{p+1-j}(0)Q_j^1 + v^{[p]}(Q_{p+1})(0)Q_j^1 + v^{[p+1]}(Q_{p+1})(0)Q_j^1 = v^{[p]}(Q_{p+1})(0)Q_j^1 + v^{[p+1]}(Q_{p+1})(0)Q_j^1$. More generally, for any integer $n \geq p+1$, one has $Q_n = \sum_{j=p+1}^{n} v^{[j]}(Q_n)(0)Q_j^1$.

$-\bullet\bullet-$ Since $v^{[n]}$ is a difference operator, one has $c \circ v^{[\ell]}(Q_n) = (id \otimes v^{[\ell]}) \circ c(Q_n) = \sum_{k=0}^{n} \binom{n}{k} Q_{n-k} \otimes v^{[\ell]}(Q_k) \Longrightarrow v^{[\ell]}(Q_n) = (id \otimes \sigma) \circ c \circ v^{[\ell]}(Q_n) = \sum_{k=0}^{n} \binom{n}{k} v^{[\ell]}(Q_k)(0)Q_{n-k} =$

$\sum_{k=\ell}^{n} \binom{n}{k} v^{[\ell]}(Q_k)(0)Q_{n-k}$.

$-\bullet\bullet\bullet-$ One has $v^{[\ell]}(Q_n) = \binom{n}{\ell} Q_{n-\ell}, \forall n, \ell \Longleftrightarrow v^{[\ell]}(Q_\ell)(0) = 1$ and $v^{[\ell]}(Q_k)(0) = 0, \ \forall k \neq \ell$. Or, equivalently, $< \mu^{[\ell]}, Q_k >= v^{[\ell]}(Q_k)(0) = \delta_{k,\ell}$, where $\mu = \sigma \circ v$. In these conditions, one sees that $Q_n^1 = Q_n, \forall n \geq 0$ and since $(Q'_n)_{n\geq 0}$ is the dual basis of $(Q_n)_{n\geq 0}$, one has $Q'_\ell = \mu^{[\ell]}$ and $< Q'_j, v^{[\ell]}(Q_n) >= \binom{n}{\ell} \delta_{j,n-\ell}$.

$-\bullet\bullet\bullet\bullet-$ Let U and U_1 be the continuous linear endomorphisms of the Banach space $\mathcal{C}(\mathbb{F}_q[[T]], K)$ such that $U(h_n) = Q_n$ and $U_1(h_n) = Q_n^1$. Since the bases $(Q_n)_{n\geq 0}$ and $(Q_n^1)_{n\geq 0}$ are both binomial divided power sequences, the operators U and U_1 are isometrical coalgebra automorphisms of the Banach coalgebra $\mathcal{C}(\mathbb{F}_q[[T]], K)$.

One deduces from Corollary 15 that $\omega_1 = {}^tU_1$ is an algebra automorphism of the dual Banach algebra $M(\mathbb{F}_q[[T]], K)$ that commutes with the divided powers operations on $M(\mathbb{F}_q[[T]], K)$. Moreover, if ν is the reverse measure of $\mu = \sigma \circ v$, then ω_1 is defined by substitution, that is for any bounded measure ψ, one has $\omega_1(\psi) = \psi \diamond \nu$. It follows that the bases $(Q_n)_{n\geq 0}$ and $(Q_n^1)_{n\geq 0}$ coincide if and only if the coalgebra automorphism U defined by $U(h_n) = Q_n$ is such that its transpose $w = {}^tU$, an algebra automorphism of the algebra $M(\mathbb{F}_q[[T]], K)$ commutes with the operations of divided powers.

Summarizing, one has proved the following proposition.

Proposition 18. *The subset of the difference operators* $v = \sum_{j\geq 1} \alpha_j A^{[j]}$ *of* $\mathcal{C}(\mathbb{F}_q[[T]], K)$ *such that* $\|v\| = |\alpha_1| = 1$ *corresponds bijectively to the orthonormal bases* $(Q_n)_{n\geq 0}$ *of* $\mathcal{C}(\mathbb{F}_q[[T]], K)$ *which are binomial divided power sequences of polynomial functions such that the transpose* $w = {}^tU$ *of the coalgebra automorphism* U *of* $\mathcal{C}(\mathbb{F}_q[[T]], K)$ *defined by* $U(h_n) = Q_n$ *is an algebra automorphism of the dual algebra* $M(\mathbb{F}_q[[T]], K)$ *that commutes with the operations of divided powers.*

N.B : $-\bullet-$ There exist orthonormal basis $(g_n)_{n\geq 0}$ of $\mathcal{C}(\mathbb{F}_q[[T]], K)$ satisfying the condition : $c(g_n) = \sum_{i+j=n} \binom{n}{i} g_i \otimes g_j$ and are not polynomial functions. Orthonormal bases satisfying the binomial divided power condition correspond bijectively to the continuous coalgebra automorphisms of $\mathcal{C}(\mathbb{F}_q[[T]], K)$.
In [7] we have characterized all continuous bialgebra endomorphisms of $\mathcal{C}(\mathbb{F}_q[[T]], K)$. The description of the set of coalgebra endomorphisms done in the algebraic setting in [11] can be adapted here, with assuming the appropriate continuity conditions.
$-\bullet\bullet-$ The umbral calculus developed here, as already said, is different from the one given by A. Kochubei in [14] which is done on the closed subspace of $\mathcal{C}(\mathbb{F}_q[[T]], K)$ of the $\mathbb{F}_q$-linear continuous functions.

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References

[1] Y. Amice, Interpolation p-adique, *Bull. Soc. Math. France* 92 (1964), 117-180.

[2] L. Carlitz, On polynomials in Galois field, *Bull. Amer. Math. Soc.* 38 (1932), 736-744

[3] L. Carlitz, On certain functions connected with polynomials in Galois field, *Duke Math. Jour.* 1 (1935), 137-168.

[4] L. Carlitz, A set of polynomials, *Duke Math. Jour.* 6 (1940), 486-504.

[5] B. Diarra, Complete ultrametric Hopf algebras which are free Banach spaces, In *"p-adic functional analysis"*, edited by W.H. Schikhof, C. Perez-Garcia and J. Kąkol, Marcel Dekker, New-York (1997), 61-80.

[6] B. Diarra, The continuous coalgebra endomorphisms of $\mathcal{C}(\mathbb{Z}_p, K)$, *Bull. Belg. Math. Soc.* -supplement - December (2002), 63-79.

[7] B. Diarra, The Hopf algebra structure of the space of continuous functions on power series over $\mathbb{F}_q$ and Carlitz polynomials - *Ultrametric Functional Analysis* - Seventh International Conference - W.H. Schikof, C. Perez-Garcia, A. Escassut, Editors - *Contemporary Mathematics*, AMS, Vol. 319, (2003), 75-97.

[8] B. Diarra, Applications exponentielles en caractéristique p, *Séminaires d'Analyse et d'Algèbre* / 1997/98 - 21 Octobre et 15 Décembre 1997 *http://math.univ-bpclermont.fr/* $\sim$ *diarra* / Expo-carc-p.

[9] D. Goss, *Basic structures of function field arithmetic*, Springer Verlag, Berlin, (1996).

[10] M. Héraoua, *Cogèbre binomiale et calcul ombral des opérateurs différentiels* -Thèse de Doctorat, Université de Limoges, 15 juillet 2004.

[11] M. Héraoua and A. Salinier, Endormorphisms of the binomial coalgebra, *J. Pure and Appl. Algebra* 193 (2004), 193-230

[12] W.F. Keigher and F.L. Pritchard, Hurwitz series as formal functions, *J. Pure and Appl. Algebra*, 146 (2000), 191-304.

[13] A.N. Kochubei, Harmonic oscillators in characteristic p, *Letters in Math. Phys.* 45 (1998), 11-20.

[14] A.N. Kochubei, Umbral calculus in characteristic p, *Adv. Applied Math.*, 34 (2005), 175-191

[15] M. van der Put, Algèbres de fonctions continues p-adiques I-II, *Indag. Math.* 30 (1968), 401-411; 412-420.

[16] M. van der Put, Difference equations over p-adic fields, *Math. Ann.* 198 (1972), 4189-203.

[17] A.C.M. van Rooij, *Non-archimedean functional analysis*, Marcel Dekker, Inc., New York, (1978).

[18] T. Satoh, On duality over a certain divided power algebra with positive characteristic, *Manuscripta Math. Vol.92*, (1997), 153-172 .

[19] J.P. Soublin, Puissances divisées en caractéristique non nulle, *Journal of Algebra,* Vol. 110, (1987), 523-529.

[20] G.C. Wagner, Interpolation series for continuous functions on π-adique completion of $GF(q, x)$, *Acta Arithm.* 17 (1971), 389-406.

[21] G.C. Wagner, Linear operators in local fields of prime characteristic, *J. Reine Angew. Math.* 251 (1971), 153-160.

[22] G.C. Wagner, Interpolation series in local fields of prime characteristic, *Duke Math. Jour.* 39 (1972), 203-210.

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