A Generalization of Bertilsson's Theorem

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Abstract

We are concerned with the following problem. Let L and K be fixed real numbers. When does the Koebe function $k(z) = z(1-z)^{-2}$ maximize the Nth Taylor coefficient of $(1/f'(z))^L (z/f(z))^K$ for f in the class S of normalized schlicht functions? A sufficient condition for $L \geq -1$ is $1 \leq N \leq 2L + K + 1$. A necessary condition is that a certain trigonometric sum involving hypergeometric functions is non–negative. These results generalize a recent theorem of Bertilsson and suggest a link between Brennan's conjecture in conformal mapping and Baernstein's theorem about integral means of functions in S.

1 Introduction

An open problem in conformal mapping, which recently received a great deal of attention [2, 3, 4, 8], is Brennan's conjecture [7]. It states

$$\iint_{\mathbb{D}} |f'(z)|^{-L} \, dxdy < \infty \tag{1}$$

for every conformal map f from the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ into the complex plane $\mathbb{C}$ and every real number $L \geq 2$. Of course, one may assume f belongs to the class S of univalent functions $f : \mathbb{D} \to \mathbb{C}$, normalized by $f(0) = 0$ and $f'(0) = 1$. In [3] Bertilsson observed Brennan's conjecture is equivalent to the

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following conjecture about integral means of the derivatives of functions f in S: for every $L \geq 2$ there exists a constant $C_L > 0$ such that

$$\int\limits_{|z|=r} \left| \frac{1}{f'(z)} \right|^L d\theta \leq C_L \int\limits_{|z|=r} \left| \frac{1}{k'(z)} \right|^L d\theta \tag{2}$$

for every $f \in S$ and every $0 \leq r < 1$, where

$$k(z) = \frac{z}{(1-z)^2}$$

is the Koebe function. It is even conjectured (see [3]) that one may take $C_L = 1$. The corresponding problem of estimating the integral means of the functions in S instead of their derivatives was completely settled by Baernstein [1], who proved

$$\int\limits_{|z|=r} \left| \frac{z}{f(z)} \right|^K d\theta \leq \int\limits_{|z|=r} \left| \frac{z}{k(z)} \right|^K d\theta \tag{3}$$

for every $0 \leq r < 1$, every $K \in \mathbb{R}$ and every $f \in S$. The purpose of the present note is to point out a possible link between Brennan's conjecture (2) and Baernstein's result (3), which might be useful in attacking Brennan's conjecture.

We first note Brennan's conjecture can be stated as a coefficient problem for univalent functions as follows. For $f \in S$ and $L, K \in \mathbb{R}$ let

$$\left(\frac{1}{f'(z)} \right)^L = \sum_{N=0}^{\infty} a_N(L,f)\, z^N, \qquad a_0(L,f) = 1,$$

and

$$\left(\frac{z}{f(z)} \right)^K = \sum_{N=0}^{\infty} b_N(K,f)\, z^N, \qquad b_0(K,f) = 1.$$

Then (see, for instance, [3]), Brennan's conjecture is equivalent to the coefficient estimate

$$|a_N(L,f)| \leq c_L\, |a_N(L,k)| \quad \text{for } N \geq 1,\ f \in S,\ L \geq 2. \tag{4}$$

Here, c_L is a constant which does not depend on f and N. Again, one might suspect $c_L = 1$.

In [2, 3] D. Bertilsson was able to prove an estimate of the form (4). Specifically, he established for $L > 0$ the inequality

$$|a_N(L,f)| \leq |a_N(L,k)| \quad \text{for } 1 \leq N \leq 2L+1, \quad f \in S. \tag{5}$$

Bertilsson's proof of (5) is based on an ingenious modification of de Branges's method [5] and is quite involved. A quick proof of Bertilsson's inequalities (5) was given in [17]. Recently, the method of [17] was adapted in [13] to establish the following similar result for the Taylor coefficients $b_N(K,f)$:

$$|b_N(K,f)| \leq |b_N(K,k)| \quad \text{for } 1 \leq N \leq K+1, \quad f \in S. \tag{6}$$

Consequently, combining (5) and (6), it is easy to see that

$$|c_N(L,K,f)| \le |c_N(L,K,k)|$$

for every $f \in S$ and every $1 \le N \le \min\{K+1, 2L+1\}$ where the coefficients $c_N(L,K,f)$ are defined by

$$\left(\frac{1}{f'(z)}\right)^L \left(\frac{z}{f(z)}\right)^K = \sum_{N=0}^{\infty} c_N(L,K,f)\, z^N, \qquad c_0(L,K,f) = 1.$$

Somewhat surprisingly much more than this is true:

Theorem 1. *Let $f \in S$. Then for every $L \ge -1$, every $K \in \mathbb{R}$ and all integers N with $1 \le N \le 2L+K+1$ the inequalities*

$$|c_N(L,K,f)| \le |c_N(L,K,k)| \tag{7}$$

are valid. Except for the cases

$$\begin{aligned} 2L+K &= 0, \qquad && N = 1, \\ 2L+K &= 1, && N = 2, \\ L+1 &= 0, && N = K-1, \qquad K \ge 2 \end{aligned}$$

equality is attained if and only if f is the Koebe function or one of its rotations.

The proof of Theorem 1 will be given in Section 2. It uses the Löwner differential equation and proceeds along similar lines as the proofs in [11, 13, 17].

Remarks.

(a) We shall see in Section 3 that in general not all of the individual coefficients $a_j(L,f)$ and $b_{N-j}(K,f)$ in the sum

$$c_N(L,K,f) = \sum_{j=0}^{N} a_j(L,f) b_{N-j}(K,f) \tag{8}$$

are maximized by the Koebe function. Nevertheless, Theorem 1 guarantees that the Koebe function does maximize the absolute value of the sum (8) itself if $1 \le N \le 2L+K+1$. So in a sense a kind of averaging phenomenon occurs, which is to be reminiscent of Milin's inequality for the weighted sums

$$\sum_{k=1}^{N} k(N-k+1)|\gamma_k|^2 \tag{9}$$

of the logarithmic coefficients of univalent functions defined by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n.$$

As in (8) the Koebe function is not extremal for the absolute value of the individual logarithmic coefficients γ_n, $n \ge 2$ (see, for instance, [9]), but by de Branges's theorem [5] the weighted sums (9) are indeed maximized by the Koebe function.

(b) In view of the equivalence of Brennan's conjecture with the coefficient problem (4), Theorem 1 strongly suggests to consider the integral means of the product

$$\left(\frac{1}{f'(z)}\right)^L \left(\frac{z}{f(z)}\right)^K$$

and makes the following generalization of Brennan's problem (2) irresistable.

Problem. *For which real numbers K and $L \geq 2$ does there exist a constant $E_{K,L}$ such that*

$$\int\limits_{|z|=r} \left|\frac{1}{f'(z)}\right|^L \left|\frac{z}{f(z)}\right|^K d\theta \leq E_{K,L} \cdot \int\limits_{|z|=r} \left|\frac{1}{k'(z)}\right|^L \left|\frac{z}{k(z)}\right|^K d\theta,$$

for every $f \in S$ and every $0 \leq r < 1$?

(c) Theorem 1 simultaneously generalizes Bertilsson's theorem (5) and the inequalities (6).

We now return to estimate (7) and derive a necessary condition for this inequality for fixed $N \in \mathbb{N}$ and fixed real parameters K and L. As in many extremal problems for univalent functions [6, 15] hypergeometric functions enter the picture. We first recall that for fixed complex numbers a, b, c with $c \neq -n$ $(n = 0, 1, 2, \ldots)$, the Gaussian hypergeometric series is defined by

$${}_2F_1(a, b, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \frac{z^n}{n!}, \qquad |z| < 1,$$

where

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

is the Pochhammer symbol. If b is a negative integer, $b = -j$, then

$${}_2F_1(a, -j, c; z) = \sum_{n=0}^{j} \frac{(a)_n\,(-j)_n}{(c)_n} \frac{z^n}{n!},$$

is a polynomial of degree j and it is easy to check that

$$\alpha_j(L, K) = \frac{\Gamma(L+j)}{j!\,\Gamma(L)}\,{}_2F_1(-2K-3L, -j, 1-L-j; -1) \tag{10}$$

are well–defined real numbers for every $L, K \in \mathbb{R}$ and $j = 0, 1, 2, \ldots$. We also note

$$\begin{aligned} \left(\frac{1}{k_0'(z)}\right)^L \left(\frac{z}{k_0(z)}\right)^K &= \frac{(1+z)^{3L+2K}}{(1-z)^L} \\ &= \sum_{N=0}^{\infty} \left(\sum_{j=0}^{N} \binom{3L+2K}{j} \binom{-L}{N-j} (-1)^{N-j} \right) z^N \qquad (11) \\ &= \sum_{N=0}^{\infty} \alpha_N(L, K) z^N, \end{aligned}$$

for $k_0(z) = -k(-z) = z/(1+z)^2$, that is,

$$\alpha_N(L, K) = c_N(L, K, k_0).$$

Theorem 2. *Let $N \geq 1$ be a fixed integer and let L, K be real numbers. If the inequality (7) holds for all functions $f \in S$, then the trigonometric sum*

$$\alpha_N(L,K) \sum_{j=1}^{N} \Big((L+K)\alpha_{N-j}(L,K) + Lj\alpha_{N-j}(L+1,K-1) \Big) \sin(j\,u) \qquad (12)$$

is non–negative for $u \in [0,\pi]$.

Condition (12) can easily be checked for fixed L, K and N with the help of a computer. It follows from Theorem 2 (see Section 3) that (7) does *not* hold for every $N \in \mathbb{N}$. In this sense Theorem 2 complements Theorem 1. The proof of Theorem 2 is given in Section 3 and is based on an elementary special case of Schiffer's method of boundary variation [16].

2 Proof of Theorem 1

We begin relating the Taylor coefficients $c_N(L,K,f)$ of

$$\left(\frac{1}{f'(z)}\right)^L \left(\frac{z}{f(z)}\right)^K$$

to the Taylor coefficients $d_n(L,K,N,f)$ defined by

$$F'(w)^{L+1} \left(\frac{F(w)}{w}\right)^{K-N-1} = 1 + \sum_{n=1}^{\infty} d_n(L,K,N,f)\, w^n, \qquad (13)$$

where F is the inverse function to $f \in S$.

Lemma 3. *Let $f \in S$ and F be the inverse function of f. For any real numbers L and K and any positive integer N let the coefficients $d_n(L,K,N,f)$ be defined by (13). Then*

$$c_N(L,K,f) = d_N(L,K,N,f).$$

In particular, $c_N(-1,N+1,f) = 0$ for any $N \geq 1$.

Proof. By Koebe's One–Quarter Theorem, $f(\mathbb{D})$ contains the disk $|w| < 1/4$, so the circle Γ of radius $1/8$, say, centered at the origin belongs to $f(\mathbb{D})$ and Cauchy's integral formula gives

$$\begin{aligned} d_N(L,K,N,f) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{F'(w)^{L+1} F(w)^{K-N-1}}{w^K}\, dw \\ &= \frac{1}{2\pi i} \int_{F(\Gamma)} \left(\frac{1}{f'(z)}\right)^L \left(\frac{z}{f(z)}\right)^K \frac{1}{z^{N+1}}\, dz = c_N(L,K,f). \end{aligned}$$

■

In the next step we apply Löwner's theory to find the sharp upper bound for the coefficients $d_n(L, K, N, f)$. The method goes back to Löwner's paper [12] and has been used before in [11], [17] and [13].

Theorem 4. *Let K and L be real numbers with $L \geq -1$ and let N be a positive integer with $1 \leq N \leq 2L + K + 1$ such that either $L \neq -1$ or $K - N - 1 \neq 0$. Moreover, let $f \in S$ and $d_n(L, K, N, f)$ be defined as in Lemma 3. Then the sharp estimate*

$$|d_n(L, K, N, f)| \leq |d_n(L, K, N, k)|$$

holds for any positive integer n. Except for the cases $2L + K + 1 = N$ and $n = 1$ or $n = 2$ equality occurs if and only if f is the Koebe function or one of its rotations.

Proof. We first recall some basics from Löwner's theory (see [12, 14] for details). Every $f \in S$ can be embedded in a normalized subordination chain $f(z, t)$, $0 \leq t < \infty$, with $f(z, 0) = f(z)$. This means $z \mapsto f(z, t) = e^t z + \cdots$ is univalent in $\mathbb{D}$ and $f(\mathbb{D}, t) \subseteq f(\mathbb{D}, \tau)$ for $0 \leq t \leq \tau < \infty$, i.e., the image domains $f(\mathbb{D}, t)$ are increasing. Since $f(z, t)$ is absolutely continuous in $t \geq 0$ for each $z \in \mathbb{D}$, (see [14, Theorem 6.2]), the function

$$p(z,t) := \frac{\dfrac{\partial f(z,t)}{\partial t}}{z\dfrac{\partial f(z,t)}{\partial z}} = \sum_{n=0}^{\infty} p_n(t) z^n, \qquad p_0(t) = 1, \tag{14}$$

is an analytic function of $z \in \mathbb{D}$ for a.e. $t \geq 0$, and a measurable function of t in $[0, \infty)$ for each fixed $z \in \mathbb{D}$. Moreover, $\operatorname{Re} p(z, t) \geq 0$. This is geometrically obvious from (14) since the image domains of the functions $f(z, t)$ are increasing. Consequently, $z \mapsto p(z, t)$ belongs to the class of normalized analytic functions with positive real part, so $|p_n(t)| \leq 2$ for every $n \geq 1$. If, moreover, $p_1(t) = 2$ (a.e.), then $p_n(t) = 2$ (a.e.) for every $n \geq 1$. The only normalized subordination chain in which the Koebe function $f(z) = k(z)$ can be embedded is $f(z, t) = e^t k(z)$, so we have $p(z, t) = (1 + z)/(1 - z)$ in this case.

Now, let $w \mapsto \Phi(w, t)$ be the inverse function of $z \mapsto f^{-1}(f(z), t)$. Then (14) implies

$$\frac{\partial \Phi(w,t)}{\partial t} = w \frac{\partial \Phi(w,t)}{\partial w} p(w,t), \tag{15}$$

for every w in some neighborhood of $w = 0$ (depending on t) and

$$\Phi(w, 0) = w, \qquad F(w) = \lim_{t \to \infty} \Phi(e^{-t} w, t).$$

Using the differential equation (15) the function

$$Q(w,t) = \left(\frac{\partial \Phi(w,t)}{\partial w}\right)^{L+1} \left(\frac{\Phi(w,t)}{w}\right)^{K-N-1} = \sum_{n=0}^{\infty} D_n(t) w^n$$

is easily seen to be a solution of the partial differential equation

$$\frac{\partial Q(w,t)}{\partial t} = \left((L+1)\frac{\partial (w p(w,t))}{\partial w} + (K - N - 1) p(w,t)\right) Q(w,t) + \frac{\partial Q(w,t)}{\partial w} w p(w,t),$$

that is,

$$\sum_{n=0}^{\infty} \frac{dD_n(t)}{dt} w^n = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} ((L+1)(n-j+1) + j - N + K - 1) D_j(t) p_{n-j}(t) \right) w^n.$$

This yields the following initial value problems for the Taylor coefficients $D_n(t)$:

$$\frac{dD_0(t)}{dt} = (L - N + K) D_0(t), \qquad D_0(0) = 1,$$

and

$$\begin{aligned} \frac{dD_n(t)}{dt} &= (L + n - N + K) D_n(t) \\ &\quad + \sum_{j=0}^{n-1} ((L+1)(n-j+1) + j - N + K - 1) D_j(t) p_{n-j}(t), \qquad D_n(0) = 0, \end{aligned}$$

for $n = 1, 2, \ldots$. These initial value problems have the solutions
$D_0(t) = e^{(L-N+K)t}$,
$D_n(t) = \int_0^t e^{(L+n-N+K)(t-\tau)} \sum_{j=0}^{n-1} ((L+1)(n-j+1) + j - N + K - 1) D_j(\tau) p_{n-j}(\tau)\, d\tau$,
for $n = 1, 2, \ldots$. In particular, $D_1(t) \equiv 0$ if $N = 2L + K + 1$.

We note

$$(L+1)(n-j+1) + j - N + K - 1 \geq 2(L+1) + j - N + K - 1 \geq 2L + K + 1 - N \geq 0$$

for $0 \leq j \leq n-1$, $L \geq -1$ and $1 \leq N \leq 2L + K + 1$. Moreover, equality occurs if and only if $n = 1$ and $2L + K + 1 = N$. It follows $\operatorname{Re} D_n(t)$ is maximized for fixed $t \geq 0$, if we choose $D_j(\tau)$ real and maximal for every $j = 1, \ldots, n-1$ and a.e. $\tau \in [0, t]$, and also $p_j(\tau) = 2$ for every $j = 1, \ldots, n$ and a.e. $\tau \in [0, t]$. These conditions are also necessary for $\operatorname{Re} D_n(t)$ to be maximal except $N = 2L + K = 1$ and either $n = 1$ or $n = 2$.

In view of the relation

$$d_n(L, K, N, f) = \lim_{t \to \infty} e^{-t(L+n-N+K)} D_n(t)$$

we conclude the functional $f \mapsto \operatorname{Re} d_n(L, K, N, f)$ attains its maximal value on the set S if

$$p(w, t) = \frac{1+w}{1-w}, \tag{16}$$

that is, if $f(z) = k(z)$. Only in the cases $2L + K + 1 = N$ and either $n = 1$ or $n = 2$, $p(w, t)$ doesn't have to be of the form (16).

Now the assertion of Theorem 4 follows immediately from the fact that a function $F \in S$ maximizes $|d_n(L, K, N, f)|$, if and only if a suitable rotation $e^{-i\theta} F(e^{i\theta} z) \in S$, $\theta \in \mathbb{R}$, maximizes $\operatorname{Re} d_n(L, K, N, f)$. ■

After this preparations, Theorem 1 is an immediate consequence of Lemma 3 and Theorem 4.

Remarks.

(a) In the first exceptional case, $2L+K=0$ and $N=1$, of Theorem 1, the estimate (7) is trivial since $c_1(L,K,f)=0$ for every $f\in S$. We next consider the second exceptional case $2L+K=1$ and $N=1$. Now, $c_2(L,K,f)=(L+1)(a_2^2-a_3)$ for every $f(z)=z+a_2z^2+a_3z^3+\cdots$ in S. Hence, if $L=-1$, then again (7) is trivially satisfied. If $L>-1$, then equality occurs in (7) if and only if f is a rotation of
$$G(z)=\frac{z}{1+cz+z^2},$$
where c is a real number with $-2\le c\le 2$. This is classical and may be found in [10, Chapter 2]. Note in Exercise 1 of [9, Chapter 2] there is the erroneous statement that equality holds only if f is a rotation of the Koebe function. Finally, the third exceptional case, $L=-1$ and $N=K-1$, of Theorem 1 is again trivial since $c_N(L,K,f)=0$ for every $f\in S$ by Lemma 3.

(b) Our method of proof can also be used to consider the cases $L<-1, K>0$. In these cases one is lead to the conclusion the Taylor coefficients $c_N(L,K,f)$ are maximized by the Koebe function if $N\le -1-K/L$.

3 Proof of Theorem 2

If the inequality (7) holds, then the Koebe function
$$k_0(z)=\frac{z}{(1+z)^2}$$
maximizes the functional
$$\phi(f)=|c_N(L,K,f)|^2$$
on the set S. We produce a one–parameter family of neighboring functions
$$k_r(z)=k_0(z)+\frac{r^2}{4}\left(1-e^{i\gamma}\right)\frac{k_0(z)^2}{\eta^2(\eta-k_0(z))}+O(r^3),\qquad r\to 0, \tag{17}$$
with $\gamma\in\mathbb{R}$ and $\eta>1/4$ as follows.

Let $\varphi(u)=u-u^{-1}+\ldots$ be the inverse of the Joukowski transform $\Psi(\xi)=\xi+1/\xi$, which maps $|\xi|>1$ conformally onto $\mathbb{C}\setminus[-2,2]$. The rotation $h(\xi)=\xi+e^{i\gamma}/\xi$, $\gamma\in\mathbb{R}$, of the Joukowski function maps $|\xi|>1$ conformally onto $\mathbb{C}$ minus a line segment of length 4. We deduce that for fixed $\eta>1/4$ and fixed $0<r<\eta-1/4$ the function
$$H_r(w)=h\left(\varphi\left(\frac{2}{r}(w-\eta)\right)\right)=\frac{2}{r}(w-\eta)-\frac{r}{2}\frac{1-e^{i\gamma}}{w-\eta}+O(r^2)$$
is univalent on $k_0(\mathbb{D})=\mathbb{C}\setminus[1/4,\infty)$. Finally, we normalize
$$G_r(w)=\frac{H_r(w)-H_r(0)}{H_r'(0)}=w-\frac{r^2}{4}\left(1-e^{i\gamma}\right)\frac{w^2}{\eta^2(w-\eta)}+O(r^3),$$
and set $k_r(z)=G_r(k_0(z))$ to obtain the variation (17).

Next, a calculation using (17) gives

$$c_N(L,K,k_r) = c_N(L,K,k_0) - \frac{1-e^{i\gamma}}{4\eta^3}\delta_N(\eta)r^2 + O(r^3), \tag{18}$$

where $\delta_N(\eta)$ is the Nth Taylor coefficient of the function

$$\left(\frac{1}{k_0'(z)}\right)^L \left(\frac{z}{k_0(z)}\right)^K \frac{\eta\, k_0(z)}{(\eta - k_0(z))^2}\left[\eta\,(K+2L) - k_0(z)(K+L)\right]. \tag{19}$$

From (18) we obtain

$$|c_N(L,K,k_r)|^2 = |c_N(L,K,k_0)|^2 - \frac{1}{2}\mathrm{Re}\left\{\frac{1-e^{i\gamma}}{\eta^3}r^2 c_N(L,K,k_0)\delta_N(\eta)\right\} + O(r^3).$$

Now k_0 maximizes $|c_N(L,K,f)|^2$ on S. This implies

$$\mathrm{Re}\left\{\frac{1-e^{i\gamma}}{\eta^3}c_N(L,K,k_0)\delta_N(\eta)\right\} \geq 0, \qquad \gamma\in\mathbb{R},\ \eta > 1/4,$$

or

$$c_N(L,K,k_0)\delta_N(k_0(\xi)) \geq 0 \tag{20}$$

for every $|\xi| = 1$.

Since the identity (19) may be written for $\eta = k_0(\xi)$ as

$$(L+K)t_\xi(z)\left(\frac{1}{k_0'(z)}\right)^L \left(\frac{z}{k_0(z)}\right)^K + Lzt_\xi'(z)\left(\frac{1}{k_0'(z)}\right)^{L+1}\left(\frac{z}{k_0(z)}\right)^{K-1},$$

with

$$t_\xi(z) = \frac{z}{1-(\xi+\overline{\xi})z+z^2} = \sum_{j=1}^{\infty}\frac{\sin(j\,u)}{\sin\,u}z^j, \qquad \xi = e^{iu},$$

and using (10), we see that (20) reduces to the fact that (12) is non–negative for $0 \leq u \leq \pi$. ■

Remarks. We take briefly a closer look at Theorem 1 and Theorem 2 in the case $N = 2$ and $L \geq -1$. It follows from these two results that

$$|b_2(K,f)| \leq |b_2(K,k)| \tag{21}$$

for every $f \in S$ if and only if $K \geq 1$, (see also [13]), that

$$|a_2(L,f)| \leq |a_2(L,k)| \tag{22}$$

for every $f \in S$ if and only if $L \geq 1/2$, and also that

$$|c_2(L,K,f)| \leq |c_2(L,K,k)| \tag{23}$$

for every $f \in S$ if $K + 2L \geq 1$. In particular, if $K = 1/2$ and $L = 1/4$, then $K + 2L \geq 1$, so (23) holds for every $f \in S$, but (21) and (22) are not fulfilled for any $f \in S$. We see that in the sum

$$c_2(L,K,f) = b_2(K,f) + b_1(K,f)a_1(L,f) + a_2(L,f)$$

not all of the individual terms are maximized by the Koebe function, but the sum itself is. Theorem 2 also implies that (23) can only hold if either $K + 2L \geq 1$ or if $K + 3L > 0$ and $(K + 2L)^2 + K + 4L < 0$. In particular, (23) fails to hold for $K + 3L < 0$. Finally we note that a similar analysis can be carried out for $N > 2$ or $L < -1$.

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