Exceptional sets with a weight in a unit ball

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Abstract

For a given number $s > -1$ and a multiindex $\alpha \in \mathbb{N}^n$ we give a proof of the following equality:

$$\int_{\|z\|<R} z^{\alpha}\overline{z^{\alpha}} \left(R^2 - \|z\|^2\right)^s dz = \frac{\pi^n \alpha! R^{2(s+|\alpha|+n)}}{\prod_{i=1}^{|\alpha|+n}(s+i)}.$$

As a result we receive different properties of the sets defined by the following formula

$$E^s(f) = \left\{z \in \partial\mathbb{B}^n : \int_{|\lambda|<1} |f(\lambda z)|^2 \left(1 - |\lambda|^2\right)^s d\mathfrak{L}^2 = \infty\right\}$$

for the holomorphic function $f \in \mathbb{O}(\mathbb{B}^n)$.

1 Preface

This paperdeals with the exceptional sets with a weight:

$$\chi_s : \mathbb{B}^n \ni z \longrightarrow \chi_s(z) = (1 - \|z\|^2)^s.$$

We denote $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. The exceptional set with a weight s for the holomorphic function $f \in \mathbb{O}(\mathbb{B}^n)$ in this paper is denoted as

$$E^s(f) = \left\{z \in \partial\mathbb{B}^n : \int_{\mathbb{D}z} |f|^2 \chi_s d\mathfrak{L}^2 = \infty\right\}.$$

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In the 80s Peter Pflug [7] posed the question whether there exists a domain $\Omega \subset \mathbb{C}^n$, a complex subspace M of $\mathbb{C}^n$ and a holomorphic, square integrable function f in Ω such that $f|_{M\cap\Omega}$ is not square integrable.

A similar question was posed by Jacques Chaumat [1] in the late 80s; whether there exists a holomorphic function f in a ball $\mathbb{B}^n$ such that for any linear, complex subspace M in $\mathbb{C}^n$ a holomorphic function $f|_{M\cap\mathbb{B}^n}$ is not square integrable.

The questions mentioned above inspired further investigation among the authors [2, 3, 4, 5, 6]. These authors consider holomorphic functions which are not square integrable along complex lines with a point 0. Due to [2, 3] we know that for a convex domain Ω with a boundary of a class C^1 it is possible to create a holomorphic function f, which is not square integrable along any real manifold M of a class C^1 crossing transversally a boundary Ω.

Let E be any circular subset of the type G_δ of $\partial\mathbb{B}^n$. In the papers [5, 6] we presented a construction of the holomorphic function $f \in \mathbb{O}(\mathbb{B}^n)$ for which $E = E^0(f)$. Additionally in the paper [6] we proved that a function f can be selected so that $\int_{\mathbb{B}^n\setminus\Lambda(E)} |f|^2 d\mathfrak{L}^{2n} < \infty$, where $\Lambda(E) = \{\lambda z : |\lambda| = 1, z \in E\}$.

In this paper we deal mainly with the exceptional sets with a non-trivial weight. The following theorem is of key importance for this paper:

Theorem 2.2. *For $k \in \mathbb{N}_+$, a number $s > -1$, a number $R > 0$ and for a multiindex $\alpha = (\alpha_1, ..., \alpha_k)$ we have the following equality*

$$\int_{|z_1|^2+...+|z_k|^2<R^2} z^\alpha \overline{z^\alpha} \left(R^2 - \|z\|^2\right)^s dz = \frac{\pi^k \alpha! R^{2(s+|\alpha|+k)}}{\prod_{i=1}^{|\alpha|+k}(s+i)}.$$

Let us define the functional:

$$\mathfrak{F}_s : \mathbb{O}(\mathbb{B}^n) \ni f = \sum_{m\in\mathbb{N}} p_m \to \sum_{m\in\mathbb{N}} \frac{p_m}{\sqrt{(m+n)^s}} \in \mathbb{O}(\Omega),$$

where p_m denote homogeneous polynomial of the degree m.

Observe that $\mathfrak{F}_{s+t} = \mathfrak{F}_s \circ \mathfrak{F}_t$. We use this property to describe the functional $\mathfrak{F}$:

Theorem 2.4. Define $s > -1$. The operator $\mathfrak{F}_s$ is properly defined and has the following properties:

1. $\mathfrak{F}_s(\mathbb{O}(\mathbb{B}^n)) = \mathbb{O}(\mathbb{B}^n)$,

2. there exist the constants $c_1, c_2 > 0$ such that:

$$\begin{aligned} c_1 \int_{\mathbb{D}z} |\mathfrak{F}_s(f)|^2 d\mathfrak{L}^2_{\mathbb{D}z} &\leq \int_{\mathbb{D}z} |f|^2 \chi_s d\mathfrak{L}^2_{\mathbb{D}z} \\ &\leq c_2 \int_{\mathbb{D}z} |\mathfrak{F}_s(f)|^2 d\mathfrak{L}^2_{\mathbb{D}z} \end{aligned}$$

for $f \in \mathbb{O}(\mathbb{B}^n)$, $z \in \partial\mathbb{B}^n$ and

$$c_1 \int_{\mathbb{B}^n} |\mathfrak{F}_s(f)|^2 d\mathfrak{L}^{2n} \leq \int_{\mathbb{B}^n} |f|^2 \chi_s d\mathfrak{L}^{2n} \leq c_2 \int_{\mathbb{B}^n} |\mathfrak{F}_s(f)|^2 d\mathfrak{L}^{2n}$$

for $f \in \mathbb{O}(\mathbb{B}^n)$.

Due to this Theorem it is possible to create the exceptional sets with a weight on the basis of the exceptional sets without a weight:

Example 2.5. *Let E be a circular set of the type G_δ of the measure zero in $\partial\mathbb{B}^n$. Define $s > -1$. Therefore there exists a holomorphic function $f \in \mathbb{O}(\mathbb{B}^n)$ such that $E = E^s(f)$ and $\int_{\mathbb{B}^n} |f|^2 \chi_s d\mathfrak{L}^{2n} < \infty$.*

Due to Theorems 2.2 and 2.4 we can prove some estimations connected with the exceptional sets with a weight:

Theorem 2.7. *If $s > -1$, the function $f \in \mathbb{O}(\mathbb{B}^n)$ is such that: $\int_{\mathbb{B}^n} |f|^2 \chi_s d\mathfrak{L}^{2n} < \infty$, then $E^{s+n-1}(f) = \emptyset$.*

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Lemma 2.1. *Let us define $R > 0$. We have the following equality*

$$\int_0^R t^m (R-t)^s dt = \frac{m! R^{s+m+1}}{\prod_{i=1}^{m+1}(s+i)}$$

for $s > -1$ and $m \in \mathbb{N}$. Additionally

$$\int_0^R t^m (R-t)^s dt = \infty$$

for $s \leq -1$ and $m \in \mathbb{N}$.

Proof. First, we assume that $s > -1$. Let $G_s^m(R) = \int_0^R t^m (R-t)^s dt$. It is easy to observe that $G_s^0 = \left[-\frac{(R-t)^{s+1}}{s+1}\right]_0^R = \frac{R^{s+1}}{s+1}$. Therefore we get the equality

$$G_s^m(R) = \frac{m! R^{s+m+1}}{\prod_{i=1}^{m+1}(s+i)} \tag{2.1}$$

for $m = 0$ and for any $s > -1$. We assume that we have (2.1) for a given $m \in \mathbb{N}$ and $s > -1$. We can calculate

$$\begin{aligned}
G_s^{m+1}(R) &= \int_0^R t^{m+1}(R-t)^s dt \\
&= \left[-\frac{t^{m+1}(R-t)^{s+1}}{s+1}\right]_0^R + \int_0^R (m+1)t^m \left(\frac{(R-t)^{s+1}}{s+1}\right) dt \\
&= \frac{m+1}{s+1} G_{s+1}^m(R) = \frac{m+1}{s+1} \frac{m! R^{s+m+2}}{\prod_{i=1}^{m+1}(s+1+i)} \\
&= \frac{(m+1)! R^{s+m+2}}{\prod_{i=1}^{m+2}(s+i)}
\end{aligned}$$

for a given $m \in \mathbb{N}$ and for any $s > -1$. Therefore, using induction, we have the equality (2.1) for every $m \in \mathbb{N}$ and for any $s > -1$.

Let $s \leq -1$. Let ϵ be such that $\max\{0, R-1\} < R - \epsilon < R$. We can calculate

$$\begin{aligned}
\int_0^R t^m (R-t)^s dt &\geq \int_{R-\epsilon}^R t^m (R-t)^{-1} dt \geq (R-\epsilon)^m \int_{R-\epsilon}^R (R-t)^{-1} dt \\
&\geq (R-\epsilon)^m \left[-\ln(R-t)\right]_{R-\epsilon}^R = \infty,
\end{aligned}$$

which finishes the proof. ■

Theorem 2.2. *For $k \in \mathbb{N}_+$, a number $s > -1$, a number $R > 0$ and for a multiindex $\alpha = (\alpha_1, ..., \alpha_k)$ we have*

$$\int_{|z_1|^2+...+|z_k|^2<R^2} z^\alpha \overline{z^\alpha} \left(R^2 - \|z\|^2\right)^s dz = \frac{\pi^k \alpha! R^{2(s+|\alpha|+k)}}{\prod_{i=1}^{|\alpha|+k}(s+i)}.$$

Proof. We define

$$G^m_{\alpha,s}(R) = \int_{|z_1|^2+...+|z_m|^2\leq R^2} z^\alpha \overline{z^\alpha} \left(R^2 - \|z\|^2\right)^s dz.$$

We prove the following equality

$$G^m_{\alpha,s}(R) = \frac{\pi^m \alpha! R^{2(s+|\alpha|+m)}}{\prod_{i=1}^{|\alpha|+m}(s+i)}. \tag{2.2}$$

If $m = 1$, then $\alpha \in \mathbb{N}$. Therefore, due to Lemma 2.1, we can calculate

$$\begin{aligned} G^1_{\alpha,s}(R) &= \int_{|z|^2\leq R^2} z^\alpha \overline{z^\alpha} \left(R^2 - |z|^2\right)^s dz = 2\pi \int_0^R r^{2\alpha+1}(R^2 - r^2)^s dr \\ &= \pi \int_0^{R^2} t^\alpha (R^2 - t)^s dt = \frac{\pi \alpha! R^{2(s+|\alpha|+1)}}{\prod_{i=1}^{|\alpha|+1}(s+i)}. \end{aligned}$$

for $R > 0$ and $s > -1$. We assume that we have (2.2) for a given $m \in \mathbb{N}_+$, any number $R > 0$, a number $s > -1$ and a multiindex $\alpha = (\alpha_1, ..., \alpha_m)$. We define a multiindex $\beta = (\alpha_1, ..., \alpha_m, \beta_{m+1})$. We have the equality

$$\begin{aligned} G^{m+1}_{\beta,s}(R) &= \int_{|z_1|^2+...+|z_{m+1}|^2\leq R^2} z^\beta \overline{z^\beta} \left(R^2 - \|z\|^2\right)^s dz \\ &= \int_{|z_{m+1}|^2\leq R^2} |z_{m+1}|^{2\beta_{m+1}} G^m_{\alpha,s}\left(\sqrt{R^2 - |z_{m+1}|^2}\right) dz_{m+1} \\ &= 2\pi \int_0^R r^{2\beta_{m+1}+1} \frac{\pi^m \alpha! (R^2 - r^2)^{(s+|\alpha|+m)}}{\prod_{i=1}^{|\alpha|+m}(s+i)} dr \\ &= \int_0^{R^2} t^{\beta_{m+1}} \frac{\pi^{m+1} \alpha! (R^2 - t)^{(s+|\alpha|+m)}}{\prod_{i=1}^{|\alpha|+m}(s+i)} dt. \end{aligned}$$

Using Lemma 2.1 we can calculate:

$$\begin{aligned} G^{m+1}_{\beta,s}(R) &= \frac{\pi^{m+1}\alpha!}{\prod_{i=1}^{|\alpha|+m}(s+i)} \int_0^{R^2} t^{\beta_{m+1}} (R^2 - t)^{(s+|\alpha|+m)} dt \\ &= \frac{\pi^{m+1}\alpha!\beta_{m+1}! R^{2(s+|\beta|+m+1)}}{\prod_{i=1}^{|\alpha|+m}(s+i) \prod_{i=1}^{\beta_m+1}(s+|\alpha|+m+i)} \\ &= \frac{\pi^{m+1}\beta! R^{2(s+|\beta|+m+1)}}{\prod_{i=1}^{|\beta|+m+1}(s+i)}. \end{aligned}$$

Therefore, using induction we have (2.2) for any $m \in \mathbb{N}_+$, a number $s > -1$, a number $R > 0$ and a multiindex $\alpha = (\alpha_1, ..., \alpha_m)$. ■

We need the following estimations:

Lemma 2.3. *Let $s > -1$. There exist the constants $C, c > 0$ such that*

$$c \leq \frac{m! m^s}{\prod_{i=1}^m (i+s)} \leq C$$

for $m \geq 1$.

Proof. Let $N \in \mathbb{N}$ be such that $\frac{|s|}{N} < 1$. Let $M \in \mathbb{N}$ be such that $N < M$.

For $|x| < 1$ we have the following inequality $x - \frac{x^2}{2} \leq \ln(1+x) \leq x$. In particular, we have $|\ln(1+x) - x| \leq \frac{x^2}{2}$. We can conclude the following estimation

$$\left| \ln \prod_{i=N}^{M} \left(1 + \frac{s}{i}\right) - \sum_{i=N}^{M} \frac{s}{i} \right| = \left| \sum_{i=N}^{M} \left(\ln\left(1 + \frac{s}{i}\right) - \frac{s}{i} \right) \right| \leq \sum_{i=1}^{\infty} \frac{s^2}{2i^2}.$$

Similarly

$$\begin{aligned} \left| \ln \frac{M}{N} - \sum_{i=N}^{M-1} \frac{1}{i} \right| &= \left| \ln \prod_{i=N}^{M-1} \left(1 + \frac{1}{i}\right) - \sum_{i=N}^{M-1} \frac{1}{i} \right| \\ &= \left| \sum_{i=N}^{M-1} \left(\ln\left(1 + \frac{1}{i}\right) - \frac{1}{i} \right) \right| \leq \sum_{i=1}^{\infty} \frac{1}{2i^2}. \end{aligned}$$

We can now estimate:

$$\begin{aligned} \left| \ln \frac{\prod_{i=N}^{M} \left(1 + \frac{s}{i}\right)}{\left(\frac{M}{N}\right)^s} \right| &= \left| \ln \prod_{i=N}^{M} \left(1 + \frac{s}{i}\right) - s \ln \frac{M}{N} \right| \\ &\leq \left| \ln \prod_{i=N}^{M} \left(1 + \frac{s}{i}\right) - \sum_{i=N}^{M} \frac{s}{i} \right| + |s| \left| \ln \frac{M}{N} - \sum_{i=N}^{M} \frac{1}{i} \right| \\ &\leq \sum_{i=1}^{\infty} \frac{s^2}{2i^2} + \sum_{i=1}^{\infty} \frac{|s|}{2i^2} + 1. \end{aligned}$$

Therefore

$$\frac{1}{C} \leq \frac{\prod_{i=N}^{M} \left(1 + \frac{s}{i}\right)}{\left(\frac{M}{N}\right)^s} \leq C$$

for

$$C = \exp\left(\sum_{i=1}^{\infty} \frac{s^2}{2i^2} + \sum_{i=1}^{\infty} \frac{|s|}{2i^2} + 1 \right)$$

and for any $M > N$. There exists $\widetilde{C} > 0$ such that

$$\frac{1}{\widetilde{C}} \leq \frac{m! m^s}{m! \prod_{i=1}^m \left(1 + \frac{s}{i}\right)} = \frac{m! m^s}{\prod_{i=1}^m (i+s)} \leq \widetilde{C}$$

for $m \in \mathbb{N}$, which finishes the proof. ■

Theorem 2.4. *We define $s > -1$. The operator $\mathfrak{F}_s$ is properly defined and has the following properties:*

1. $\mathfrak{F}_s(\mathbb{O}(\mathbb{B}^n)) = \mathbb{O}(\mathbb{B}^n)$,

2. *there exists the constants $c_1, c_2 > 0$ such that:*

$$\begin{aligned} c_1 \int_{\mathbb{D}z} |\mathfrak{F}_s(f)|^2 d\mathfrak{L}^2_{\mathbb{D}z} &\leq \int_{\mathbb{D}z} |f|^2 \chi_s d\mathfrak{L}^2_{\mathbb{D}z} \\ &\leq c_2 \int_{\mathbb{D}z} |\mathfrak{F}_s(f)|^2 d\mathfrak{L}^2_{\mathbb{D}z} \end{aligned}$$

for $f \in \mathbb{O}(\mathbb{B}^n)$, $z \in \partial\mathbb{B}^n$ and

$$c_1 \int_{\mathbb{B}^n} |\mathfrak{F}_s(f)|^2 d\mathfrak{L}^{2n} \leq \int_{\mathbb{B}^n} |f|^2 \chi_s d\mathfrak{L}^{2n} \leq c_2 \int_{\mathbb{B}^n} |\mathfrak{F}_s(f)|^2 d\mathfrak{L}^{2n}$$

for $f \in \mathbb{O}(\mathbb{B}^n)$.

Proof. Observe that due to Lemma 2.3 there exist the constants $c_1, c_2 > 0$ such that

$$c_1 \leq \frac{(m+n)!(m+n)^s}{\prod_{i=1}^{m+n}(s+i)} \leq c_2$$

and

$$c_1 \leq \frac{(m+1)!(m+n)^s}{\prod_{i=1}^{m+1}(s+i)} \leq c_2$$

for $m \in \mathbb{N}$.

As $\lim_{m\to\infty} (m^s)^{\frac{1}{m}} = 1$, therefore the operator $\mathfrak{F}_s$ is properly defined and $\mathfrak{F}_s(\mathbb{O}(\mathbb{B}^n)) = \mathbb{O}(\mathbb{B}^n)$.

Let us take any function

$$f(z) = \sum_{\alpha\in\mathbb{N}^n} b_\alpha z^\alpha \in \mathbb{O}(\mathbb{B}^n).$$

Observe that

$$\mathfrak{F}_s(f)(z) = \sum_{\alpha\in\mathbb{N}^n} \frac{b_\alpha z^\alpha}{\sqrt{(|\alpha|+n)^s}}$$

and due to Theorem 2.2

$$\int_{\mathbb{B}^n} |\mathfrak{F}_s(f)|^2 d\mathfrak{L}^{2n} = \sum_{\alpha\in\mathbb{N}^n} \frac{|b_\alpha|^2 \pi^n \alpha!}{(|\alpha|+n)!(|\alpha|+n)^s}.$$

Using Theorem 2.2 we can again calculate

$$\begin{aligned} \int_{\mathbb{B}^n} |f|^2 \chi_s d\mathfrak{L}^{2n} &= \sum_\alpha \frac{|b_\alpha|^2 \pi^n \alpha!}{\prod_{i=1}^{|\alpha|+n}(s+i)} \\ &= \sum_\alpha \frac{d_\alpha |b_\alpha|^2 \pi^n \alpha!}{(|\alpha|+n)!(|\alpha|+n)^s}, \end{aligned}$$

where

$$c_1 \le d_\alpha = \frac{(|\alpha|+n)!(|\alpha|+n)^s}{\prod_{i=1}^{|\alpha|+n}(s+i)} \le c_2.$$

Therefore

$$c_1 \int_{\mathbb{B}^n} |\mathfrak{F}_s(f)|^2 d\mathfrak{L}^{2n} \le \int_{\mathbb{B}^n} |f|^2 \chi_s d\mathfrak{L}^{2n} \le c_2 \int_{\mathbb{B}^n} |\mathfrak{F}_s(f)|^2 d\mathfrak{L}^{2n}.$$

There exists a sequence of homogeneous polynomials p_m of a degree m such that $f(z) = \sum_{m\in\mathbb{N}} p_m(z)$. Observe that due to Lemma 2.1 for $s > -1$ we have:

$$\begin{aligned}
\int_{\mathbb{D}_z} |p_m|^2 \chi_s d\mathfrak{L}^2_{\mathbb{D}_z} &= \int_{|\lambda|<1} |p_m(z)|^2 |\lambda|^{2m} \chi_s(\lambda z) d\mathfrak{L}^2(\lambda) \\
&= |p_m(z)|^2 \pi \int_0^1 t^m (1-t)^s dt \\
&= \frac{|p_m(z)|^2 \pi m!}{\prod_{i=1}^{m+1}(s+i)}.
\end{aligned}$$

Therefore

$$\int_{\mathbb{D}_z} |\mathfrak{F}_s(f)|^2 d\mathfrak{L}^2_{\mathbb{D}_z} = \sum_{m\in\mathbb{N}} \frac{|p_m(z)|^2 \pi m!}{(m+1)!(m+n)^s}$$

and

$$\begin{aligned}
\int_{\mathbb{D}_z} |f|^2 \chi_s d\mathfrak{L}^2_{\mathbb{D}_z} &= \sum_{m\in\mathbb{N}} \frac{|p_m(z)|^2 \pi m!}{\prod_{i=1}^{m+1}(s+i)} \\
&= \sum_{m\in\mathbb{N}} \frac{k_{m,s} |p_m(z)|^2 \pi m!}{(m+1)!(m+n)^s}
\end{aligned}$$

for $z \in \partial\mathbb{B}^n$, where

$$c_1 \le k_{m,s} = \frac{(m+1)!(m+n)^s}{\prod_{i=1}^{m+1}(s+i)} \le c_2.$$

In particular:

$$\begin{aligned}
c_1 \int_{\mathbb{D}_z} |\mathfrak{F}_s(f)|^2 d\mathfrak{L}^2_{\mathbb{D}_z} &\le \int_{\mathbb{D}_z} |f|^2 \chi_s d\mathfrak{L}^2_{\mathbb{D}_z} \\
&\le c_2 \int_{\mathbb{D}_z} |\mathfrak{F}_s(f)|^2 d\mathfrak{L}^2_{\mathbb{D}_z}
\end{aligned}$$

for $z \in \partial\mathbb{B}^n$, which finishes the proof. ∎

Example 2.5. Let E be a circular set of the type G_δ of the measure zero in $\partial\mathbb{B}^n$. We define $s > -1$. There exists therefore a holomorphic function $f \in \mathbb{O}(\mathbb{B}^n)$ such that $E = E^s(f)$ and $\int_{\mathbb{B}^n} |f|^2 \chi_s d\mathfrak{L}^{2n} < \infty$.

Proof. On the basis of the paper [6] there exists a holomorphic function g such that $E = E^0(g)$ and $\int_{\mathbb{B}^n \setminus \Lambda(E)} |g|^2 \, d\mathfrak{L}^{2n} < \infty$. On the basis of Theorem 2.4 there exists a holomorphic function $f \in \mathbb{O}(\mathbb{B}^n)$ such that $g = \mathfrak{F}_s(f)$. Therefore, due to Theorem 2.4 function f has the required properties. ∎

The following question can be posed: is it possible that a holomorphic function $f \in \mathbb{O}(\mathbb{B}^n)$ is square integrable with a given weight χ_s and $E^t(f) \neq \emptyset$ for $t > -1$. The answer to this question is negative.

Lemma 2.6. *There exists a constant $C > 0$ such that*

$$\sup_{z \in \partial\mathbb{B}^n} \int_{\mathbb{D}z} |p_m|^2 \chi_{n-1} d\mathfrak{L}^2_{\mathbb{D}z} \leq C \int_{\mathbb{B}^n} |p_m|^2 d\mathfrak{L}^{2n} \tag{2.3}$$

for any natural number m and for any homogeneous polynomial p_m of a degree m.

Proof. Let $e_1 = (1, 0, ...0) \in \partial\mathbb{B}^n$. By β_m we denote a multiindex such that $\beta_m = (m, 0, ..., 0) \in \mathbb{N}^n$ for $m \in \mathbb{N}$.

We prove that there exists a constant $C > 0$ such that:

$$\int_{|\lambda|<1} |p_m(\lambda e_1)|^2 \chi_{n-1}(\lambda e_1) d\mathfrak{L}^2(\lambda) \leq C \int_{\mathbb{B}^n} |p_m|^2 d\mathfrak{L}^{2n} \tag{2.4}$$

for any natural number m and for any homogeneous polynomial p_m of a degree m. There exists a constant $c_1 > 0$ such that:

$$\frac{n!m!m^n}{(m+n)!} \leq c_1$$

and a constant c_2 such that:

$$c_2 \leq \frac{m!m^n}{(m+n)!}$$

for $m \in \mathbb{N}$. Let

$$p_m(z) = \sum_{|\alpha|=m} b_\alpha z^\alpha$$

be a homogeneous polynomial of a degree m. Let us estimate using Theorem 2.2 (for $s = n-1$, $k = 1$) that:

$$\begin{aligned} \frac{c_1 \pi |b_{\beta_m}|^2 m!}{m!m^n} &\geq \frac{\pi |b_{\beta_m}|^2 m!}{\prod_{i=1}^{m+1}(n-1+i)} \\ &= \int_{|\lambda|<1} |p_m(\lambda e_1)|^2 \chi_{n-1}(\lambda e_1) d\mathfrak{L}^2(\lambda). \end{aligned}$$

Therefore, again due to Theorem 2.2 (for $s = 0$, $k = n$), we can estimate

$$\begin{aligned} \int_{\mathbb{B}^n} |p_m|^2 d\mathfrak{L}^{2n} &= \sum_{|\alpha|=m} \frac{\pi^n |b_\alpha|^2 \alpha!}{(m+n)!} \\ &\geq \sum_{|\alpha|=m} \frac{c_2 \pi^n |b_\alpha|^2 \alpha!}{m!m^n} \\ &\geq \frac{c_2 \pi^n |b_{\beta_m}|^2 m!}{m!m^n} \\ &\geq \frac{c_2 \pi^{n-1}}{c_1} \int_{|\lambda|<1} |p_m(\lambda e_1)|^2 \chi_{n-1}(\lambda e_1) d\mathfrak{L}^2(\lambda). \end{aligned}$$

Constant C can be defined as $C = \frac{c_1}{c_2\pi^{n-1}}$, which finishes the proof of the inequality (2.4).

We show that such a constant C is appropriate. Therefore, let us select any point $z \in \partial\mathbb{B}^n$. There exists linear isometry (a geometric turn around a point 0) $\Theta : \mathbb{C}^n \to \mathbb{C}^n$ such that $\Theta(e_1) = z$. Let us take any homogeneous polynomial p_m of a degree m. Let us observe that

$$\int_{\mathbb{B}^n} |p_m \circ \Theta|^2 d\mathfrak{L}^{2n} = \int_{\mathbb{B}^n} |p_m|^2 d\mathfrak{L}^{2n}.$$

Moreover

$$\int_{|\lambda|<1} |p_m(\lambda z)|^2 \chi_{n-1}(\lambda z) d\mathfrak{L}^2(\lambda) = \int_{|\lambda|<1} |p_m \circ \Theta(\lambda e_1)|^2 \chi_{n-1}(\lambda e_1) d\mathfrak{L}^2(\lambda).$$

In particular when using (2.4) for a homogeneous polynomial $p_m \circ \Theta$ of a degree m we get:

$$\int_{|\lambda|<1} |p_m(\lambda z)|^2 \chi_{n-1}(\lambda z) d\mathfrak{L}^2(\lambda) \leq C \int_{\mathbb{B}^n} |p_m|^2 d\mathfrak{L}^{2n},$$

which finishes the proof. ∎

Theorem 2.7. *If $s > -1$, the function $f \in \mathbb{O}(\mathbb{B}^n)$ is such that: $\int_{\mathbb{B}^n} |f|^2 \chi_s d\mathfrak{L}^{2n} < \infty$, then $E^{s+n-1}(f) = \emptyset$.*

Proof. Assume that $\int_{\mathbb{B}^n} |f|^2 \chi_s d\mathfrak{L}^{2n} < \infty$ for a holomorphic function $f \in \mathbb{O}(\mathbb{B}^n)$. Observe that due to Theorem 2.4 there is also the inequality $\int_{\mathbb{B}^n} |\mathfrak{F}_s(f)|^2 d\mathfrak{L}^{2n} < \infty$ for the function $\mathfrak{F}_s(f)$. There exists a sequence of homogeneous polynomials p_m of a degree m such that

$$\mathfrak{F}_s(f)(z) = \sum_{m\in\mathbb{N}} p_m(z).$$

On the basis of Lemma 2.6, there exists a constant $C > 0$ such that

$$\begin{aligned} C\int_{\mathbb{B}^n} |\mathfrak{F}_s(f)|^2 d\mathfrak{L}^{2n} &= C\int_{\mathbb{B}^n} |p_m|^2 d\mathfrak{L}^{2n} \\ &\geq \sum_{m\in\mathbb{N}} \int_{\mathbb{D}z} |p_m|^2 \chi_{n-1} d\mathfrak{L}^2_{\mathbb{D}z} \\ &= \int_{\mathbb{D}z} |\mathfrak{F}_s(f)|^2 \chi_{n-1} d\mathfrak{L}^2_{\mathbb{D}z}. \end{aligned}$$

Therefore $E^{n-1}(\mathfrak{F}_s(f)) = \emptyset$. As $\mathfrak{F}_{s+n-1} = \mathfrak{F}_{n-1} \circ \mathfrak{F}_s$, on the basis of Theorem 2.4, we have

$$E^{s+n-1}(f) = E(\mathfrak{F}_{s+n-1}(f)) = E(\mathfrak{F}_{n-1}(\mathfrak{F}_s(f))) = E^{n-1}(\mathfrak{F}_s(f)) = \emptyset,$$

which finishes the proof. ∎

It appears that Theorem 2.7 cannot be strengthened.

Example 2.8. Let $e_1 = (1, 0, ...0) \in \partial\mathbb{B}^n$. If

$$f(z_1, ..., z_n) = \sum_{m=2}^{\infty} \frac{\sqrt{m^{n-1}}}{\ln m} z_1^m,$$

then $\int_{\mathbb{B}^n} |f|^2 d\mathfrak{L}^{2n} < \infty$ and $E^{n-1-\varepsilon}(f) = \mathbb{S}e_1$, where ε is any number such that $0 < \varepsilon < n$.

Proof. Let us select ε such that $0 < \varepsilon < n$. There exists a constant $c > 0$ such that

$$\frac{m^n}{\prod_{i=1}^n (m+i)} < c$$

for $m \in \mathbb{N}$. Due to Theorem 2.2, we can calculate:

$$\begin{aligned}
\int_{\mathbb{B}^n} |f|^2 d\mathfrak{L}^{2n} &= \sum_{m=2}^{\infty} \frac{m^{n-1}\pi^n m!}{(\ln m)^2 (m+n)!} \leq \sum_{m=2}^{\infty} \frac{m^n \pi^n m!}{m(\ln m)^2 (m+n)!} \\
&\leq \sum_{m=2}^{\infty} \frac{c\pi^n}{(\ln m)^2 m} \leq c\pi^n \int_2^{\infty} \frac{dt}{t(\ln t)^2} = \frac{c\pi^n}{\ln 2} < \infty.
\end{aligned}$$

Let $e_1 = (1, 0, ...0) \in \partial\mathbb{B}^n$. On the basis of Lemma 2.3, there exist the constants $q_1, q_2 > 0$ such that

$$\frac{q_1}{m^{n-\varepsilon}} = \frac{q_1 m!}{m! m^{n-\varepsilon}} \leq \frac{m!}{\prod_{i=1}^{m+1}(n-1-\varepsilon+i)} \leq \frac{q_2 m!}{m! m^{n-\varepsilon}} = \frac{q_2}{m^{n-\varepsilon}}.$$

There also exists a constant $p > 0$ such that

$$t^{\varepsilon} \geq p(\ln t)^2$$

for $t \geq 2$. Therefore, due to Theorem 2.2, we can estimate:

$$\begin{aligned}
\int_{\mathbb{D}e_1} |f|^2 \chi_{n-1-\varepsilon} d\mathfrak{L}^2_{\mathbb{D}e_1} &= \sum_{m=2}^{\infty} \int_{|\lambda|<1} \frac{|\lambda|^{2m}(1-|\lambda|^2)^{n-1-\varepsilon}}{m^{-n+1}(\ln m)^2} d\mathfrak{L}^2(\lambda) \\
&= \sum_{m=2}^{\infty} \frac{m^{n-1}\pi m!}{(\ln m)^2 \prod_{i=1}^{m+1}(n-1-\varepsilon+i)} \\
&\geq \sum_{m=2}^{\infty} \frac{q_1 m^{n-1}\pi}{(\ln m)^2 m^{n-\varepsilon}} = \sum_{m=2}^{\infty} \frac{q_1 \pi}{(\ln m)^2 m^{1-\varepsilon}} \\
&\geq q_1 \pi \int_3^{\infty} \frac{dt}{t^{1-\varepsilon}(\ln t)^2} \\
&\geq q_1 p \pi \int_3^{\infty} \frac{dt}{t} = \infty.
\end{aligned}$$

It follows that $\lambda e_1 \subset E^{n-1-\varepsilon}(f)$, when $|\lambda| = 1$.

However, if $z \in \partial\mathbb{B}^n$ and $z \notin \mathbb{S}e_1$, then $|z_1| < 1$, which results in

$$\begin{aligned}
\int_{\mathbb{D}z} |f|^2 \chi_{n-1-\varepsilon} d\mathfrak{L}^2_{\mathbb{D}z} &\leq \sum_{m=2}^{\infty} \frac{|z_1|^{2m} m^{n-1}\pi m!}{(\ln m)^2 \prod_{i=1}^{m+1}(n-1-\varepsilon+i)} \\
&\leq \sum_{m=2}^{\infty} \frac{|z_1|^{2m} q_2 \pi}{(\ln m)^2 m^{1-\varepsilon}} < \infty,
\end{aligned}$$

because

$$\lim_{m\to\infty}\sqrt[m]{\frac{|z_1|^{2m}q_2\pi}{(\ln m)^2 m^{1-\varepsilon}}}=|z_1|^2<1.$$

Therefore $z \notin E^{n-1-\varepsilon}(f)$, which finishes the proof. ■

References

[1] J. Chaumat, oral communication, Seminar on Complex Analysis at the Institute of Mathematics at Jagiellonian University, 1988.

[2] J. Globevnik, Holomorphic functions which are highly nonintegrable at the boundary, Israel J. Math. 115 (2000), 195-203.

[3] P. Jakóbczak, Highly nonintegrable functions in the unit ball. Israel J. Math 97 (1997), 175-181.

[4] P. Jakóbczak, Exceptional sets of slices for functions from the Bergman Space in the ball, Canad. Math. Bull. 44(2), (2001), 150-159

[5] P. Kot, Description of simple exceptional sets in the unit ball, Czechoslovak Mathematical Journal, 54 (129) (2004), 55-63.

[6] P. Kot, Boundary functions in $L^2H(\mathbb{B}^n)$, Czechoslovak Mathematical Journal, submitted.

[7] P. Pflug, oral communication, 4-th Symposium on Classical Analysis, Kazimierz 1987.

[8] P. Wojtaszczyk, On highly nonintegrable functions and homogeneous polynomials, Annales Polonici Mathematici no. 65, (1997), 245-251.

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