New geometries for finite groups and polytopes

Christiane Lefèvre-Percsy   Nicolas Percsy   Dimitri Leemans

Abstract

We describe two methods to obtain new geometries from classes of geometries whose diagram satisfy given conditions. This gives rise to lots of new geometries for finite groups, and in particular for sporadic groups. It also produces new thin geometries related to polytopes.

1 Introduction

Starting from some particular geometries, we can obtain new geometries, by using two general constructions described here. These constructions also show that some geometries are strongly related to others.

The main theorems obtained in this paper were presented at Oberwolfach's session "Finite Geometries" in May 1989 (title: "New geometries for finite groups"). Recent investigations by D. Leemans on geometries for finite groups show the interest in detailed proofs of these results and in their applications. Note that Pasini presents in [12] (§8.2.2 and §8.3), a construction similar to that used in theorem 4.1 (but he uses an extra hypothesis); he illustrates it by some interesting examples.

The results proved in this paper are significant in two ways :
First, the methods produce new geometries from known geometries. In particular, one obtains new thin geometries related to polytopes and new geometries for sporadic groups.
Secondly, by exhibiting which geometries are derived from which, the classification process may be simplified. Rather than listing all geometries satisfying a set of axioms, one needs only list the essential ones from which other are derived. For example, if we look at all the regular thin geometries on which the group $Alt(5)$ acts flag-transitively [9], we obtain, up to isomorphism, four rank 3 geometries. If

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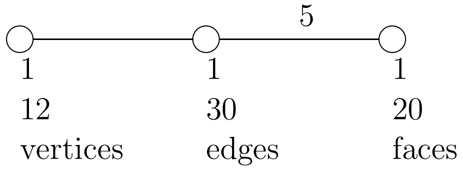
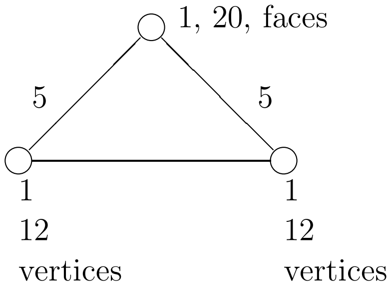
we decide to count only those that cannot be obtained from another by using the construction described in this paper, we only have two. For $PSL(2,8)$, we get 16 instead of 24.

To illustrate our purpose, let us take a simple example related to the icosahedron and to the group $2 \times Alt(5)$. Let Γ_1 be the geometry of rank 3 consisting of vertices, edges and faces of the icosahedron. Then Γ_1 has the following well known diagram usually called H_3.



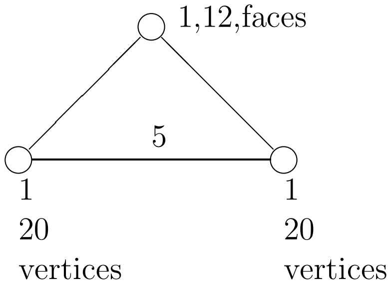
We can construct a new geometry $\tilde{\Gamma}_1$ starting from this one. The idea consists in withdrawing the edges of the icosahedron and replacing them by a copy of the vertices. We say that two vertices of distinct type are incident if and only if they are copies of distinct vertices of the icosahedron which lie on one of its edges. The diagram of $\tilde{\Gamma}_1$, consisting of the faces and two copies of the vertices, is as follows.



Let us mention that $\tilde{\Gamma}_1$ admits a nice representation on the "great dodecahedron". This star polyhedron has the same vertices and edges as the icosahedron and its faces are the convex pentagons whose five vertices are "adjacent" to a common vertex. Then $\tilde{\Gamma}_1$ may be described as follows: the elements of type 0 are the vertices of the great dodecahedron, the elements of type 1 are its pentagonal faces, and the elements of type 2 are the "cells" consisting of three pentagonal faces delimiting a face of the corresponding icosahedron; incidence is containment.

A similar representation of $\tilde{\Gamma}_1$ can be obtained on the "small stellated dodecahedron"; this is left to the interested reader.

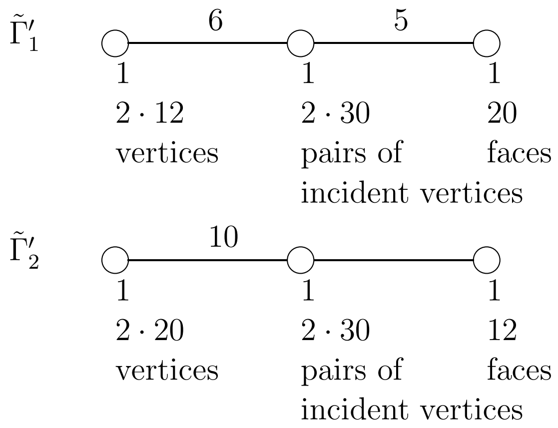
Similarly, starting from the geometry Γ_2 of vertices, edges and faces of a dodecahedron (Γ_2 is the dual of Γ_1), we get a geometry $\tilde{\Gamma}_2$ whose diagram is as follows.



Evidently, the group $2 \times Alt(5)$ acts flag-transitively on Γ_1, $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$: these three geometries may be considered as an "equivalence class" of geometries for this group. Similarly, the quotients of Γ_1, $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ by the automorphism permuting the pairs of opposite vertices provide an "equivalence class" of geometries for $Alt(5)$, with the same diagrams as above (i.e. three of the four geometries listed in [9] for $Alt(5)$).

The second idea of the present paper is to combine the previous construction to an earlier one, due to Neumaier (see [10], theorem 4). Actually, we need and prove an extension of Neumaier's theorem (see section 6). This will allow one to build further geometries for polytopes and groups.

Let us illustrate this with the geometries $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ obtained above. The idea is to bring together, as 0-elements, the two kinds of vertices, to add as new 1-elements the pairs of incident vertices (or flags of type {0,1} in $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$) and to keep the faces as 2-elements; incidence is containment. We get respectively the geometries



Now the group $2 \times Sym(5)$ acts chamber-transitively on these geometries.

The paper is organized as follows. In section 2, we fix some terminology and notation about incidence geometry. In section 3, we introduce the neighbourhood geometry of a graph and give a list of properties for this geometry. In section 4, we state our main result concerning our constructions and a corollary useful in the applications. Section 5 explains the construction and proves the main result. Section 6 is devoted to our extension of Neumaier's theorem. Applications of both the main construction of section 4 and the theorem of section 6 are presented in the two final sections: section 7 is concerned with thin geometries related to polytopes and section 8 presents geometries for finite groups, in particular for sporadic groups.

2 Terminology and notation

We assume familiarity with the basic notions about geometries as introduced in [14] or [1] (incidence, type, rank, flag, residue, shadow, ...). We use notation about diagrams as it appears in [3] or [13] together with [2] unless stated otherwise. For a general introduction on the subject, we refer to [4].

We recall that an *incidence structure* over a *set of types* Δ is a triple $\Gamma = (X, I, t)$ where:

- X is a set of *elements* also called *varieties*;

- $t : X \to \Delta$ is an application called the *type function*, such that $t(x) = i \in \Delta$ is the *type* of the element x for every $x \in X$. An element of type i is also called an i-*element*;
- I is an *incidence relation*, that is a relation that is reflexive and symmetric over the set of elements, and such that $\forall x, y \in X, t(x) = t(y)$ and $xIy \Rightarrow x = y$.

A *flag* of an incidence structure Γ is a set of pairwise incident elements of Γ. The *rank* of a flag F is the cardinality of $t(F)$ and the cardinality of Δ is called the *rank* of Γ. A flag F of type $t(F) = \Delta$ is called a *chamber*. If Γ is an incidence structure over Δ, and F is a flag of type $t(F)$, we call $\Delta \setminus t(F)$ the *cotype* of F and its cardinality the *corank* of F.
We call Γ a *geometry* when every flag of Γ is contained in a chamber.

The *shadow* of a flag F on the elements of type i, for any $i \in \Delta$, is denoted $\sigma_i(F)$.
The *residue* of F is denoted by $Res(F)$. Given distinct $i, j \in \Delta$, we denote by Γ_{ij} the class of all residues of Γ with type $\{i, j\}$.
Finally, the $\{i, j\}$-*truncation* of Γ, for distinct $i, j \in \Delta$, is the incidence structure over $\{i, j\}$ of all i-elements and j-elements of Γ, with the induced incidence relation of Γ.

According to [14], a geometry Γ is *residually connected* if every residue of rank at least two of Γ is a connected geometry.
A geometry Γ is *strongly connected* [1] if every truncation of rank at least two of Γ is a connected geometry and if the same property holds for every residue of rank at least two.

3 The neighbourhood geometry of a graph

Throughout this paper, every graph is assumed to be undirected, without loops and multiple edges; its vertex set is non-empty and each vertex is on at least one edge; the graph is not reduced to a unique edge.

Let G be a graph; its set of vertices (resp. edges) is denoted by G_0 (resp. G_1). For distinct p, $q \in G_0$, we say that p and q are *adjacent* - and we write $p \sim q$ - whenever $\{p, q\} \in G_1$. A circuit of length m of G is a sequence $p_1, p_2, \ldots, p_m$ of pairwise distinct vertices such that $p_i \sim p_{i+1}$ for $i = 1, \ldots, m \pmod m$. Recall that G may be thought of as a rank 2 geometry $(G_0 \cup G_1, t, I)$ over the set $\{0, 1\}$, where t is the type function defined by $t(G_i) = i$ for $i = 0, 1$, and I is the natural incidence relation between vertices and edges. By abuse of language, we identify the graph G and its associated rank 2 geometry.

Definition 3.1. To any graph $G = (G_0 \cup G_1, t, I)$, we associate a new rank 2 geometry $\tilde{G}$, called the *neighbourhood geometry* of G, whose elements are, roughly speaking, the vertices and the neighbourhoods of vertices of G. More precisely, we define $\tilde{G} = (G_0 \times \{0\} \cup G_0 \times \{1\}, \tilde{t}, \tilde{I})$ with
(i) $\tilde{t}(G_0 \times \{i\}) = i$, for $i = 0, 1$;
(ii) $(p, 0)\tilde{I}(q, 1)$ iff $p \sim q$, for $p, q \in G_0$.

If $\mathcal{G}$ is a class of graphs, $\tilde{\mathcal{G}}$ denotes the class of all $\tilde{G}$, for $G \in \mathcal{G}$.

We list some straightforward, but useful properties of $\tilde{G}$. The proofs are left to the reader.

Lemma 3.1. *$\tilde{G}$ is a self-dual geometry admitting the polarity $\Pi : (p, i) \to (p, i+1$ (mod 2)), for $p \in G_0$ and $i = 0, 1$.*

Lemma 3.2. *$\tilde{G}$ is connected if and only if G is connected and admits a circuit of odd length.*

Lemma 3.3. *$\tilde{G}$ has no repeated blocks - i.e. no two distinct 1-elements of $\tilde{G}$ are incident with the same set of 0-elements - if and only if G is irreducible - i.e. no two distinct vertices are both adjacent to the same set of vertices.*

Lemma 3.4. *$\tilde{G}$ satisfies the intersection property - i.e. (for non trivial rank 2 geometries) no two distinct 0-elements are both incident with two distinct 1-elements - if and only if G has no circuit of length 4.*

Lemma 3.5. *Any automorphism α of G induces a Δ-automorphism $\tilde{\alpha}$ of $\tilde{\mathcal{G}}$ defined by: $\tilde{\alpha}(p, i) = (\alpha(p), i)$ for $p \in G$ and $i \in \{0, 1\}$. Conversely, any Δ-automorphism of $\tilde{G}$ induces, by restriction on the 0-elements of $\tilde{G}$, an automorphism of G. Thanks to lemma 3.1, we get $Aut(\tilde{G}) = Aut(G) : 2$.*

Note that in general $Aut(\tilde{G}) \neq Aut(G) \times 2$ since the polarity Π of lemma 3.1 is not in $Z(Aut(\tilde{G}))$, for all G.

The parameters of $\tilde{G}$, such as the diameters (see [2]), cannot be calculated by a general formula from the parameters of G. However, the following lemma says something about the gonality.

Lemma 3.6. *If the gonality g of the geometry G is an even number, then the gonality $\tilde{g}$ of $\tilde{G}$ is $\frac{g}{2}$. If the gonality g of G is odd, then:*
- either $\tilde{g} = g$, when the graph G has no circuit([1])of even length $< 2g$;
- or $\tilde{g} = \frac{k}{2}$, when the graph G admits a circuit of even length $k < 2g$ and k is minimal for that property (obviously, $g < k$).

In view of their use as rank 2 residues of geometries for groups, we list in table 1 the rank 2 diagrams of some graphs G together with their associated geometry $\tilde{G}$ (the table is limited to graphs G whose $\tilde{G}$ is connected). Note that the geometry $\tilde{G}$ associated to a $2n$-gon or to the graph on the 16 points of the Minkowsky space $Mk(4, 2)$ (see group (111) in [3]) is not connected by lemma 3.2.

4 The main result

The main theorem applies to the geometries that satisfy one of the following two sets of conditions, whose equivalence is established by the next lemma.

Lemma 4.1. *Let Γ be a geometry of (possibly infinite) rank $n \geq 3$ over a set Δ and let $\{0, 1\} \subset \Delta$. Then conditions (A_1), (A_2) and (A_3) are equivalent to conditions (B_1) and (B_2) given below.*

(A_1) The class Γ_{01} of all residues of type $\{0, 1\}$ is a class $\mathcal{G}$ of (rank two) geometries of graphs (in the sense of section 3).

G	$\tilde{G}$
for $n \geq 1$, the $(2n+1)$-gon $\underset{1}{\bigcirc}\overset{2n+1}{\rule[0.6ex]{6em}{0.4pt}}\underset{1}{\bigcirc}$	the $(2n+1)$-gon $\underset{1}{\bigcirc}\overset{2n+1}{\rule[0.6ex]{6em}{0.4pt}}\underset{1}{\bigcirc}$
for $q \geq 3$, the complete graph K_{q+1} $(g_{01} = 3 = d_{01}, d_{10} = 4)$ $\underset{1}{\bigcirc}\overset{\subset}{\rule[0.6ex]{6em}{0.4pt}}\underset{\text{q-1}}{\bigcirc}$	the trivial structure of a set of $q+1$ points provided with all its q-subsets $(g_{01} = 2,\ d_{01} = d_{10} = 3)$. $\underset{\text{q-1}}{\bigcirc}\overset{T}{\rule[0.6ex]{6em}{0.4pt}}\underset{\text{q-1}}{\bigcirc}$
the Petersen graph (see group (84) in [3]) $(g_{01} = 5 = d_{01}, d_{10} = 6)$ $\underset{1}{\bigcirc}\overset{P}{\rule[0.6ex]{6em}{0.4pt}}\underset{2}{\bigcirc}$	the points and lines of the Desargues configuration (the group $S_5 = 2 \cdot O_4^-(2)$ acts chamber-transitively) $(g_{01} = 3,\ d_{01} = d_{10} = 5)$. $\underset{2}{\bigcirc}\overset{\bigtriangledown}{\rule[0.6ex]{6em}{0.4pt}}\underset{2}{\bigcirc}$
the diagram (34) in [3] associated to the group J_1 $(g_{01} = 5, d_{01} = 7, d_{10} = 8)$ $\underset{1}{\bigcirc}\overset{7\quad 5\quad 8}{\rule[0.6ex]{6em}{0.4pt}}\underset{11}{\bigcirc}$	the diagram (35) in [3] associated to the group J_1 $(g_{01} = 3,\ d_{01} = d_{10} = 5)$. $\underset{11}{\bigcirc}\overset{5\quad 3\quad 5}{\rule[0.6ex]{6em}{0.4pt}}\underset{11}{\bigcirc}$
the graph of the orthogonality relation on the 28 points exterior to a conic in $PG(2,7)$ [7] (see group (99) in [3]) $(g_{01} = 7, d_{01} = 9, d_{10} = 10)$ $\underset{1}{\bigcirc}\overset{\perp}{\rule[0.6ex]{6em}{0.4pt}}\underset{2}{\bigcirc}$	the point-line geometry induced by the projective plane on the 28 points exterior to a conic in $PG(2,7)$ (the group $PGL_2(7)$ acts chamber-transitively) $(g_{01} = 4,\ d_{01} = d_{10} = 7)$. $\underset{2}{\bigcirc}\overset{7\quad 4\quad 7}{\rule[0.6ex]{6em}{0.4pt}}\underset{2}{\bigcirc}$

Table 1: Some examples of graphs G and their associated geometry $\tilde{G}$. All $\tilde{G}$ are connected, without repeated block and all - provided T - satisfy the intersection property (see lemma 3.4).

(A_2) Let e be a 1-element and x an i-element, with $i \in \Delta - \{0,1\}$. If there are two distinct 0-elements p, q which are incident with both e and x, then eIx.
(A_3) Let e be a 1-element and x be an i-element, with $i \in \Delta - \{0,1\}$. Then eIx implies that $\sigma_0(e)$ is contained in $\sigma_0(x)$.

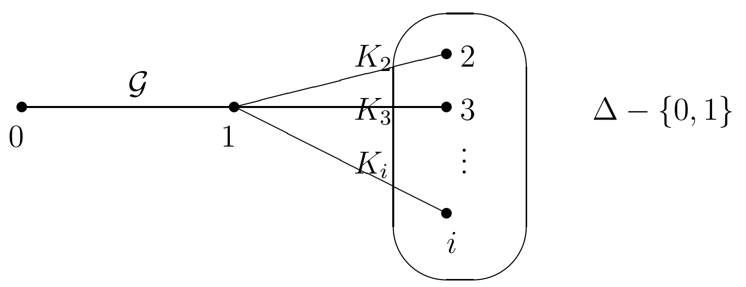
(B_1) The $\{0,1\}$-truncation of Γ is the geometry of a graph.
(B_2) Let e be a 1-element and x an i-element, with $i \in \Delta - \{0,1\}$. Then eIx if and only if $\sigma_0(e)$ is contained in $\sigma_0(x)$.

Proof: It is clear that (B_1) and (B_2) imply (A_1), (A_2) and (A_3). We prove the converse.
Proof of (B_1): let e be a 1-element. Since Γ is a geometry, there is a chamber C containing e. Thanks to (A_3), any 0-element incident with e is also incident with x, for $x \in C$ and $t(x) \neq 0,1$. Then, considering the flag of cotype $\{0,1\}$ contained in C and applying (A_1) to its residue, we can conclude that $\sigma_0(e)$ has cardinality two. We still have to show that if $\sigma_0(e) = \sigma_0(e')$ then $e = e'$. Using (A_2) and (A_3), we can say that e and e' are both incident with the same i-elements x (for $i \neq 0,1$). Hence, if F is a flag of cotype $\{0,1\}$ incident with e, then F is also incident with e'. So thanks to (A_1), we get $e = e'$. The existence of vertices and of edges through each vertex follows from the fact that Γ is a geometry.
Proof of (B_2): since we saw that $\sigma_0(e)$ has cardinality two, (B_2) is an immediate consequence of (A_2) and (A_3). ∎

Lemma 4.2. *Let Γ be as in lemma 4.1. Condition (A_3) implies that all Γ_{0i}, for $i \in \Delta - \{0,1\}$, are classes of generalized digons.*

Proof: Condition (A_3) means that, in the residue of any 1-element e, every i-element x (for $i \in \Delta - \{0,1\}$) is incident with every 0-element. Hence, any rank two residue of type $\{0,i\}$ is a generalized digon. ∎

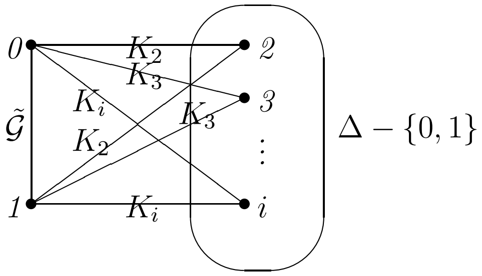
A partial converse will be proved below (lemma 4.4). In view of the previous lemma, the diagram of a geometry satisfying (A_1), (A_2) and (A_3) can be drawn as follows:



where the K_i are arbitrary classes of rank 2 geometries and where the diagram on $\Delta - \{0,1\}$ is arbitrary.

Note that, by (B_1), any 1-element e of Γ may be identified with the pair $\{p,q\}$ of the 0-elements incident with e and that, by (B_2), the residues of the flags $\{p,e\}$ and $\{q,e\}$ necessarily coincide. Moreover, any residue whose type contains $\{0, 1\}$ obviously satisfies (B_1), (B_2) (and hence (A_1), (A_2), (A_3)).

Theorem 4.1. *Let Γ be a geometry of (possibly infinite) rank $n \geq 3$ over a set Δ. If Γ satisfies (A_1), (A_2) and (A_3), or equivalently (B_1) and (B_2), for some $\{0, 1\} \subset \Delta$ and if the diagram of Γ is denoted as above, then there exists a geometry $\tilde{\Gamma}(0, 1)$ with diagram*



where $\tilde{\mathcal{G}}$ is the class of neighbourhood geometries of the members of $\mathcal{G}$ (see definition 3.1) and where the diagram on $\Delta - \{0, 1\}$ is the restriction of that of Γ. Furthermore, the following main properties hold:
(i) $\tilde{\Gamma}(0, 1)$ is residually (resp. strongly) connected if and only if the following holds: $\tilde{\mathcal{G}}$ is a class of connected geometries and Γ is residually (resp. strongly) connected.
(ii) If Σ is a group of Δ-automorphisms of Γ acting transitively on the chambers, then Σ is also a chamber-transitive group of Δ-automorphisms of $\tilde{\Gamma}(0, 1)$.
(iii) If there is some pair $\{k, l\} \subset \Delta - \{0, 1\}$ for which conditions (B_1) and (B_2) hold in Γ, then $\tilde{\Gamma}(0, 1)$ also satisfies (B_1) and (B_2) for that pair.

The construction of $\tilde{\Gamma}(0, 1)$ and the proof of theorem 4.1 are given in section 5. We mention here some remarks on this theorem and a corollary.

Note first that condition (A_2) appears as a very special case of the intersection property (defined in [1] for instance). Actually, we prove that it follows from the following weak version of the intersection property:

(I_{01}) Let e be a 1-element and x an i-element, with $i \in \Delta - \{0, 1\}$. Then, either $\sigma_0(x) \cap \sigma_0(e) = \emptyset$, or there exists a flag F, incident with e and x, such that $\sigma_0(F) = \sigma_0(x) \cap \sigma_0(e)$.

Lemma 4.3. *Let Γ be as in lemma 4.1. If Γ satisfies (A_1) and (I_{01}), then Γ satisfies (A_2).*

Proof: Let e be a 1-element and x an i-element, with $i \in \Delta - \{0, 1\}$ such that $| \sigma_0(x) \cap \sigma_0(e) | \geq 2$; we must prove that eIx. Assume the contrary; then the flag F, incident with e and x, with the property mentioned in (I_{01}), may not contain e. Since Γ is a geometry, F is contained in some flag F' of cotype $\{0, 1\}$ also incident with e. Since $\sigma_0(F) \subset \sigma_0(e)$ by (I_{01}), all 0-elements in $Res(F')$ are incident with e. But $Res(F')$ is a graph by (A_1), with no multiple edge and not reduced to a unique edge as assumed in section 3, a contradiction. ■

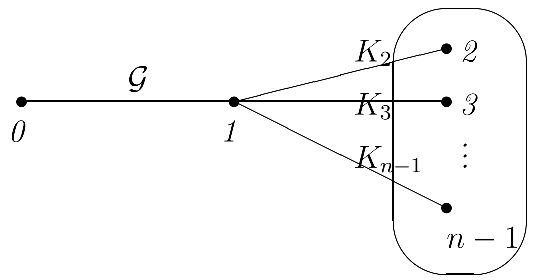
In the particular case of a residually connected geometry Γ with finite rank, condition (A_3) is equivalent to requiring that all rank two residues of type $\{0, i\}$, for $i \in \Delta - \{0, 1\}$, are generalized digons.

Lemma 4.4. *Let Γ be a residually connected geometry over a finite set Δ. Then (A_3) holds if and only if all Γ_{0i}, for $i \in \Delta - \{0, 1\}$, are classes of generalized digons.*

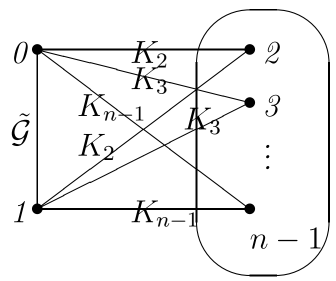
Proof: By lemma 4.2, we know already that (A_3) implies the mentioned property. Conversely, assume all Γ_{0i} are classes of digons. Since Γ is residually connected with finite rank, a theorem of Tits (see [1], lemma 7.2) applies to $Res(e)$, for any 1-element e, proving (A_3). ■

In view of applications, we restrict ourselves to residually connected geometries of finite rank satisfying the intersection property. In that case, thanks to lemmas 4.3 and 4.4 above, the main theorem may be stated as follows.

Corollary 4.1. *Let Γ be a residually connected geometry of finite rank $n \geq 3$, satisfying the intersection property (or the above mentioned weaker version (I_{01})). Suppose the diagram of Γ is as follows.*



where $\mathcal{G}$ is a class of graphs (in the sense of section 3). Then, there exists a geometry $\tilde{\Gamma}(0,1)$ with diagram as follows.



where $\tilde{\mathcal{G}}$ is the class of neighbourhood geometries of $\mathcal{G}$ (see definition 3.1) and where the diagram on $\{2, 3, \ldots, n-1\}$ is the restriction of that of Γ. Furthermore, (i), (ii) and (iii) of theorem 4.1 hold.

5 Construction and properties of $\tilde{\Gamma}(0,1)$: proof of theorem 4.1

Construction 5.1. *Let Γ be a geometry satisfying the hypothesis of theorem 4.1, and put $\Gamma = (\bigcup_{i=0}^{n-1} S_i, t, I)$ where $S_i = t^{-1}(i)$ for all $i \in \Delta$. The geometry $\tilde{\Gamma}(0,1)$ over Δ - whose existence is claimed in theorem 4.1 - is defined as follows: $\tilde{\Gamma}(0,1) = (\bigcup_{i=0}^{n-1} \tilde{S}_i, \tilde{t}, \tilde{I})$ where*

1. *$\tilde{S}_i = S_i$ for $i \neq 0, 1$ and $S_0 \times \{i\}$ for $i = 0, 1$;*

2. *$\tilde{t}(\tilde{S}_i) = i$;*

3. *given $x \in \tilde{S}_i$ and $y \in \tilde{S}_j$, with $i \neq j$,*
   - if $i, j \notin \{0,1\}$, then $x \in S_i$ and $y \in S_j$; we define $x\tilde{I}y$ if and only if xIy.
   - if $i \in \{0,1\}$ and $j \notin \{0,1\}$, then $x = (p,i)$ for some $p \in S_0$; we define $x\tilde{I}y$ if and only if pIy.
   - if $i, j \in \{0,1\}$, then $x = (p,i)$ and $y = (q,j)$ for some $p, q \in S_0$; we define $x\tilde{I}y$ if and only if $p \sim q$ in the $\{0,1\}$-truncation of Γ (which is a graph by condition (B_1)).

The flags of $\tilde{\Gamma}(0,1)$

It is usefull to establish a correspondence between the flags of Γ and those of $\tilde{\Gamma}(0,1)$.
If F (resp. $\tilde{F}$) is a flag of Γ (resp. $\tilde{\Gamma}(0,1)$), let H (resp. $\tilde{H}$) be the maximal subflag of F (resp. $\tilde{F}$) whose type is disjoint from $\{0,1\}$. Then H (resp. $\tilde{H}$) is also a flag of $\tilde{\Gamma}(0,1)$ (resp. Γ), whose cotype contains $\{0,1\}$.

Construction 5.2. *To each flag F of Γ is associated a flag $\tilde{F}$ of $\tilde{\Gamma}(0,1)$, with the same type, as follows.*
(i) if $F = H$, then $\tilde{F} = H$.
(ii) if $F = \{p\} \cup H$ for some $p \in S_0$, then define $\tilde{F} = \{(p,0)\} \cup H$.
(iii) if $F = \{e\} \cup H$ for some $e \in S_1$, then define $\tilde{F} = \{(p,1)\} \cup H$ where pIe; $\tilde{F}$ is a flag, thanks to (A_3).
(iv) if $F = \{p, e\} \cup H$ for some $p \in S_0$ and $e \in S_1$, then define $\tilde{F} = \{(p,0),(q,1)\} \cup H$, where $e = \{p, q\}$ (see (B_1)).

Construction 5.3. *Conversely, to each flag $\tilde{F}$ of $\tilde{\Gamma}(0,1)$ is associated a flag F of Γ as follows.*
$(\tilde{i})$ if $\tilde{F} = \tilde{H}$, then $F = \tilde{H}$.
$(\tilde{ii})$ if $\tilde{F} = \{(p,i)\} \cup \tilde{H}$ for some $p \in S_0$ and $i \in \{0,1\}$, then define $F = \{p\} \cup \tilde{H}$, which is clearly a flag of Γ.
$(\tilde{iii})$ if $\tilde{F} = \{(p,0),(q,1)\} \cup \tilde{H}$ for $p, q \in S_0$, the p, q are distinct and incident with some common $e \in S_1$. By (A_2), we deduce that $F = \{p, e\} \cup \tilde{H}$ is a flag of Γ.

The latter construction allows to prove a first property of $\tilde{\Gamma}(0,1)$.

Proposition 5.1. *$\tilde{\Gamma}(0,1)$ is a geometry.*

Proof: We have to show that any flag $\tilde{F}$ of $\tilde{\Gamma}(0,1)$ can be completed into a chamber. Let F be the flag of Γ associated to $\tilde{F}$ by construction 5.3. Since Γ is a geometry, there is a chamber C of Γ containing the flag F. Put $C = \{p', e'\} \cup H$, with $p' \in S_0$, $e' \in S_1$ and H a flag of cotype $\{0,1\}$. Since (B_1) shows that $\sigma_0(e') = \{p', q'\}$ for some $q' \in S_0$, the unions $\{(p',0),(q',1)\} \cup H$ and $\{(q',0),(p',1)\} \cup H$ are chambers of $\tilde{\Gamma}(0,1)$. At least one of them contains $\tilde{F}$ because if F has been deduced from $\tilde{F}$ following $(\tilde{ii})$ (resp. $(\tilde{iii})$) above, then $p = p'$ (resp. $p' = p$ and $q' = q$). Hence the proof is finished. ■

The residues of $\tilde{\Gamma}(0,1)$

Given a flag F of Γ, we denote by $Res(F)$ the residue of F in Γ. Similarly, $\tilde{Res}(\tilde{F})$ is the residue in $\tilde{\Gamma}(0,1)$ of a flag $\tilde{F}$ of $\tilde{\Gamma}(0,1)$.

Lemma 5.1. *Let $x = (p,i) \in \tilde{S}_i$ for $p \in S_0$ and $i \in \{0,1\}$. Then there exists an isomorphism α from the geometry $\tilde{Res}(x)$ over $\Delta - \{i\}$ to the geometry $Res(p)$ over $\Delta - \{0\}$; α induces the identity on $\Delta - \{0,1\}$ and maps $i+1$ (mod 2) on 1.*

Proof: Let $y \in \tilde{Res}(x)$. If $\tilde{t}(y)$ is different from 0 and 1, then we can define $\alpha(y) = y$. If $\tilde{t}(y) = i+1 \pmod 2$, then $y = (q, i+1 \pmod 2)$ for some $q \in S_0$ with $p \sim q$. Thanks to (B$_1$), there is a unique $e \in S_1$ such that $\sigma_0(e) = \{p, q\}$ and we can define $\alpha(y) = e \in Res(p)$. The function α is bijective because, given $e' \in S_1$ contained in $Res(p)$, condition (B$_1$) implies the unicity of $q' \in S_0$ such that $e' = \{p, q'\}$; so there is a unique $z = (q', i+1 \pmod 2) \in \tilde{Res}(x)$ such that $\alpha(z) = e'$. Furthermore, α is an incidence preserving map. This is trivial for an incident pair whose type is disjoint from $\{0,1\}$. Now, if $y = (q, i+1 \pmod 2) \in \tilde{Res}(x)$ and $z \in \tilde{Res}(x)$ with $\tilde{t}(z) \notin \{0,1\}$, then, by definition, $y\tilde{I}z$ if and only if qIz. By (B$_2$), the latter incidence is equivalent to $e = \alpha(y)Iz$. ∎

Lemma 5.2. *Let $\tilde{F}$ be a flag of $\tilde{\Gamma}(0,1)$ of cotype $\{0,1\}$. Then $\tilde{F}$ is also a flag F in Γ and $\tilde{Res}(\tilde{F})$ is the neighbourhood geometry $\tilde{Res}(F)$ of $Res(F)$*

Proof: By the definition of $\tilde{\Gamma}(0,1)$, we have $\tilde{Res}(\tilde{F}) = \{(p,0),(q,1) \mid p,q \in S_0 \cap Res(F)\}$ and $(p,0)\tilde{I}(q,1)$ if and only if $p \sim q$. This shows that $\tilde{Res}(F)$ is the neighbourhood geometry of $Res(F)$, in the sence of definition 3.1. ∎

Proposition 5.2. *The geometry $\tilde{\Gamma}(0,1)$ has the diagram announced in theorem 4.1.*

Proof: Obvious thanks to lemmas 5.1 and 5.2. ∎

Lemma 5.3. *Let $\tilde{F} = \{(p,0),(q,1)\}$, with $p,q \in S_0$, $p \sim q$, be a flag of type $\{0,1\}$ in $\tilde{\Gamma}(0,1)$. If F is the flag $\{p,e\}$, with $e = \{p,q\}$, then there is an isomorphism from $\tilde{Res}(\tilde{F})$ to $Res(F)$ inducing the identity on $\Delta - \{0,1\}$.*

Proof: This is an immediate consequence of lemma 5.1. Note that $Res(\{p,e\}) = Res(\{q,e\})$ as it was mentioned in section 4. ∎

Lemma 5.4. *Let $\tilde{F}$ be a flag of cotype strictly containing $\{0,1\}$ in $\tilde{\Gamma}$. Then $\tilde{Res}(\tilde{F})$ (obviously satisfying (B$_1$) and (B$_2$) as mentioned in section 4) is the geometry $\tilde{Res}(F)(0,1)$ obtained from $Res(F)$ following construction 5.1.*

Proof: Similar to that of lemma 5.2. ∎

Lemma 5.5. *Let $e \in S_1$ and $e = \{p,q\}$, with $p,q \in S_0$. Let F be a flag of Γ, of cotype containing $\{0,1\}$ and incident with p and q. Then for $i \in \{0,1\}$, the residues in $\tilde{\Gamma}(0,1)$ of $F \cup \{(p,i)\}$ are isomorphic to $Res(F \cup \{p\})$ and the residues in $\tilde{\Gamma}(0,1)$ of $F \cup \{(p,i),(q,i+1 \pmod 2)\}$ are isomorphic to $Res(F \cup \{p,e\})$.*

Proof: Immediate consequence of lemma 5.1. ∎

The connectedness of $\tilde{\Gamma}(0,1)$

In order to study the connectedness of $\tilde{\Gamma}(0,1)$, we have to find a correspondence between paths of Γ and paths of $\tilde{\Gamma}(0,1)$. For our purpose, a *path* of a geometry is a sequence of pairwise distinct elements $(x_1,\ldots,x_m)$, with $m \geq 2$ such that x_l and x_{l+1} are incident for all $l \in \{1,\ldots,m-1\}$. Such a path is called a $\{i,j\}$-*path*, for two distinct $i,j \in \Delta$, if all x_l's are of type i or j. Moreover, if x_m is incident to x_1, the path is called a *circuit*.

Construction 5.4. *To any path P of Γ, we associate a path $\tilde{P}$ of $\tilde{\Gamma}(0,1)$ as follows:*
(i) each element of type $\neq 0,1$ in P is unchanged;
(ii) each 1-element between two 0-elements in P is deleted;
(iii) each other 1-element e in P is replaced by a 0-element p of Γ such that pIe and p does not occur next to e in P;
(iv) rules (i), (ii) and (iii) give rise to a sequence of elements of Γ, which is transformed into a sequence in $\tilde{\Gamma}(0,1)$ by replacing each 0-element p of Γ by a (p,i) of $\tilde{\Gamma}(0,1)$ in such a way that no two consecutive (p,i)'s have the same type i;
(v) finally, if a same element (p,i) occurs twice in that sequence of $\tilde{\Gamma}(0,1)$, the sequence is shortened by deleting one (p,i) and all elements between the two (p,i)'s, except of course if the two (p,i)'s are the extremities of the sequence, in which case we just delete the last (p,i).

We claim that the sequence $\tilde{P}$ obtained in that way is a path of $\tilde{\Gamma}(0,1)$. The elements of $\tilde{P}$ are distinct, thanks to *(v)* and $\mid \tilde{P} \mid \geq 2$. We have to check that any two consecutive elements x_l and x_{l+1} of $\tilde{P}$ are incident in $\tilde{\Gamma}(0,1)$.
If the types of x_l and x_{l+1} are both different from 0 and 1, this follows from *(i)*. If exactly one of the elements x_l and x_{l+1} is of type 0 or 1, say $x_l = (p,i)$, then x_l corresponds, in P, either to the 0-element p or to a 1-element e incident with p; in the first case, the incidence between x_l and x_{l+1} follows from the incidence between p and x_{l+1}; in the second case, thanks to (A_3), the incidence of e with x_{l+1} implies the incidence of p with x_{l+1} and that of x_l with x_{l+1}. Finally, if both x_l and x_{l+1} are of type 0 or 1, the incidence follows from *(iv)*, *(ii)* and *(iii)*.

Construction 5.5. *Conversely, to each path $\tilde{P}$ of $\tilde{\Gamma}(0,1)$ is associated a path P of Γ:*
$\widetilde{(i)}$ elements of type different from 0,1 in $\tilde{P}$ are unchanged;
$\widetilde{(ii)}$ any element (p,i) of $\tilde{P}$ is replaced by the 0-element p of Γ;
$\widetilde{(iii)}$ if two elements (p,i) and (q,j) are consecutive in $\tilde{P}$, then insert in P, between p and q, the 1-element e incident with p and q (such an element exists, since $p \sim q$ by the definition of the incidence in $\tilde{\Gamma}(0,1)$).
$\widetilde{(iv)}$ if some 0-element or 1-element appears twice, then shorten the sequence by deleting one of the occurences and the elements between them.

Proposition 5.3. $\tilde{\Gamma}(0,1)$ *is connected if and only if* Γ *is connected.*

Proof: Assume Γ is connected. By proposition 5.1, each element of $\tilde{\Gamma}$ is incident to some element of type $i \notin \{0,1\}$. So, for proving $\tilde{\Gamma}$ is connected, it is enough to exhibit a path of $\tilde{\Gamma}$ joining any two i-elements, with $i \neq 0,1$. Such a path P exists in Γ; the associated path $\tilde{P}$ of $\tilde{\Gamma}$, defined by construction 5.4, is the required path.
Conversely, if $\tilde{\Gamma}(0,1)$ is assumed to be connected, the correspondence defined by construction 5.5 allows to prove in a similar way that Γ must also be connected. ■

Proposition 5.4. $\tilde{\Gamma}(0,1)$ *is residually connected if and only if the following holds : the class $\tilde{\mathcal{G}}$ of neighbourhood geometries is a class of connected geometries and Γ is residually connected.*

Proof: By proposition 5.3, $\tilde{\Gamma}(0,1)$ is connected if and only if Γ is connected. We have to prove the same equivalence for residues of nonempty flags with corank ≥ 2.

This follows immediately from lemmas 5.1, 5.3, 5.5, for flags of $\tilde{\Gamma}(0,1)$ whose type intersects $\{0,1\}$ and flags of Γ whose type contains 0. It is clear also for flags (of Γ or $\tilde{\Gamma}$) of cotype $\{0,1\}$ by lemma 5.4 together with proposition 5.3 or by lemma 5.2. Finally, for residues in Γ of flags whose type contains 1 and not 0, the property is obvious by condition (A_3). ∎

Proposition 5.5. $\tilde{\Gamma}(0,1)$ *is strongly connected if and only if the following holds : $\tilde{\mathcal{G}}$ is a class of connected geometries and Γ is strongly connected.*

Proof: It is enough to prove that any $\{i,j\}$-truncation of $\tilde{\Gamma}(0,1)$ (resp. Γ) is connected if Γ is strongly connected and $\tilde{\mathcal{G}}$ is a class of connected geometries (resp. $\tilde{\Gamma}(0,1)$ is strongly connected). The corresponding property for the residues follows by applying 5.1 up to 5.5 and condition (A_3), as in the proof of proposition 5.4.
The $\{i,j\}$-truncations of Γ and $\tilde{\Gamma}(0,1)$ coincide when $i,j \notin \{0,1\}$. They are obviously isomorphic, for $i=0$ and $j \neq 1$. Also, the $\{1,j\}$-truncation of $\tilde{\Gamma}(0,1)$ is isomorphic to the $\{0,j\}$-truncation of Γ, for any $j \notin \{0,1\}$.
Look at the $\{0,1\}$-truncation of $\tilde{\Gamma}(0,1)$. If the $\{0,1\}$-truncation of Γ is connected, then applying *(ii)* and *(iv)* of construction 5.4, we see that any 0-element $(p,0)$ of $\tilde{\Gamma}(0,1)$ is linked by a $\{0,1\}$-path to either $(q,0)$ or $(q,1)$, for any element q of Γ. Now, consider a flag F of cotype $\{0,1\}$ incident with q in Γ: the residue $\widetilde{Res}(F)$ of F in $\tilde{\Gamma}(0,1)$ contains $(q,0)$ and $(q,1)$; hence, if we assume that $\tilde{\mathcal{G}}$ is a class of connected geometries, then $\widetilde{Res}(F)$ is connected and the elements $(q,0)$ and $(q,1)$ are linked by a $\{0,1\}$-path. Thus we conclude that the $\{0,1\}$-truncation of $\tilde{\Gamma}(0,1)$ is connected. Hence, every $\{i,j\}$-truncation of $\tilde{\Gamma}(0,1)$ (and of its residues) is connected and so, the strong connectedness of $\tilde{\Gamma}(0,1)$ follows from that of Γ and from the connectedness of the neighbourhood geometries in $\tilde{\mathcal{G}}$.

Conversely, assume $\tilde{\Gamma}(0,1)$ is strongly connected. Then the $\{0,1\}$-truncation of Γ is connected: indeed, given two 0-elements p and q of Γ, there is a $\{0,1\}$-path of $\tilde{\Gamma}(0,1)$ from $(p,0)$ to $(q,0)$; this path leads to a$\{0,1\}$-path of Γ from p to q (see $(\widetilde{ii})$ and $(\widetilde{iii})$ of construction 5.5). Note the existence of this path from p to q is sufficient to ensure the connectedness of the $\{0,1\}$-truncation, thanks to the fact that Γ is a geometry.
It remains to look at the connectedness of the $\{1,j\}$-truncations of Γ, for all $j \notin \{0,1\}$. Let e, f be two 1-elements of Γ. Assume first that e and f belong to $Res(p)$ for some 0-element p. Since the $\{1,j\}$-truncation of $\widetilde{Res}(p,0)$ is connected by hypothesis, it is easy to obtain a $\{1,j\}$-path linking e to f in Γ, by applying construction 5.5, lemma 5.1 and condition (A_2). Now if e and f are not incident to a same 0-element, then there is a $\{0,1\}$-path $(e,p_1,e_1,p_2,e_2,\ldots,p_m,f)$ in Γ, since the $\{0,1\}$-truncation of Γ is connected as proved above. By the previous argument, each pair $\{e,e_1\}$, $\{e_1,e_2\}$, …, $\{e_m,f\}$ is linked by a suitable path and so the strong connectedness of Γ is proved. ∎

The Automorphism Group of $\tilde{\Gamma}(0,1)$

The reader can easily verify that $\tilde{\Gamma}(0,1)$ admits the following canonical duality.

Proposition 5.6. *The function α, defined on the elements of $\tilde{\Gamma}(0,1)$ by $\alpha(p,i) = (p, i+1 \ (mod\ 2))$ for $i=0,1$ and $\alpha(x)=x$ for $x \in \tilde{S}_i$ with $i \notin \{0,1\}$, is a duality of $\tilde{\Gamma}(0,1)$.*

Let σ be any Δ-automorphism of Γ. Then σ induces a function $\tilde{\sigma}$ on $\tilde{\Gamma}(0,1)$ as follows: if $x \in \tilde{S}_i$ with $i \notin \{0,1\}$, then $x \in S_i$ and $\tilde{\sigma}(x) = \sigma(x)$; if $x \in \tilde{S}_i$ with $i \in \{0,1\}$, then $\tilde{\sigma}(p,i) = (\sigma(p), i)$. It is an easy exercise to verify that the function $\tilde{\sigma}$ is actually a Δ-automorphism of $\tilde{\Gamma}(0,1)$.

Proposition 5.7. *Let Σ be a group of Δ-automorphisms of Γ acting transitively on the chambers. Then Σ is also a chamber-transitive group of Δ-automorphisms of $\tilde{\Gamma}(0,1)$.*

Proof: Consider the action of Σ on $\tilde{\Gamma}(0,1)$ as defined above. Let $\tilde{C}$ be a chamber of $\tilde{\Gamma}(0,1)$: $\tilde{C}$ is a set $\{(p,0),\ (q,1),\ x_2,\ \ldots,\ x_i,\ \ldots\}$ where $p, q \in S_0$, $x_i \in \tilde{S}_i = S_i$ for $i \notin \{0,1\}$ and $e = \{p,q\} \in S_1$, $x_i I p$, $x_i I q$, $x_i I x_j$, for $i, j \notin \{0,1\}$ and $i \neq j$. To $\tilde{C}$ is associated a chamber C of Γ: $C = \{p, e, x_2, \ldots, x_i, \ldots\}$ (condition (A_2)). Let $\tilde{C'} = \{(p',0), (q',1), \ldots, x'_i, \ldots\}$ be another chamber of $\tilde{\Gamma}(0,1)$ and let C' be the one associated in Γ. Since Σ acts transitively on the chambers of Γ; there is an automorphism σ mapping C onto C' such that $\sigma(p) = p'$, $\sigma(e) = e'$ and $\sigma(x_i) = x'_i$. Thanks to condition (B_1), this implies that $\sigma(q) = q'$ and so the automorphism σ induces an automorphism $\tilde{\sigma}$ of $\tilde{\Gamma}(0,1)$ mapping $\tilde{C}$ onto $\tilde{C}'$. ■

Does $\tilde{\Gamma}(0,1)$ satisfy (B_1) and (B_2) for some $k, l \notin \{0,1\}$?

Suppose Γ satisfy (B_1) and (B_2) for some $k, l \notin \{0,1\}$. We show that the same holds for $\tilde{\Gamma}(0,1)$. This result is usefull for a repetitive application of the main theorem (see sections 7 and 8).

Proposition 5.8. *Suppose Γ satisfies conditions (B_1) and (B_2) for $\{0,1\} \subset \Delta$ (as in lemma 4.1) but also for some pair $\{k,l\} \subset \Delta - \{0,1\}$. Then $\tilde{\Gamma}(0,1)$ satisfies (B_1) and (B_2) for the pair $\{k,l\}$.*

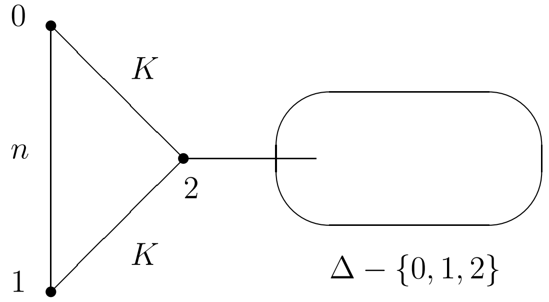
Proof: Condition (B_1) is trivially satisfied in $\tilde{\Gamma}(0,1)$ if it holds in Γ since the $\{k,l\}$-truncations of these geometries coincide. We now prove (B_2). Let e be a l-element of $\tilde{\Gamma}(0,1)$. If x is a i-element with $i \notin \{0,1,k,l\}$, then e and x are also elements of Γ and the assertion holds in both Γ and $\tilde{\Gamma}(0,1)$. Now if x is a i-element with $i \in \{0,1\}$, then $x = (p,i)$ for some $p \in S_0$ and $e\tilde{I}x$ is equivalent to eIp; as (B_2) holds in Γ, this means $\sigma_k(e) \subset \sigma_k(p)$ in Γ; by the definition of the incidence in $\tilde{\Gamma}(0,1)$, this leads to $\sigma_k(e) \subset \sigma_k(x)$ and the proposition is proved. ■

This completes the proof of theorem 4.1

6 A generalization of Neumaier's theorem

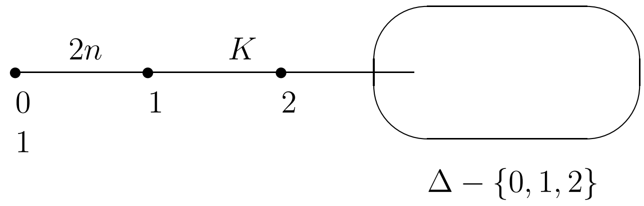
In 1982, Neumaier ([10], theorem 4) established an equivalence between the class of geometries with diagram

(1)



and the class of geometries with diagram

(2)



where K is an arbitrary class of rank two geometries, the $2n$-gon is a graph and the diagram on $\Delta - \{0, 1, 2\}$ is arbitrary.

The application of our main theorem 4.1 leads to new geometries whose diagram is (1) or somewhat more general. A result like Neumaier's theorem is thus of great interest here: by applying it successively to theorem 4.1, we get again new geometries whose diagram is like (2). Moreover, if a chamber-transitive automorphism group acts on the initial geometry, it still acts on the last. Actually, in order to cover all possible diagrams generated by application of theoren 4.1, we need an extended version of Neumaier's result, which is proved below (see theorem 6.1). This extension can be deduced from the general results on shadows ([1] and [11], chap. 5). In order to combine it with theorem 4.1, we need some additional properties given in theorem 6.1. Hence it is fair to give an elementary proof of the whole theorem.

Finally, note that the rank two truncations of the geometries with diagram (2) (or, more generally, diagram (4) below) are geometries of graphs: theorem 4.1 or corollary 4.1 can again be applied to them. Unfortunately, we get then non residually connected geometries by lemma 3.2 and theorem 4.1(i) (see lemma 6.1).

Definition 6.1. *Let $R = (R_0 \cup R_1, t, I)$ be any rank two geometry. R is described by its incidence graph whose set of vertices is $R_0 \cup R_1$ and whose edges are the incident pairs (x, y), with $x \in R_0$ and $y \in R_1$. The latter graph can be thought of as a new rank two geometry, called the* flag geometry *of R and denoted by $\mathcal{F}(R)$: its 0-elements set is $R_0 \cup R_1$, its 1-elements are the chambers of R and the incidence is containment.*

Conversely, any bipartite graph $\mathcal{F}$ (with no vertex of degree zero, as assumed in section 2) defines a rank two geometry R: the two sets of vertices are the two types of elements and the edges are the chambers. Obviously, $\mathcal{F} = \mathcal{F}(R)$.

We list some useful properties: the straightforward proofs are left to the reader.

Lemma 6.1. *For any rank two geometry R (not reduced to a unique chamber), the flag geometry $\mathcal{F}(R)$ is a bipartite graph. Hence, the neighbourhood geometry $\tilde{\mathcal{F}}(R)$ is not connected: it is the union of two disjoint copies of $\mathcal{F}(R)$.*

Proof: Follows from definition 6.1 and lemma 3.2. ■

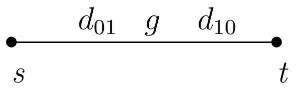
Lemma 6.2. *$\mathcal{F}(R)$ satisfies the intersection property.*

Lemma 6.3. *$\mathcal{F}(R)$ is connected if and only if R is connected.*

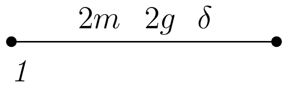
Lemma 6.4. *The automorphism group $Aut(R)$ of the geometry R acts on $\mathcal{F}(R)$ as a group of Δ-automorphisms. Moreover, if R admits a chamber-transitive Δ-automorphism group Ω and if R admits a duality (an automorphism permuting the* *types), then $\Omega : 2$ acts as a chamber-transitive Δ-automorphism group on $\mathcal{F}(R)$. In particular, if Γ is a graph with a chamber-transitive Δ-automorphism group Ω, then $\Omega : 2$ acts similarly on $\mathcal{F}(\tilde{\Gamma})$, where $\tilde{\Gamma}$ is the neighbourhood geometry of Γ.*

Proof: The last property follows from lemma 3.4. ∎

Lemma 6.5. *If R has diagram*



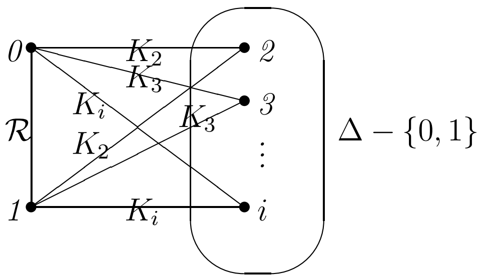
then $\mathcal{F}(R)$ has diagram



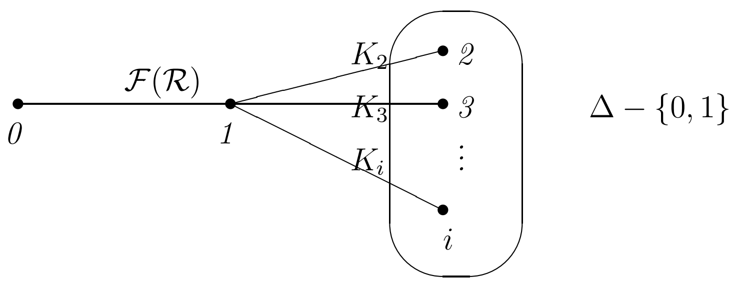
where $m = max\{d_{01}, d_{10}\}$ and $\delta \in \{2m, 2m-1\}$. Moreover, $d_{01} \neq d_{10}$ implies $\delta = 2m - 1$ and $d_{01} = d_{10} = g$ implies $\delta = 2m$.

We list in Table 2, some rank two geometries together with their flag geometry, useful for the applications. With help of lemma 6.5, the proofs are straightforward. Note that the first example is well known. The numbers appearing below a parameter is the total number of elements of that type.

Theorem 6.1. *(extension of Neumaier ([10], theorem 4)) Let Γ be a geometry of (possibly infinite) rank $n \geq 3$ admitting the following diagram*



where $\mathcal{R}$ and the K_i are arbitrary classes of rank two geometries and where the diagram on $\Delta - \{0,1\}$ is arbitrary. Then there is a geometry $\Gamma'(0,1)$, satisfying conditions (B_1) and (B_2) of lemma 4.1 for the pair $\{0,1\}$, with diagram



R	$\mathcal{F}(R)$
for $n \geq 2$, the generalized n-gon $\bigcirc\overset{n}{\rule{3em}{0.4pt}}\bigcirc$ In particular, the n-gon $\underset{1}{\bigcirc}\overset{n}{\rule{3em}{0.4pt}}\underset{1}{\bigcirc}$	the generalized $2n$-gon $\underset{1}{\bigcirc}\overset{2n}{\rule{3em}{0.4pt}}\bigcirc$ the $2n$-gon $\underset{1}{\bigcirc}\overset{2n}{\rule{3em}{0.4pt}}\underset{1}{\bigcirc}$
for $q \geq 3$, the geometry of the complete graph K_{q+1} $(g_{01} = d_{01} = 3,\ d_{10} = 4)$ $\underset{1}{\bigcirc}\overset{\subset}{\rule{3em}{0.4pt}}\underset{q-1}{\bigcirc}$	$(g_{01} = 6,\ d_{01} = 8,\ d_{10} = 7)$ $\underset{1}{\bigcirc}\overset{8\quad 6\quad 7}{\rule{3em}{0.4pt}}\underset{\{1,q-1\}}{\bigcirc}$
for $q \geq 3$, the neighbourhood geometry of the complete graph K_{q+1} (see table 1). $(g_{01} = 2,\ d_{01} = d_{10} = 3)$ $\underset{\substack{q-1\\q+1}}{\bigcirc}\overset{T}{\rule{3em}{0.4pt}}\underset{\substack{q-1\\q+1}}{\bigcirc}$	the dual geometry of a $(q+1) \times (q+1)$ grid with a maximal set of pairwise non-collinear points deleted; $2 \times S_{q+1}$ acts chamber-transitively. $(g_{01} = 4,\ d_{01} = d_{10} = 6)$ $\underset{\substack{1\\2(q+1)}}{\bigcirc}\overset{6\quad 4\quad 6}{\rule{3em}{0.4pt}}\underset{\substack{q-1\\q(q+1)}}{\bigcirc}$
the points and lines of the Desargues configuration (see table 1); the group S_5 acts chamber-transitively. $(g_{01} = 3,\ d_{01} = d_{10} = 5)$ $\underset{\substack{2\\10}}{\bigcirc}\overset{\bigtriangledown}{\rule{3em}{0.4pt}}\underset{\substack{2\\10}}{\bigcirc}$	the group $2 \cdot S_5$ acts chamber-transitively $(g_{01} = 6,\ d_{01} = d_{10} = 10)$ $\underset{\substack{1\\20}}{\bigcirc}\overset{10\quad 6\quad 10}{\rule{3em}{0.4pt}}\underset{\substack{2\\30}}{\bigcirc}$
diagram (35) in [3] associated to J_1 (see table 1) $\underset{\substack{11\\266}}{\bigcirc}\overset{5\quad 3\quad 5}{\rule{3em}{0.4pt}}\underset{\substack{11\\266}}{\bigcirc}$	the group $2 \cdot J_1$ acts chamber-transitively $\underset{\substack{1\\532}}{\bigcirc}\overset{10\quad 6\quad 10}{\rule{3em}{0.4pt}}\underset{\substack{11\\3192}}{\bigcirc}$
$\underset{\substack{4\\8}}{\bigcirc}\overset{3\quad 2\quad 3}{\rule{3em}{0.4pt}}\underset{\substack{4\\8}}{\bigcirc}$	diagram (111) in [3] associated to $2^4 \cdot A_5$ $\underset{\substack{1\\16}}{\bigcirc}\overset{5\quad 4\quad 6}{\rule{3em}{0.4pt}}\underset{\substack{4\\40}}{\bigcirc}$

Table 2: Some examples of geometries R and their flag geometry $\mathcal{F}(R)$. All $\mathcal{F}(R)$ are connected and satisfy the intersection property.

where $\mathcal{F}(\mathcal{R})$ is the class of flag geometries of the members of $\mathcal{R}$.

Furthermore, the following properties hold:
(i) $\Gamma'(0,1)$ *is connected (resp. residually or strongly connected) if and only if* Γ *is;*
(ii) $\Gamma'(0,1)$ *satisfies conditions (B$_1$) and (B$_2$) for some pair* $\{k,l\} \subset \Delta - \{0,1\}$ *if and only if* Γ *does;*
(iii) if Σ *is a group of* Δ*-automorphisms of* Γ *acting transitively on the chambers and if there is an automorphism* σ *of* Γ *permuting the types 0 and 1 and fixing* $\Delta - \{0,1\}$ *pointwise, then* $< \Sigma, \sigma >$ *acts as a chamber-transitive* Δ*-automorphism group of* $\Gamma'(0,1)$. *Conversely, if* Ω *is a chamber-transitive* Δ*-automorphism group of* $\Gamma'(0,1)$*, then* $\Omega = \Sigma : 2$ *for some* Σ *acting as a chamber transitive* Δ*-automorphism group of* Γ.

Before giving the rather straightforward proof of this theorem, we make the construction of $\Gamma'(0,1)$ more explicit.

Construction 6.1. *Let* Γ *be a geometry satisfying the hypotheses of theorem 6.1. Put* $\Gamma = (\cup_{i=0}^{n-1} S_i, t, I)$. *We define a geometry* $\Gamma'(0,1) = (\cup_{i=0}^{n-1} S'_i, t', I')$ *as follows:*
(a)

$$S'_i = \begin{cases} S_i, & \textit{for } i \neq 0,1 \\ S_0 \cup S_1, & \textit{for } i = 0 \\ \textit{set of all flags of type } \{0,1\} \textit{ of } \Gamma, & \textit{for } i = 1 \end{cases}$$

(b) $t'(S'_i) = i$
(c) given $x \in S'_i$ *and* $y \in S'_j$*, with* $i \neq j$,
- if $1 \notin \{i,j\}$ *then* $xI'y$ *if and only if* xIy;
- if $i = 1$ *(and similarly if* $j = 1$*), then* x *is a flag* $\{x_0, x_1\}$ *of* Γ *and* $xI'y$ *if and only if both* x_0Iy *and* x_1Iy *(note that this means* $y \in \{x_0, x_1\}$ *when* $j = 0$*).*

Proof of theorem 6.1: We denote $\Gamma'(0,1)$ briefly Γ'.
(1) <u>The flags of Γ'.</u> Let F be a flag of Γ, $\overline{F}$ the maximal subflag of F whose type is disjoint from $\{0,1\}$, $p \in S_0$ and $q \in S_1$ with pIq. One of the following cases occur:
(a) if $F = \overline{F}$, then F is also a flag on Γ' with same type;
(b) iff $F = \overline{F} \cup \{x\}$, where $x \in \{p,q\}$, then F is also a flag of Γ' with type $t(F) \cup \{0\}$;
(c) if $F = \overline{F} \cup \{p,q\}$, then $\overline{F} \cup \{p, \{p,q\}\}$ and $\overline{F} \cup \{q, \{p,q\}\}$ are flags of Γ' with same type; moreover, $\overline{F} \cup \{\{p,q\}\}$ is a flag of Γ' with type $t(F) \backslash \{0\}$.

Obviously, any flag of Γ' can be obtained from a unique flag F of Γ by applying (a), (b) or (c). Furthermore, any chamber of Γ leads to two chambers of Γ' by (c). Hence, any flag of Γ' is contained in a chamber and Γ' is a geometry as claimed.

Moreover, by construction 6.1, any 1-element of Γ' is incident with exactly two 0-elements: condition (B$_1$) follows immediately and construction 6.1(c) implies (B$_2$) for the pair $\{0,1\}$. In view of construction 6.1, the proof of property (ii) is straightforward.

(2) <u>The residues of Γ'.</u> If p is a 0-element of Γ' then, by construction 6.1, the residue of p is clearly isomorphic to the residue of p in Γ. Consequently, the residue of any flag of Γ' containing a 0-element is isomorphic to a residue of Γ. Moreover, any residue in Γ of a flag containing a 0- or a 1-element is of that type.
If $f = \{p,q\}$ is a 1-element of Γ' then, by construction 6.1, the residue of f is the direct sum of the set $\{p,q\}$ and of the residue of the flag $\{p,q\}$ in Γ. Consequently, the residue of any flag of Γ' containing a 1-element and no 0-element is a similar

direct sum.
Finally, if D is a flag of type $\Delta - \{0,1\}$ then, by construction 6.1, the residue of D in Γ' is the flag geometry $\mathcal{F}(R)$ of the residue R of D in Γ. Consequently, the residue of any flag F with $t(F) \subset \Delta - \{0,1\}$ is the geometry obtained from $Res(F)$ in Γ by applying construction 6.1. Now Γ' has clearly the diagram announced.

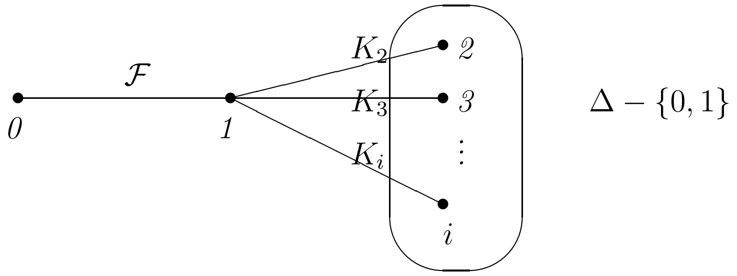
(3) The paths of Γ'. Let $(x_1, \dots x_k)$ be a path P of Γ. This path gives rise to a path P' of Γ' as follows:
(a) each x_i, $i = 1, \dots, k$, may be considered as an element of P';
(b) if $t(x_j) = 0$ or 1 and $t(x_{j+1}) = 1$ or 0, for some $j = 1, \dots, k-1$, then insert in P', between x_j and x_{j+1}, the 1-element $\{x_j, x_{j+1}\}$ of Γ'.

Conversely, to each path $P' = (x'_1, \dots, x'_k)$ of Γ' are associated two paths P_0 and P_1 as follows ($i = 1, \dots, k$):
(a') if $t(x'_i) \neq 1$, then x'_i is an element of P_0 and P_1;
(b') if $t(x'_i) = 1$, i.e. if $x'_i = \{x_{i_1}, x_{i_2}\}$ is a flag of type $\{0,1\}$ in Γ, then one of the following cases occurs:
(*) either $i \neq 1$ and $x'_{i-1} \in \{x_{i_1}, x_{i_2}\}$
or $i \neq k$ and $x'_{i+1} \in \{x_{i_1}, x_{i_2}\}$, then x'_i is deleted in P_0 and P_1;
(**) both $t(x'_{i-1}) \neq 0$ when $i = 1$ and $t(x'_{i+1}) \neq 0$ when $i = k$, then x'_i is replaced by the element of type 0 (resp. 1) of $\{x_{i_1}, x_{i_2}\}$ in P_0 (resp. P_1).

Now, the connectedness of Γ becomes obviously equivalent to that of Γ'. The same holds for residual connectedness, by a straightforward application of (2).
Let us prove finally that Γ' is strongly connected in and only if Γ is. By construction 6.1, the truncations Γ_{ij} and Γ'_{ij} coincide for distinct $i, j \notin \{0,1\}$. Moreover, the connectedness of Γ_{01} and Γ'_{01} are clearly equivalent if we consider the associated paths defined above. It remains to consider $\{0,j\}$- and $\{1,j\}$-truncations, for $j \notin \{0,1\}$. For any path P contained in Γ_{0j} or Γ_{1j}, the associated path P' (coinciding with P) is contained in Γ'_{0j}; the connectedness of Γ'_{0j} follows thus from that of Γ_{0j} and Γ_{1j}. Conversely, any path P' in Γ'_{1j} gives rise to paths P_0 and P_1 in Γ_{0j} and Γ_{1j}; the connectedness of the latter truncations follows then from that of Γ'_{1j}. It remains to prove that the connectedness of Γ'_{1j} follows from the strong connectedness of Γ. Let x, y be distinct 1-elements of Γ': we must exhibit a path of Γ'_{1j} joining x to y. We know already that a path $x = s_0, t_1, s_1, t_2, \dots, t_n, s_n = y$ exists, where the t_i are 0-elements and the s_i are 1-elements of Γ'. By (2), $Res(t_i)$ in Γ' is isomorphic with $Res(t_i)$ in Γ for any $i = 1, \dots, n$; hence, there is a path in $Res(t_i)$ joining s_{i-1} to s_i containing only 1- and j-elements of Γ'. Putting all these paths together, we join x to y as required. Now the proof of property (i) in theorem 6.1 can be completed straightforwardly by showing the connectedness of truncations in residues, thanks to (2).

(4) Automorphisms. Using (1), the first part of (iii) is straightforward. For the second part, consider the subgroup Σ of Ω fixing the partition $\{S_0, S_1\}$ of S'_0 elementwise (see construction 6.1). This completes the proof of theorem 6.1. ∎

Theorem 6.2. *(Converse of theorem 6.1). Let Γ' be a geometry of (possibly infinite) rank $n \geq 3$, satisfying (B_1) and (B_2) for the pair $\{0,1\}$ (see lemma 4.1) and admitting the diagram*



where $\mathcal{F}$ is a class of graphs, where the K_i are arbitrary classes of rank two geometries and where the diagram on $\Delta - \{0,1\}$ is arbitrary. Assume furthermore that the $\{0,1\}$-truncation of Γ' is a graph with no odd circuit (i.e. a bipartite graph). Then there is a geometry Γ satisfying the hypotheses of theorem 6.1 and such that $\Gamma' = \Gamma'(0,1)$ according to construction 6.1.

Before proving this theorem, we explicitly define the construction of the geometry Γ.

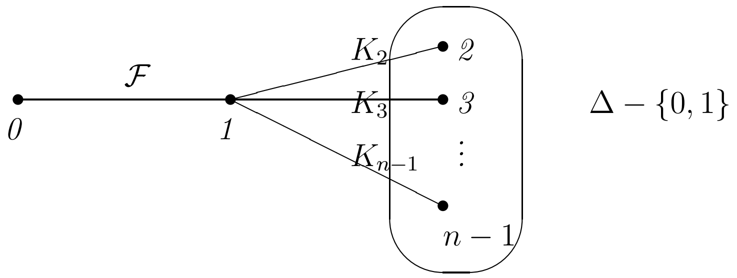
Construction 6.2. *Assume $\Gamma' = (\cup_{i=0}^{n-1} S'_i, t', I')$ is a geometry satisfying the hypotheses of theorem 6.2. Then the $\{0,1\}$-truncation is a bipartite graph with partition $\{X_0, X_1\}$ of the vertex set S'_0. We define a geometry $\Gamma = (\cup_{i=0}^{n-1} S_i, t, I)$ as follows:*
(a')

$$S_i = \begin{cases} S'_i & \text{for } i \notin \{0,1\} \\ X_i & \text{for } i \in \{0,1\} \end{cases}$$

(b') $t(S_i) = i$;
(c') given $x \in S_i$ and $y \in S_j$, with $i \neq j$:
- if $\{i,j\} \neq \{0,1\}$, then xIy if and only if $xI y'$;
- if $\{i,j\} = \{0,1\}$, then xIy if and only if $x \sim y$ in Γ'.

Proof of theorem 6.2: It is straightforward to check that Γ is actually a geometry. By construction, the (rank two) residue of any flag F of cotype $\{0,1\}$ in Γ' is the flag geometry $\mathcal{F}(G)$ of the residue G of F in Γ. Now it is easy to show that Γ has the diagram required in theorem 6.1: apply (B_1), lemma 4.1 and conditions (A_2), (A_3) in it to the residues of type $\Delta - \{0,1\}$ and $\{0,i\}$, $\{1,i\}$, for $i \in \Delta - \{0,1\}$. Finally, compare constructions 6.1 and 6.2 to conclude that Γ' can be obtained from Γ by construction 6.1. ■

Corollary 6.1. *Let Γ' be a residually connected geometry with finite rank $n \geq 3$, satisfying the intersection property (or its weaker version (I_{01}) mentioned in section 4) and with diagram*



where $\mathcal{F}$ is a class of graphs with no odd circuit, where the K_i are arbitrary classes of rank two geometries and where the diagram on $\Delta - \{0,1\}$ is arbitrary. Then there is a geometry Γ satisfying the hypotheses of theorem 6.1 such that Γ' can be obtained as $\Gamma'(0,1)$ from Γ by construction 6.1.

Proof: Conditions (B_1) and (B_2) follow from lemmas 4.1, 4.3, 4.4. Hence, the $\{0,1\}$-truncation T of Γ' is a graph and theorem 6.2 applies to prove this corollary if we show that T has no odd circuit (i.e. is bipartite).
Assume R_1, R_2 are two residues with type $\{0,1\}$ of flags F_1, F_2 in Γ' such that they contain some common 1-element e. Since R_1 and R_2 are bipartite by hypothesis, $R_1 \cup R_2$ must be bipartite by (B_2).
The proof is completed by using the residual connectedness property of Γ'. ■

7 Applications to polytopes

In this section, we apply our main result (corollary 4.1) and our extension of Neumaier's theorem (theorem 6.1) to some well known geometries related to polytopes. The list of possible applications is not exhaustive. The interested reader can refer to Grünbaum [8] for further examples to which our constructions apply.

Corollary 4.1 is used here for geometries whose $\{0,1\}$-residues are n-gons, with lines of two points. Moreover, in view of lemma 3.2 and theorem 4.1(i), n will be choosen odd. As shown in table 3 below, there may be different ways to choose the vertices 0 and 1 in certain diagrams, providing different resulting geometries. The latter are good candidates for applying theorem 6.1, getting again other geometries. For interesting results, it is good to keep lemma 6.5 in mind and choose those cases where the $\{0,1\}$-residues have their parameters s and t equal. Finally, note that no new application of corollary 4.1 is possible to certain geometries in column $\tilde{\Gamma}'$ of table 3, since axiom (I_{01}) does not hold for the two rightmost vertices. Remark that all geometries appearing in table 3 are thin geometries, i.e. every residue of corank 1 is of cardinality two. Thus the orders appearing on the diagrams are always equal to one. This is why we decide not to write the orders on the diagrams of this table.

Another family of thin geometries on which corollary 4.1 may be applied several times is the family of inductively minimal geometries [5]. Roughly speaking, these are rank $n-1$ geometries with a connected diagram on which a symmetric group $Sym(n)$ acts flag-transitively.
In $Sym(4)$, there are two rank 3 inductively minimal geometries. Their diagrams are given in [6] as geometries number 1 and 2. Clearly, the second one is obtained from the first one by applying corollary 4.1. So, given the first geometry, we can reconstruct all these inductively minimal geometries.
In $Sym(5)$, there are three rank 4 inductively minimal geometries, namely geometries 1, 2 and 3 in [6]. Again, the second (resp. third) one is obtained from the first (resp. second) one by applying corollary 4.1. So again, given the first geometry, we can reconstruct all these inductively minimal geometries.
In $Sym(6)$, there are six rank 5 inductively minimal geometries. Their diagrams are given in [6] as geometries number 1 to 6. Applying corollary 4.1 to these geometries yields that geometry number 3 is obtained from geometry 1, geometry 4 from geometry 3, geometry 5 from geometry 2 and geometry 6 from geometry 5. So here

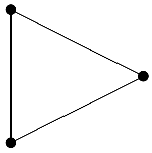
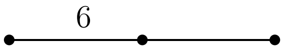
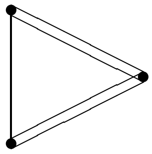
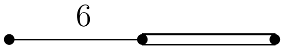
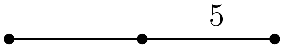
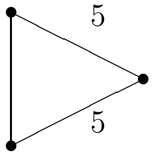
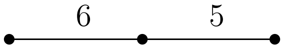
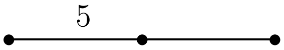
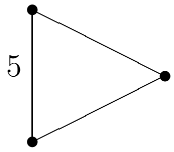
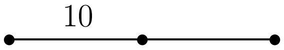
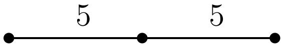
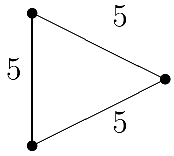
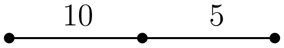
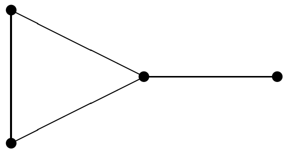
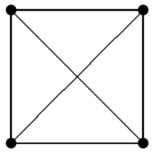
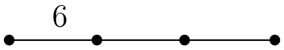
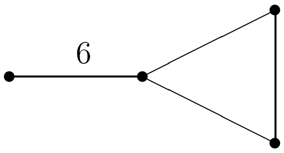
Polytope	Associated geometry Γ	$\tilde{\Gamma}$ (using corollary 4.1)	$\tilde{\Gamma}'$ (using theorem 6.1)
Tetrahedron			
Octahedron			
Icosahedron			
Dodecahedron			
Small stellated dodecahedron			
Hypertetrahedron		 using corollary 4.1 again 	 using corollary 4.1 again  using theorem 6.1 again 

Table 3: Some applications to thin geometries related to polytopes.

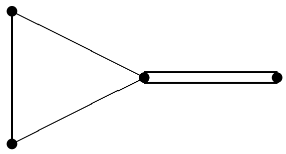
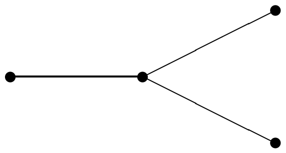
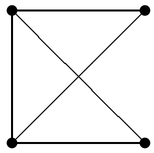
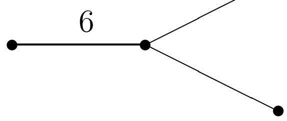
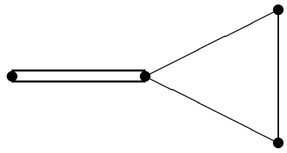
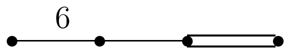
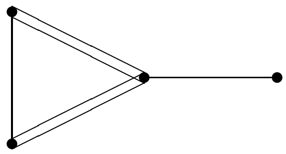
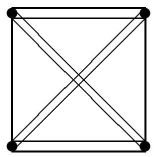
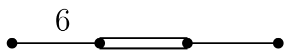
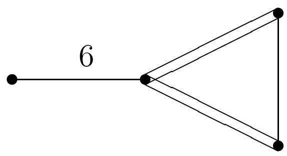
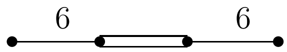
Polytope	Associated geometry Γ	$\tilde{\Gamma}$ (using corollary 4.1)	$\tilde{\Gamma}'$ (using theorem 6.1)
Hyperoctahedron			
Coloured hyperoctahedron			 and  using theorem 6.1 again (on both) 
24-cells		 using corollary 4.1 again 	 using corollary 4.1 again  using theorem 6.1 again 

Table 3: Some applications to thin geometries related to polytopes (continued).

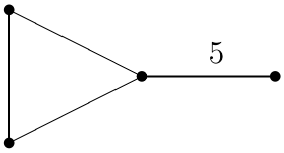
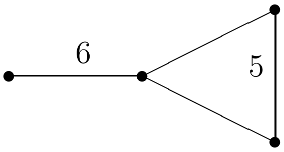
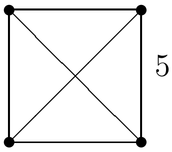
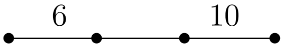
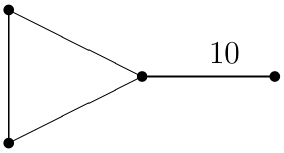
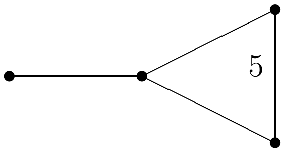
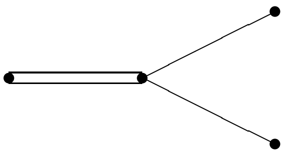
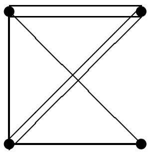
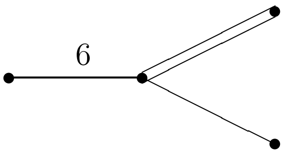
Polytope	Associated geometry Γ	$\tilde{\Gamma}$ (using corollary 4.1)	$\tilde{\Gamma}'$ (using theorem 6.1)
600-cells			
		using corollary 4.1 again	using corollary 4.1 again
			
		using theo 6.1	using theorem 6.1
			
		using theo 6.1	using theorem 6.1
			
		using corollary 4.1 again	using corollary 4.1 again
120-cells			
Coloured cubic tiling			

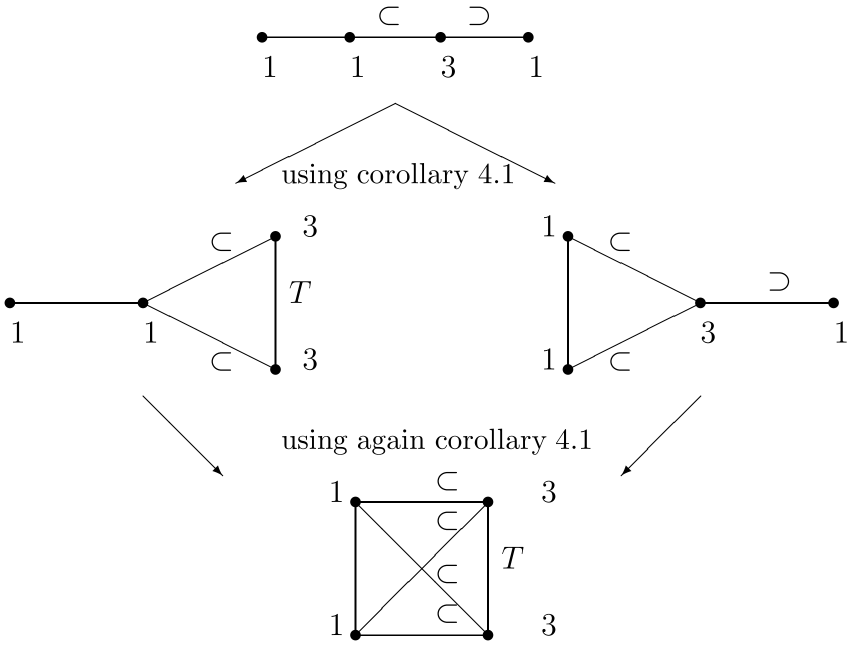
Table 3: Some applications to thin geometries related to polytopes (continued).

we need two of the six geometries to reconstruct all of them with our construction. In $Sym(7)$, there are eleven rank 6 inductively minimal geometries. Their diagrams are given in [6] as geometries number 1 to 11. Applying corollary 4.1 to these geometries yields that geometry number 2 is obtained from geometry 1, geometry 4 from geometry 2, geometries 5 and 6 from geometry 3, geometry 7 from geometry 6, geometry 9 from geometry 8, geometry 10 from geometry 9, and geometry 11 from geometry 10. So, given only geometry 1, 3 and 8, we can reconstruct all these inductively minimal geometries using corollary 4.1. A good question is: "how many inductively minimal geometries of rank n are needed to reconstruct all rank n inductively minimal geometries?" We just gave in the preceding discussion that answer for n up to 6.

8 Applications to geometries related to finite groups

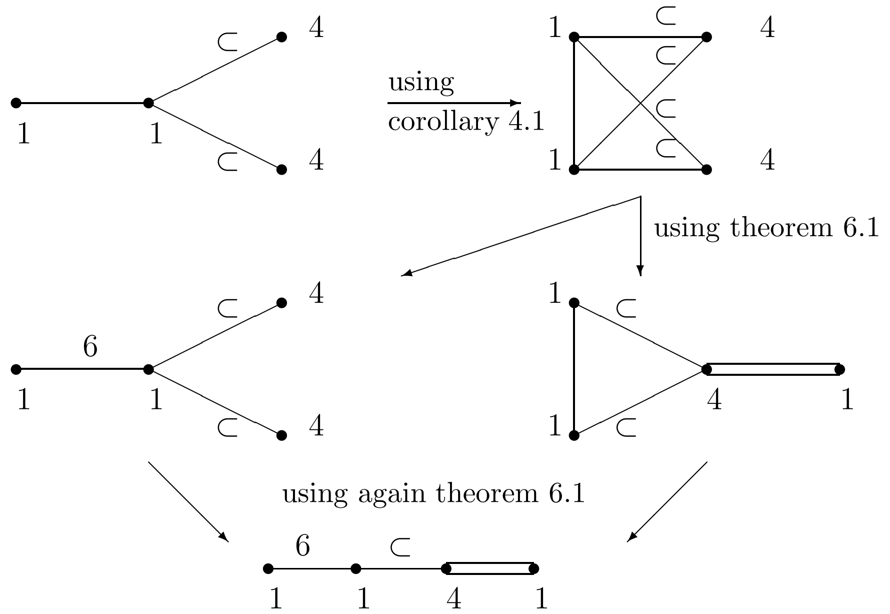
Of course, many new geometries can be obtained with our constructions. As geometries for the sporadic groups are of great interest, we go through the list of geometries for sporadic groups given in [3] and in table 4 we list those on which we can apply corollary 4.1. This table is organized as follows. In the first column, we give the name of the sporadic group for which a new (residually connected) geometry appears by using corollary 4.1, and on which the group acts chamber-transitively. In the second column, we give a list of numbers that are the numbers of the geometries as they appear in [3]. Sometimes we add a number n into parenthesis. This means that corollary 4.1 can be applied several times to get n new geometries. For instance, starting from geometry (27) of [3], we get the following three new geometries on which M_{11} acts chamber-transitively.

(27) M_{11}

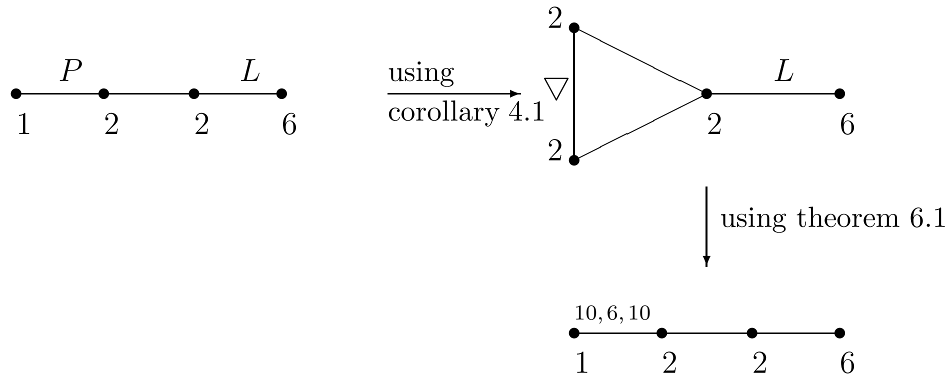


Each of the 41 new geometries listed in table 4 leads again to (at least one) new geometry by applying theorem 6.1 (more than one in the cases number 27, 49, 52, 77 and 91). Indeed, the initial group acts not necessarily chamber-transitively now. Let us detail, for instance, the cases McL (geometry number 52) and J_4 (geometry number 92).

(52) McL



(92) J_4



(P = Petersen graph and $\bigtriangledown$ = Desargues configuration, see tables 1 and 2 respectively).

Group	Geometry number as appearing in [3]
M_{11}	4, 27 (3), 89
M_{12}	5, 32
M_{22}	43, 77 (2), 90
M_{23}	44
M_{24}	45, 91 (3)
J_1	28, 34
J_2	94, 104
J_3	54
J_4	92
HS	49 (2)
McL	22, 52
LyS	10
Ru	98
Sz	6, 12, 21
Co_1	7, 13
Co_2	73
Co_3	23
Fi_{22}	46
Fi_{23}	47
Fi_{24}	24, 48
$BM = F_2$	74
$M = F_1 = FG$	8, 14

Table 4: New geometries for sporadic groups

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Université Libre de Bruxelles
Département de Mathématiques - C.P.216
Boulevard du Triomphe
B-1050 Bruxelles