Harmonic multivector fields and the Cauchy integral decomposition in Clifford analysis

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Abstract

In this paper we study the problem of decomposing a Hölder continuous k-grade multivector field F_k on the boundary Γ of an open bounded subset Ω in Euclidean space $\mathbb{R}^n$ into a sum $F_k = F_k^+ + F_k^-$ of harmonic k-grade multivector fields $F_k^\pm$ in $\Omega_+ = \Omega$ and $\Omega_- = \mathbb{R}^n \setminus (\Omega \cup \Gamma)$ respectively. The necessary and sufficient conditions upon F_k we thus obtain complement those proved by Dyn'kin in [20,21] in the case where F_k is a continuous k-form on Γ. Being obtained within the framework of Clifford analysis and hence being of a pure function theoretic nature, they once more illustrate the importance of the interplay between Clifford analysis and classical real harmonic analysis.

1 Introduction

As is well known, a k-vector in $\mathbb{R}^n$ can be interpreted as a directed k-dimensional volume. Such entities were first considered by H. Grassmann in the second half of the 19-th century. He thus created an algebraic structure which is now commonly known as the exterior algebra (see [10]). At about the same time, Sir W. Hamilton invented his quaternion algebra which a. o. enabled him to represent rotations in three dimensional space. In his 1878-paper, W. K. Clifford united both systems into

*This research was supported by the FWO Research Network WO. 003. 01N
†This research was supported by the FWO"Krediet aan Navorsers: 1.5.106.02"
Received by the editors February 2003.
Communicated by H. Van Maldeghem.
1991 *Mathematics Subject Classification :* 30G35, 45B20.
Key words and phrases : Clifford analysis, Multivector fields, Cauchy transform.

a single geometric algebra named after him (see [11]).
Clifford analysis, a function theory associated with the Dirac operator, has become an autonomous mathematical discipline since the 1980's. It comprehends several other analytic theories that have been developed for solving problems in higher dimensional space. For more detailed information, we refer the reader to [7,23,24,26,29,30] and the many references therein.
For a survey on recent research in Clifford analysis and its applications, we refer to [6,31,33,37]. In a series of papers (see e.g. [13,18,19,32,36]) one can find historical notes on Clifford analysis.
It was shown in [34] how Clifford analysis can be used to describe boundary values of harmonic fields which are in one-to-one correspondence with a subclass of Clifford algebra valued functions. In [21] E. Dyn'kin studied the following problem: Given an open bounded domain Ω of $\mathbb{R}^n$ with C_1-boundary Γ and a continuous k-form ω on Γ, under which conditions can one represent ω as a sum $\omega = \omega_+ + \omega_-$, where the forms $\omega_\pm$ are harmonic inside Ω and outside Ω, respectively? The proof of the necessary and sufficient conditions given in [21] Theorem 2, is essentially based on an asymptotically harmonic extension of ω to the whole of $\mathbb{R}^n$. In [20], the case of harmonic vector fields, i.e. the case $k = 1$, was already dealt with.
As harmonic k-forms and monogenic k-grade multivector fields are intimately related to each other (see §2.2), the previous problem formulated by Dyn'kin for k-forms may as well be outlined for k-grade multivector fields. This idea suggests the problem to be posed within the framework of Clifford analysis. It is exactly this view-point which underlies the writing out of the present paper.

2 Preliminaries

We thought it to be helpful to recall some well known, though not necessarily familiar, basic properties in Clifford algebras and Clifford analysis such as: geometric properties resulting from multiplication in Clifford algebras (§2.1); the equivalence between the (d, d^*) Hodge-de Rham system for k-forms and the monogenicity of k-grade multivector fields (§2.2); results about the Cauchy transform in Clifford analysis (§2.3).

2.1 Clifford algebras and multivectors

Let $\mathbb{R}^{0,n}$ $(n \in \mathbb{N})$ be the real vector space $\mathbb{R}^n$ endowed with a non-degenerate quadratic form of signature $(0, n)$ and let $(e_j)_{j=1}^n$ be a corresponding orthogonal basis for $\mathbb{R}^{0,n}$. Then $\mathbb{R}_{0,n}$, the universal Clifford algebra over $\mathbb{R}^{0,n}$, is a real linear associative algebra with identity such that the elements e_j, $j = 1, \dots, n$, satisfy the basic multiplication rules

$$e_j^2 = -1,\ j = 1, \dots, n;$$

$$e_i e_j + e_j e_i = 0,\ i \neq j.$$

For $A = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, put $e_A = e_{i_1} e_{i_2} \cdots e_{i_k}$, while for $A = \emptyset$, $e_\emptyset = 1$ (the identity element in $\mathbb{R}_{0,n}$). Then $(e_A : A \subset$

$\{1,\ldots,n\}$) is a basis for $\mathbb{R}_{0,n}$. For $1 \leq k \leq n$ fixed, the space $\mathbb{R}_{0,n}^{(k)}$ of k vectors or k-grade multivectors in $\mathbb{R}_{0,n}$, is defined by

$$\mathbb{R}_{0,n}^{(k)} = \mathrm{span}_{\mathbb{R}}(e_A : \ |A| = k).$$

Clearly

$$\mathbb{R}_{0,n} = \sum_{k=0}^{n} \oplus \mathbb{R}_{0,n}^{(k)}.$$

Any element $a \in \mathbb{R}_{0,n}$ may thus be written in a unique way as

$$a = [a]_0 + [a]_1 + \cdots + [a]_n$$

where $[\,]_k : \mathbb{R}_{0,n} \longrightarrow \mathbb{R}_{0,n}^{(k)}$ denotes the projection of $\mathbb{R}_{0,n}$ onto $\mathbb{R}_{0,n}^{k}$.
It is customary to identify $\mathbb{R}$ with $\mathbb{R}_{0,n}^{(0)} = \mathbb{R}1$, the so-called set of scalars in $\mathbb{R}_{0,n}$, and $\mathbb{R}^n$ with $\mathbb{R}_{0,n}^{(1)} \cong \mathbb{R}^{0,n}$, the so-called set of vectors in $\mathbb{R}_{0,n}$. The elements of $\mathbb{R}_{0,n}^{(2)}$ are also called bivectors.
Notice that for any two vectors x and y, their product is given by

$$xy = x \bullet y + x \wedge y$$

where

$$x \bullet y = \frac{1}{2}(xy + yx) = -\sum_{j=1}^{n} x_j y_j$$

is -up to a minus sign- the standard inner product between x and y, while

$$x \wedge y = \frac{1}{2}(xy - yx) = \sum_{i<j} e_i e_j (x_i y_j - x_j y_i)$$

represents the standard outer product between them.
More generally, for a 1-vector x and a k-vector Y_k, their product xY_k splits into a $(k-1)$-vector and a $(k+1)$-vector, namely:

$$xY_k = [xY_k]_{k-1} + [xY_k]_{k+1},$$

where

$$[xY_k]_{k-1} = \frac{1}{2}(xY_k - (-1)^k Y_k x)$$

and

$$[xY_k]_{k+1} = \frac{1}{2}(xY_k + (-1)^k Y_k x).$$

The inner and outer products between x and Y_k are then defined by

$$x \bullet Y_k = [xY_k]_{k-1} \text{ and } x \wedge Y_k = [xY_k]_{k+1}. \tag{1}$$

For further properties concerning inner and outer products between multivectors, we refer to [25].
Finally we recall the definition of the conjugation $a \to \overline{a}$ and the norm $|.|$ on $\mathbb{R}_{0,n}$.
For each $j = 1, \ldots, n$,

$$\overline{e_j} = -e_j$$

while for $a,\ b \in \mathbb{R}_{0,n}$,

$$\overline{(ab)} = \bar{b}\,\bar{a}.$$

Notice that for any basic element e_A with $|A| = k$,

$$\overline{e_A} = (-1)^{\frac{k(k+1)}{2}} e_A.$$

For $a,\ b \in \mathbb{R}_{0,n}$, we put

$$(a,\ b) = [a\bar{b}]_0 = \sum_A a_A b_A.$$

An inner product is thus obtained, leading to the norm $|.|$ given by

$$|a|^2 = [a\bar{a}]_0 = \sum_A a_A^2.$$

2.2 Clifford analysis and harmonic multivector fields

Let $\Omega \subset \mathbb{R}^n$ be open and let f be an $\mathbb{R}_{0,n}$-valued function in Ω, say

$$f(x) = \sum_A f_A(x) e_A,\ x \in \Omega,$$

all f_A thus being real valued .
Such a function is said to belong to some classical class of functions on Ω if each of its components belongs to that class.
Furthermore, let D be the Dirac operator in $\mathbb{R}^n$:

$$D = \sum_{j=1}^{n} e_j \partial_{x_j}.$$

Then

$$D^2 = -\Delta,$$

Δ being the Laplacian in $\mathbb{R}^n$.
For $f \in C_1(\Omega)$, we define the left and right action of D on f by

$$Df = \sum_{j,A} e_j e_A \frac{\partial f_A}{\partial x_j}$$

and

$$fD = \sum_{j,A} e_A e_j \frac{\partial f_A}{\partial x_j}.$$

We say that f is left (resp. right) monogenic in Ω if $Df = 0$ (resp. $fD = 0$) in Ω.
Notice in particular that if $f = F$ is scalar valued, then DF is the vector valued function

$$DF = \sum_{j=1}^{n} e_j \frac{\partial F}{\partial x_j}.$$

The action of D on F may thus be identified with the action of the gradient ∇ on F, i.e. DF may be identified with the classical vector field

$$\nabla F = (\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}).$$

If $f = F$ is vector valued, i.e.

$$F = \sum_{i=1}^{n} e_i F_i,$$

then the action of D on F is given by

$$DF = -\sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} + \sum_{i<j} e_i e_j \left(\frac{\partial F_j}{\partial x_i} - \frac{\partial F_i}{\partial x_j}\right),$$

i.e., using inner and outer products,

$$DF = D \bullet F + D \wedge F.$$

Clearly, identifying F with the vector field $F' = (F_1, \dots, F_n)$, we obtain that

$$D \bullet F = -\mathrm{div} F'$$

and

$$D \wedge F = \mathrm{curl} F'.$$

Consequently, a vector valued function F is left monogenic if and only if F' satisfies the system

$$\begin{cases} \mathrm{div} F' = 0 \\ \mathrm{curl} F' = 0, \end{cases} \tag{2}$$

known as the Riesz system in $\mathbb{R}^n$ (see [38]).
Notice that a vector field F' satisfying (2) in Ω was called by Stein-Weiss a system of conjugate harmonic functions in Ω. As is well known, if Ω is simply connected, a vector field F' satisfies (2) if and only if F' is the gradient of a real valued harmonic function H in Ω, i.e. $F' = \nabla H$.
Finally, if $f = F_k$ is k-vector valued, i.e.

$$F_k = \sum_{|A|=k} e_A F_{k,A}\,,$$

then, by using the inner and outer products (1), the action of D on F_k is given by

$$DF_k = D \bullet F_k + D \wedge F_k, \tag{3}$$

where

$$D \bullet F_k = \frac{1}{2}(DF_k - (-1)^k F_k D) = [DF_k]_{k-1}$$

and

$$D \wedge F_k = \frac{1}{2}(DF_k + (-1)^k F_k D) = [DF_k]_{k+1}.$$

As $\overline{DF_k} = \overline{F_k}\;\overline{D}$ with $\overline{D} = -D$ and $\overline{F_k} = (-1)^{\frac{k(k+1)}{2}} F_k$, it follows that if F_k is left monogenic, then it is right monogenic as well.
Furthermore, in view of (3), F_k is left monogenic in Ω if and only if F_k satisfies in Ω the system of equations

$$\begin{cases} D \bullet F_k = 0 \\ D \wedge F_k = 0 \end{cases} \tag{4}$$

A k-vector valued function F_k satisfying (4) in Ω is called a *harmonic k-grade multivector field* in Ω.
Now identify the k-grade multivector field F_k in Ω, where

$$F_k = \sum_{|A|=k} F_{k,A}\, e_A,$$

with the k-form ω_k in Ω, where

$$\omega_k = \sum_{|A|=k} F_{k,A}\, dx_A.$$

Hereby, as usual, for $A = \{i_1, \ldots, i_k\}$,

$$dx_A = dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Furthermore, consider the Hodge-de Rham (d, d^*)-operators, where

$$d = \sum_{k=1}^{n} \mu_k \frac{\partial}{\partial x_k}\ ,\ d^* = \sum_{k=1}^{n} \mu_k^* \frac{\partial}{\partial x_k}$$

with

$$\mu_k(1) = dx_k\ ,\ \mu_k(dx_i) = dx_k \wedge dx_i\ ,\ i = 1, \ldots, n$$

and

$$\mu_k^*(dx_i) = \delta_{ki}\ ,\ \mu_k^*(dx_i \wedge dx_j) = \delta_{ki} dx_j - \delta_{kj} dx_i.$$

Then a straightforward calculation shows that for the k-vector field F_k and its corresponding k-form ω_k, $DF_k = 0$ in Ω if and only if $(d - d^*)\omega_k = 0$ in Ω, i.e. the k-grade multivector field F_k is harmonic in Ω if and only if ω_k satisfies in Ω the Hodge-de Rham system

$$\begin{cases} d\omega_k = 0 \\ d^*\omega_k = 0 \end{cases} \tag{5}$$

In other words, after the identification between F_k and ω_k, the systems (4) and (5) are equivalent.
As is well known, k-forms ω_k satisfying (5) in Ω are called harmonic in Ω.
Obviously, the systems (4) or (5) reduce to the Riesz system (2) in the case $k = 1$.
Solutions to one of these three equivalent systems are called *harmonic vector fields.*
For a more detailed discussion concerning the Hodge-de Rham (d, d^*)-operators and the standard Dirac operator D in $\mathbb{R}^n$, we refer to [23].
The following lemma will be much useful in the proof of our main result in section 4.

Lemma 2.1. *Let u be an $\mathbb{R}_{0,n}$-valued C_1-function in Ω admitting the decomposition*

$$u = \sum_{k=0}^{n} [u]_k.$$

Then u is both left and right monogenic in Ω if and only if for each $k = 0, 1, \ldots, n$, $[u]_k$ is a harmonic k-grade multivector field in Ω.

An important example of a function which is both left and right monogenic is the fundamental solution of the Dirac operator, given by

$$e(x) = \frac{1}{A_n}\frac{\overline{x}}{|x|^n}\,,\ x \in \mathbb{R}^n \setminus \{0\}.$$

Hereby A_n stands for the surface area of the unit sphere in $\mathbb{R}^n$.
In the sense of distributions we have:

$$De(x-y) = e(x-y)D = \delta_x(y),$$

$\delta_x(y)$ being the generalized function in $\mathbb{R}^n$ such that for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$< \delta_x(y), \varphi(y) >= \varphi(x).$$

The function $e(x-y)$ plays the same role in Clifford analysis as the Cauchy kernel does in complex analysis. For this reason it is called the Cauchy kernel in $\mathbb{R}^n$. For a more general notion of the Cauchy kernel related to the Dirac operator on a manifold, we refer to [12].

2.3 Cauchy's Integral Formula

Throughout this paper, surface integration will be with respect to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ in $\mathbb{R}^n$. This measure is defined in terms of the diameters of various efficient coverings and it agrees with ordinary "$(n-1)$-dimensional surface area" on nice surfaces (see [22,27]).
It is well known that some basic results in Clifford analysis rely heavily on Stokes' formula, which is usually stated on domains with a sufficiently smooth boundary. The validity of this formula when the boundary is geometrically very complicated is not at all obvious. Research on the problem of finding the most general form of Stokes' formula has much contributed to the development of Geometric Measure Theory. The concept of the exterior normal ν as defined in [22] was crucial in establishing the following version of Stokes' formula:

$$\int_\Gamma \varphi(x) \bullet \nu(x) d\mathcal{H}^{n-1}(x) = \int_\Omega \mathrm{div}\varphi(x) d\mathcal{L}^n(x).$$

Hereby Ω is an open subset of $\mathbb{R}^n$ with boundary Γ such that $\mathcal{H}^{n-1}(\Gamma) < \infty$, $\varphi \in C_1(\Omega \cup \Gamma)$ and $\mathcal{L}^n$ denotes Lebesgue measure in $\mathbb{R}^n$.
Using Stokes' formula, basic integral formulae in Clifford analysis may thus be obtained (see e.g. [7]). For the sake of completeness, we here recall Cauchy's Integral Formula.
In what follows we suppose that:

(i) Ω is a bounded open domain in $\mathbb{R}^n$ with boundary Γ such that $\mathcal{H}^{n-1}(\Gamma) < \infty$;

(ii) $\Omega_+ = \Omega$, $\Omega_- = \mathbb{R}^n \setminus (\Omega \cup \Gamma)$

(iii) u is an $\mathbb{R}_{0,n}$-valued function

(iv) $\nu(x)$ is the vector valued exterior normal at $x \in \Gamma$.

Notice that multiplications appearing in the formulae below are performed in $\mathbb{R}_{0,n}$.

Theorem 2.1 (Cauchy's Integral Formula). *(i) Let u belong to $C_1(\Omega) \cap C(\overline{\Omega})$. If u is left monogenic in Ω, then for $x \in \Omega$,*

$$u(x) = \int_\Gamma e(y-x)\nu(y)u(y)d\mathcal{H}^{n-1}(y)$$

(ii) Let u belong to $C_1(\Omega_-)$ with $u(\infty) = 0$ and suppose that u is continuously extendable to Γ. If u is left monogenic in Ω, then for $x \in \Omega_-$,

$$u(x) = -\int_\Gamma e(y-x)\nu(y)u(y)d\mathcal{H}^{n-1}(y)$$

Cauchy's Integral Formula applied to a k-grade harmonic multivector field F_k in Ω which is continuously extendable to Γ implies that

$$0 = \int_\Gamma e(y-x) \bullet (\nu(y) \bullet F_k(y))d\mathcal{H}^{n-1}(y);$$

$$F_k(x) = \int_\Gamma \{e(y-x) \wedge (\nu(y) \bullet F_k(y)) + e(y-x) \bullet (\nu(y) \wedge F_k(y))\}d\mathcal{H}^{n-1}(y);$$

$$0 = \int_\Gamma e(y-x) \wedge (\nu(y) \wedge F_k(y))d\mathcal{H}^{n-1}(y). \tag{6}$$

Of course, the application of the Cauchy Integral Formula to the case of a k-grade harmonic multivector field F_k in Ω_-, which is continuously extendable to Γ and is such that $F_k(\infty) = 0$, leads to relations similar to those in (6).
We end up this subsection with the following results.

Theorem 2.2 (Painlevé). *Let u, left monogenic in $\Omega_+ \cup \Omega_-$, be continuously extendable to Γ. Then u is left monogenic in $\mathbb{R}^n$.*

Theorem 2.3 (Liouville). *Let u, left monogenic in $\mathbb{R}^n$, be bounded in $\mathbb{R}^n$. Then u is a constant function.*

3 Cauchy transforms on AD-regular surfaces

In this section we state some important results related to the Cauchy transform $\mathcal{C}$ on Γ:

$$(\mathcal{C}u)(x) = \int_\Gamma e(y-x)\nu(y)u(y)d\mathcal{H}^{n-1}(y)\,,\ x \notin \Gamma,$$

and its singular version, the singular Cauchy transform $\mathcal{S}$ (also called the Hilbert transform) on Γ:

$$(\mathcal{S}u)(x) = 2\int_\Gamma e(y-x)\nu(y)(u(y)-u(x))d\mathcal{H}^{n-1}(y) + u(x)\,,\ x \in \Gamma,$$

the integral being taken in the sense of the principal value.

3.1 Ahlfors-David regular surfaces

We will say that the set $\mathbf{E}$ in $\mathbb{R}^n$ is an $(n-1)$-set if $\mathcal{H}^{n-1}(\mathbf{E}) < +\infty$, where as above $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.
The geometric condition $\mathcal{H}^{n-1}(\mathbf{E}) < +\infty$ represents a natural condition without any quantitative estimates on the size of the set $\mathbf{E}$. Among $(n-1)$-sets, the rectifiable sets of H. Federer (see [22]) form essentially the largest class where many basic properties of smooth surfaces have reasonable analogues such as, for example, the existence of tangent planes (defined in a measure-theoretic approximate way); parametrization by Lipschitz maps, and an analogue of Lebesgue's density point theorem. All these properties are qualitative, without any estimates.
A curve γ in the complex plane such that $\mathcal{H}^1(\gamma) < +\infty$ can be parametrized nicely by a Lipschitz function. For $(n-1)$-dimensional surfaces $(n > 2)$ one can not, in general, find such a nice parametrization.

Definition 3.1. *A closed set $\mathbf{E}$ in $\mathbb{R}^n$ is said to be Ahlfors-David regular (AD-regular) with dimension $n-1$ if there exists a constant $c > 0$ such that*

$$c^{-1}r^{n-1} \leq \mathcal{H}^{n-1}(\mathbf{E} \cap B(x,r)) \leq c\, r^{n-1}, \tag{7}$$

for all $x \in \mathbf{E}$ and $r > 0$, where $B(x,r)$ stands for the closed ball with center x and radius r

The requirement that the set $\mathbf{E}$ is AD-regular can be viewed as a quantitative version of the property of having positive and finite upper and lower densities with respect to $\mathcal{H}^{n-1}$. For further information concerning AD-regular sets, the reader is referred to [15,16,17,28].
The AD-regular curves in the plane are closely related with the boundedness of the singular integral operator

$$f(x) \to 2\int_{\gamma} e(y-x)\nu(y)f(y)d\mathcal{H}^1(y),$$

acting on $L^p(\gamma)$.
Based on the works of Calderón [9], Coifman, McIntosh and Meyer [14], David [15] proved that the $L^p(\gamma)$ - boundedness of this singular integral operator holds if and only if γ is an AD-regular curve (in fact, only the inequality $\mathcal{H}^1(\gamma \cap B(x,r)) \leq c\, r$ is essential, since in this case the lower bound is clear). For $(n-1)$-dimensional surfaces, $n > 2$, such a simple characterization seems impossible.
In [17] the authors studied singular integral operators on general finite dimensional AD-regular sets. They showed that a large class of Calderón-Zygmund singular integral operators are bounded on $L^p(\mathbf{E})$, $1 < p < +\infty$, if and only if $\mathbf{E}$ is uniformly rectifiable.
For the sake of completeness, we now recall some recently obtained results concerning the boundedness of the Cauchy transform and its singular version on spaces of Hölder continuous functions on AD-regular surfaces. For their proofs, we refer the reader to [1,2,3,4,5]

3.2 Boundary values of Cauchy integrals on Hölder spaces

Let K be a compact set in $\mathbb{R}^n$ and let $u(x)$ be a continuous $\mathbb{R}_{0,n}$-valued function defined on K. The modulus of continuity of the function u is the non-negative function $w(u,t)$, $t>0$, defined by the formula

$$w(u,t) = \sup_{|x-y|\leq t} \{|u(x)-u(y)| : x,y \in K\}.$$

The Hölder space $C^{0,\alpha}(K)$, $0<\alpha<1$, consists of those functions $u \in C(K)$ for which

$$\|u\|_\alpha = \|u\|_{C(K)} + \sup_{0<t\leq\sigma} \frac{w(u,t)}{t^\alpha} < \infty.$$

Hereby $\sigma = \max_{x,y\in K} |x-y|$.
Provided with the norm $\|.\|_\alpha$, the space $C^{0,\alpha}(K)$ becomes a real Banach space.
In what follows, Ω is a bounded open subset of $\mathbb{R}^n$ such that its boundary Γ is an AD-regular surface.

Theorem 3.1. *The singular Cauchy transform $\mathcal{S}$ is bounded on $C^{0,\alpha}(\Gamma)$. Moreover $\mathcal{S}^2 = \mathcal{I}$, where $\mathcal{I}$ is the identity operator.*

Theorem 3.2. *Let $u \in C^{0,\alpha}(\Gamma)$, $0<\alpha<1$. Then*

(i) $\mathcal{C}u \in C^{0,\alpha}(\Omega_\pm \cup \Gamma)$ with $\mathcal{C}u(\infty) = 0$.

(ii) $\mathcal{C}u$ is left monogenic in $\mathbb{R}^n \setminus \Gamma$.

(iii) (Plemelj-Sokhotzki Formula) For all $z \in \Gamma$,

$$(\mathcal{C}^\pm u)(z) = \lim_{\Omega_\pm \ni x \to z} (\mathcal{C}u)(x) = \frac{1}{2}(\mathcal{S}u(z) \pm u(z)).$$

Theorem 3.2 implies that $u \in C^{0,\alpha}(\Gamma)$ can be represented as

$$u(z) = u^+(z) + u^-(z),\ z \in \Gamma, \tag{8}$$

where $u^+(z) = (\mathcal{C}^+u)(z)$ and $u^-(z) = -(\mathcal{C}^-u)(z)$.
Moreover, the functions $u^\pm$ have left monogenic extensions in $\Omega_\pm$, respectively, such that

$$u^\pm \in C^{0,\alpha}(\Omega_\pm \cup \Gamma),\ u^-(\infty) = 0.$$

Finally notice that by means of the Painlevé Theorem (see Theorem 2.2), the decomposition (8) is unique. It may thus be considered as a natural multidimensional analogue of the well known Cauchy integral decomposition for a Hölder continuous complex valued function on a closed curve in the plane.
Let us now consider the Cauchy transform acting on a k-grade multivector field F_k. It is easy to see that $\mathcal{C}F_k$ splits into

$$\mathcal{C}F_k = [\mathcal{C}F_k]_{k-2} + [\mathcal{C}F_k]_k + [\mathcal{C}F_k]_{k+2}, \tag{9}$$

where

$$[\mathcal{C}F_k(x)]_{k-2} = \int_\Gamma e(y-x) \bullet (\nu(y) \bullet F_k(y)) d\mathcal{H}^{n-1}(y);$$

$$[\mathcal{C}F_k(x)]_k = \int_\Gamma \{e(y-x) \wedge (\nu(y) \bullet F_k(y)) + e(y-x) \bullet (\nu(y) \wedge F_k(y))\} d\mathcal{H}^{n-1}(y);$$

$$[\mathcal{C}F_k(x)]_{k+2} = \int_\Gamma e(y-x) \wedge (\nu(y) \wedge F_k(y)) d\mathcal{H}^{n-1}(y).$$

In view of (9) and the expression of $\mathcal{S}$, we thus obtain by Theorem 3.2:

Theorem 3.3. *Let F_k be a Hölder continuous k-grade multivector field on Γ. Then for all $z \in \Gamma$:*

$$[\mathcal{C}^\pm F_k(z)]_{k-2} = \frac{1}{2} \int_\Gamma e(y-z) \bullet (\nu(y) \bullet (F_k(y) - F_k(z))) d\mathcal{H}^{n-1}(y);$$

$$[\mathcal{C}^\pm F_k(z)]_k = \frac{1}{2} \int_\Gamma e(y-z) \wedge (\nu(y) \bullet (F_k(y) - F_k(z))) d\mathcal{H}^{n-1}(y) +$$

$$+\frac{1}{2} \int_\Gamma e(y-z) \bullet (\nu(y) \wedge (F_k(y) - F_k(z))) d\mathcal{H}^{n-1}(y) + \frac{1}{2}(F_k(z) \pm F_k(z));$$

$$[\mathcal{C}^\pm F_k(z)]_{k+2} = \frac{1}{2} \int_\Gamma e(y-z) \wedge (\nu(y) \wedge (F_k(y) - F_k(z))) d\mathcal{H}^{n-1}(y).$$

4 Harmonic decomposition for Hölder continuous multivector fields

As we already pointed out in the introduction, E. Dyn'kin studied in [20,21] the problem of generalizing to Euclidean space $\mathbb{R}^n$ the classical Cauchy integral decomposition for continuous functions on the boundary γ of a bounded open domain in the complex plane. The framework he used was the theory of harmonic differential forms.

Now let Ω be a bounded open set in $\mathbb{R}^n$ such that its boundary Γ is an AD-regular surface.

As we have seen in (8), any $u \in C^{0,\alpha}(\Gamma)$, $0 < \alpha < 1$, admits a decomposition of the form

$$u(z) = u^+(z) + u^-(z),\ z \in \Gamma,$$

where $u^\pm$ have left monogenic extensions to $\Omega_\pm$, respectively.

It therefore seems natural to study Dyn'kin's problem directly within the framework of Clifford analysis.

We are thus led to consider the following problem: Under which conditions does a k-grade multivector field F_k belonging to $C^{0,\alpha}(\Gamma)$ admit the decomposition

$$F_k(z) = F_k^+(z) + F_k^-(z),\ z \in \Gamma, \tag{10}$$

where $F_k^{\pm}$ are harmonic multivector fields in $\Omega_{\pm}$, respectively, such that

$$F_k^{\pm} \in C^{0,\alpha}(\Omega_{\pm} \cup \Gamma),\ F_k^{-}(\infty) = 0?$$

The main result of this paper (Theorem 4.1) gives necessary and sufficient conditions upon F_k in terms of its Cauchy transform $\mathcal{C}F_k$. As such, these conditions are of a pure function theoretic nature, thus illustrating once more the powerfulness and elegance of Clifford analysis techniques in dealing with higher dimensional problems.

Theorem 4.1. *Let $F_k \in C^{0,\alpha}(\Gamma)$, $0 < \alpha < 1$. Then the following assertions are equivalent:*

i) The multivector field F_k admits on Γ a decomposition of the form (10)

ii) The Cauchy transform $\mathcal{C}F_k(x)$ is both left and right monogenic in $\mathbb{R}^n \setminus \Gamma$

iii) The k-grade multivector field $[\mathcal{C}F_k(x)]_k$ is harmonic in $\mathbb{R}^n \setminus \Gamma$

iv) The multivector fields $[\mathcal{C}F_k(x)]_{k-2}$ and $[\mathcal{C}F_k(x)]_{k+2}$ vanish in $\mathbb{R}^n$.

Proof: *i)* $\rightarrow$ *ii)*
Assume that

$$F_k(z) = F_k^{+}(z) + F_k^{-}(z),\ z \in \Gamma,$$

where the k-grade multivector fields $F_k^{\pm}$ are as in (10).
Then

$$\mathcal{C}F_k(x) = \mathcal{C}F_k^{+}(x) + \mathcal{C}F_k^{-}(x).$$

In view of the assumptions made on $F_k^{\pm}$, we have that $\mathcal{C}F_k^{+} = 0$ in Ω_-, $\mathcal{C}F_k^{-} = 0$ in Ω_+ and that $\mathcal{C}F_k^{\pm}(x) = F_k^{\pm}(x)$ for $x \in \Omega_{\pm}$.
Consequently

$$\mathcal{C}F_k(x) = \begin{cases} F_k^{+}(x), & x \in \Omega_+ \\ F_k^{-}(x), & x \in \Omega_- \end{cases}$$

But, as $F_k^{\pm}$ is a harmonic k-grade multivector field in $\Omega_{\pm}$, $\mathcal{C}F_k$ is as well left as right monogenic in $\mathbb{R}^n \setminus \Gamma$.
ii) $\rightarrow$ *iii)*
Let $\mathcal{C}F_k(x)$ be left and right monogenic in $\mathbb{R}^n \setminus \Gamma$ and consider its decomposition (9) in multivector fields

$$\mathcal{C}F_k = [\mathcal{C}F_k]_{k-2} + [\mathcal{C}F_k]_k + [\mathcal{C}F_k]_{k+2}. \tag{11}$$

Then, by Lemma 2.1, $[\mathcal{C}F_k]_l$, $l = k-2,\ k,\ k+2$, are harmonic l-grade multivector fields in $\mathbb{R}^n \setminus \Gamma$ and so is in particular $[\mathcal{C}F_k]_k$.
iii) $\rightarrow$ *iv)*
Consider again the decomposition (11) of $\mathcal{C}F_k$.
As $\mathcal{C}F_k$ itself and as moreover by assumption $[\mathcal{C}F_k]_k$ are left monogenic in $\mathbb{R}^n \setminus \Gamma$, we have that in $\mathbb{R}^n \setminus \Gamma$, by letting act D from the left on (11):

$$D([\mathcal{C}F_k]_{k-2}) + D([\mathcal{C}F_k]_{k+2}) = 0.$$

Furthermore, as $D([\mathcal{C}F_k]_{k-2})$ and $D([\mathcal{C}F_k]_{k+2})$ split into a $(k-3)$ and a $(k-1)$, respectively, into a $(k+1)$ and a $(k+3)$ multivector, we obtain by the uniqueness of the decomposition into multivectors, that in $\mathbb{R}^n \setminus \Gamma$:

$$D([\mathcal{C}F_k]_{k-2}) = 0 \text{ and } D([\mathcal{C}F_k]_{k+2}) = 0.$$

Consequently, the multivectors $[\mathcal{C}F_k]_{k-2}$ and $[\mathcal{C}F_k]_{k+2}$ are harmonic in $\mathbb{R}^n \setminus \Gamma$. Moreover, in view of Theorem 3.3, for all $z \in \Gamma$,

$$[\mathcal{C}^+F_k]_{k-2}(z) = [\mathcal{C}^-F_k]_{k-2}(z)$$

and

$$[\mathcal{C}^+F_k]_{k+2}(z) = [\mathcal{C}^-F_k]_{k+2}(z).$$

The Painlevé and Liouville Theorems 2.2 and 2.3 then imply that

$$[\mathcal{C}F_k]_{k-2} \equiv 0 \text{ and } [\mathcal{C}F_k]_{k+2} \equiv 0 \text{ in } \mathbb{R}^n.$$

iv) $\rightarrow$ *i)*
First notice that, as $F_k \in C^{0,\alpha}(\Gamma)$, then by means of the Plemelj-Sokhotzki formula, for $z \in \Gamma$

$$F_k(z) = \mathcal{C}^+F_k(z) - \mathcal{C}^-F_k(z). \tag{12}$$

In view of the assumption (iv) made, the functions $F_k^\pm$ defined in, respectively, $\Omega_\pm$ by $F_k^\pm = \pm\mathcal{C}F_k$ are obviously k-grade harmonic multivector fields which satisfy all required properties. ∎

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