INTERSECTIONS OF LINES AND CIRCLES

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Abstract. A method is presented for determining barycentric coordinates of points of intersection of a line and a circle. The method is applied specifically to the Euler line, the line of the circumcenter and incenter, the Brocard axis, and several circles, including the circumcircle, incircle, nine-point circle, and Brocard circle. The method also applies to intersections of certain pairs of lines, harmonic conjugate pairs, and to centers of similitude of pairs of circles.

1. Introduction. Let ABC be a triangle with vertices A, B, C, vertex angles A, B, C, sidelengths a, b, c, circumradius R, inradius r, area Δ, semiperimeter s, Brocard angle ω, and line $\mathcal{L}^{\infty}$ at infinity. Let $S_A = (b^2 + c^2 - a^2)/2$, so that $S_A = bc\cos A$, and define S_B and S_C defined cyclically. The circumcenter, incenter, orthocenter, nine-point center, centroid, and symmedian point are denoted by O, I, H, N, G, and K, respectively, and the notation $X(n)$ refers to points indexed in the *Encyclopedia of Triangle Centers - ETC* [4].

A key idea in this paper is that of linear combinations of triangle centers. It is helpful to use the notation $\lambda P + \mu Q$ for such a combination, but, we shall soon see, this notation must be understood in terms of normalized barycentric coordinates. Consider, for example,

$$P = G = 1 : 1 : 1 = abc : abc : abc \quad \text{and} \quad Q = I = a : b : c.$$

The notation "$2P+3Q$" could be taken to mean either $2+3a : 2+3b : 2+3c$ or $2abc+3a : 2abc+3b : 2abc+3c$, two distinct points. In order to establish a single-point meaning for $\lambda P + \mu Q$, recall that the notation $u : v : w$ represents an equivalence class of ordered triples (hu, hv, hw), where h is any nonzero function of the variable (a, b, c). For any point $P = u : v : w$ not on $\mathcal{L}^{\infty}$, there is a member (uh, vh, wh) of $u : v : w$ such that uh, vh, wh are the oriented areas of the triangles PBC, PCA, PAB, respectively. Indeed, $h = \Delta/(u + v + w)$. Now suppose $P = u : v : w$ and $Q = x : y : z$ are points, neither on $\mathcal{L}^{\infty}$, which is to say that $u + v + w \neq 0 \neq x + y + z$. Define $\lambda P + \mu Q$ as the point R for which the oriented areas of the triangles RBC, RCA, RAB are

$$\lambda uh + \mu xk, \lambda vh + \mu yk, \lambda wh + \mu zk,$$

respectively, where $h = \Delta/(u+v+w)$ and $k = \Delta/(x+y+z)$. Barycentrics are given by

$$\lambda P + \mu Q = \lambda u\Sigma_Q + \mu x\Sigma_P : \lambda v\Sigma_Q + \mu y\Sigma_P : \lambda w\Sigma_Q + \mu z\Sigma_P, \qquad (1)$$

where Σ_P and Σ_Q are *coordinate sums*, given by $\Sigma_P := u + v + w$ and $\Sigma_Q := x + y + z$. In this manner, given *any* barycentrics for points not on $\mathcal{L}^\infty$, the linear combination $\lambda P + \mu Q$ is now unambiguously given by (1).

Another geometric significance of $\lambda P + \mu Q$ is as the point R satisfying

$$|PR| : |RQ| = \mu : \lambda, \text{ or equivalently, } \overrightarrow{WR} = \overrightarrow{WP} + \frac{\mu}{\lambda+\mu}\overrightarrow{PQ}$$

for any point W not on $\mathcal{L}^\infty$. Taking $W = P$ identifies R as the point "$\mu/(\lambda+\mu)$ of the way from P to Q", or, writing $\mu/(\lambda+\mu)$ as f, R is the image of Q under the homothety centered at P with ratio f, denoted by $\mathbb{H}(P, f)(Q)$. If f is a fraction m/n, then the point R, alias $\mathbb{H}(P, m/n)(Q)$ is the point $(n-m)P + mQ$. Special cases follow:

$$\begin{aligned}
\mathbb{H}(P, 1/2)(Q) &= \text{midpoint of } P \text{ and } Q\\
\mathbb{H}(P, 2)(Q) &= \text{reflection of } P \text{ in } Q\\
\mathbb{H}(P, -1)(Q) &= \text{reflection of } Q \text{ in } P\\
\mathbb{H}(P, 3)(Q) &= \text{complement of } P\\
\mathbb{H}(P, 3/2)(Q) &= \text{anticomplement of } P.
\end{aligned}$$

A further note on notation will be helpful. If X is a triangle center given by a center function $f(a, b, c)$, then the notation $X = f(a, b, c) ::$ abbreviates the homogeneous barycentric (or trilinear) representation $f(a, b, c) : f(b, c, a) : f(c, a, b)$. When a specific ordered triple of coordinates is required, we replace the colons by commas and enclose the triple by parentheses, like this: $(f(a, b, c), f(b, c, a), f(c, a, b))$, abbreviated as $(f(a, b, c), ,)$. For example,

$$\begin{aligned}
\lambda(f(a, b, c), ,) + \mu(g(a, b, c), ,) &= (\lambda f(a, b, c) + \mu g(a, b, c), ,)\\
&= \lambda f(a, b, c) + \mu g(a, b, c) :: \ .
\end{aligned}$$

This double-comma notation, $(x, ,)$, will be used in the sequel.

Regarding S_A, S_B, S_C in the first paragraph, define $S = 2\Delta$. Then

$$S_B S_C + S_C S_A + S_A S_B = S^2,$$

and if $P - Q$ has normalized barycentrics (u, v, w), then

$$|PQ|^2 = u^2 S_A + v^2 S_B + w^2 S_C.$$

These and related identities are given by Yiu in [7]. See also [8].

2. Incenter, Nagel Point, and Incircle. Let P be the incenter, $X(1)$, and Q the Nagel point, $X(8)$. Then $P = a$ :: and $Q = b + c - a$ ::, and $P_\Sigma = Q_\Sigma = a + b + c$. The linear combination $2P + Q$ is the centroid, $X(2)$, and $P + Q$, the Spieker center, $X(10)$.

In order to find barycentrics for the points where the line PQ meets the incircle, note that the points are at directed distances $\pm r$ from P, so that we seek λ and μ satisfying $|PQ|\mu/(\lambda + \mu) = \pm r$. We choose $\mu = r$ and obtain $\lambda = \pm|PQ| - r$. Putting this together with $|PQ| = 3|IG|$ leads to barycentrics for the points of intersection:

$$\pm 3|IG|a + r(b + c - 2a) ::,$$

where

$$|IG|^2 = \frac{-9abc + 2\sum a^2(b+c) - \sum a^3}{9(a+b+c)}$$

$$= \frac{1}{9}(5r^2 - 16Rr + s^2).$$

The method just exemplified can be applied in many other settings, and it is the main purpose of this paper to do so. First, however, we describe a method already found in the literature. Suppose we wish to formulate the intersection points of a line PQ and a circle Λ. Let L be the line through the center of Λ and perpendicular to PQ. Let $U = PQ \cap L$. Then PQ meets $\mathcal{L}^\infty$ in the point $P - Q$, and the required points of intersection are a pair of harmonic conjugates with respect to U and $P - Q$, so that the pair are $U \pm t(P - Q)$ for some t.

3. Euler Line and Some of Its Points. The Euler line, perhaps the most famous line in triangle geometry, passes through O and H. Writing

O as $P = (a^2S_A,,)$ and H as $Q = (S_BS_C,,)$, we have $\Sigma_P = 2S^2$ and $\Sigma_Q = S^2$. These are combined using (1) to form

$$\lambda a^2S_A + \mu S_BS_C :: \quad = \mathbb{H}(O, \mu/(2\lambda+\mu))(H), \tag{2}$$

where

$$|OH|^2 = 9R^2 - a^2 - b^2 - c^2.$$

A trigonometric form can be obtained starting with $O = (a\cos A,,)$ and $H = (a\cos B\cos C,,)$, for which the coordinate sums are S/R and $S/2R$, respectively. Using (1) and canceling a, we obtain the *trilinear* representation

$$\lambda(\cos A,,) + \mu(\cos B\cos C,,) = \mathbb{H}(O, \mu/(2\lambda+\mu))(H).$$

For example, centers $X(631)$ and $X(632)$ are easily formulated in this manner, using appropriate m and n in the identity

$$(3n-m)(a^2S_A,,) + 2n(S_BS_C,,) = \mathbb{H}(O, m/n)(G),$$

where $3|OG| = 2|ON| = |OH|$. Indeed, using distances from O to selected points on the Euler line, we can easily determine barycentrics.

It is helpful to adopt the symbols J and e as defined here:

$$\begin{aligned}
J &= |OH|/R \\
&= \frac{1}{abc}\left(\sum a^6 - \sum a^2b^4 + 3a^2b^2c^2\right)^{1/2} \\
&= \frac{1}{abc}(a^2S_BS_C + b^2S_CS_A + c^2S_AS_B - 6S_AS_BS_C)^{1/2} \\
e &= (1-4\sin^2\omega)^{1/2} = \left(\frac{\sum a^4 - \sum a^2b^2}{\sum a^2b^2}\right)^{1/2},
\end{aligned}$$

as given by Gallatly [3].

4. Euler Line and Circumcircle. The Euler line passes through the center of the circumcircle, so that there are two real points of intersection: $X(1113)$ and $X(1114)$. To find barycentrics, consider the distance associated with the right-hand side of equation (2):

$$|OH|\mu/(2\lambda+\mu) = \pm R.$$

Choosing $\mu = 2R$, we find $\lambda = \pm|OH| - R$, so that the points of intersection are given by

$$(\pm|OH| - R)(a^2 S_A, ,) + 2R(S_B S_C, ,). \qquad (3)$$

The point given by $\lambda = |OH| - R$ is the one nearer to H. The points in (3) are also clearly given by

$$X(1113) = (1 - J)a^2 S_A - 2S_B S_C :: \ , \qquad (4)$$

$$X(1114) = (1 + J)a^2 S_A - 2S_B S_C :: \ . \qquad (5)$$

Equations (4) and (5) show that the two points of intersection may be regarded as linear combinations of the points $X(3) = a^2 S_A$ :: and $X(30) = 2S_B S_C - a^2 S_A$ ::, this latter point being on $\mathcal{L}^\infty$. The representations in (4) and (5) also bring to mind the following well-known connection [1] between inverse pairs and harmonic conjugate pairs.

Theorem. Suppose L is a line passing through the center W of a circle Λ. Let P and Q be the points where L meets Λ. If V = inverse-in-Λ of U, then $V = \{P, Q\}$-harmonic conjugate of U.

When $\Lambda =$ circumcircle and $L =$ Euler line, the theorem yields $X(j) = \{X(1113), X(1114)\}$-harmonic conjugate of $X(i)$ for these pairs (i, j): $(2, 23)$, $(4, 186)$, $(22, 858)$, $(24, 403)$, $(25, 468)$, $(237, 1316)$, and in a limiting sense, $(3, 30)$.

5. Euler Line and Nine-Point Circle. The nine-point circle has center $N = X(5)$, situated on the Euler line halfway between O and H. The radius is $R/2$, and again there are two points of intersection, situated at directed distances $|OH|/2 \pm R/2$ from O. We obtain $|OH|\mu/(2\lambda + \mu) = (|OH| \pm R)/2$ by choosing $\mu = 2(|OH| \pm R)$ and $\lambda = |OH| \mp R$, so that the points of intersection are given by

$$(|OH| \mp R)(a^2 S_A, ,) + 2(|OH| \pm R)(S_B S_C, ,).$$

Alternatively, we can choose $\mu = -1$ and obtain, for the same two points,

$$(1 - J)a^2 S_A - 2(1 + J)S_B S_C :: \text{ and } (1 + J)a^2 S_A - 2(1 - J)S_B S_C ::,$$

which are $X(1312)$ and $X(1313)$.

The radical axis of the circumcircle and the nine-point circle cuts the Euler line in the point $X = X(468)$ satisfying $|OX|^2 - R^2 = |NX|^2 - (R/2)^2$, which yields $|OX| = (|OH|^2 + 3R^2)/(4|OH|)$ and

$$\begin{aligned} X(468) &= 3(|OH|^2 - R^2)a^2 S_A + 2(|OH|^2 + 3R^2)S_B S_C :: \\ &= 3(J^2 - 1)a^2 S_A + 2(3 + J^2)S_B S_C :: \\ &= (a^2 - 2S_A)S_B S_C :: . \end{aligned}$$

As the points $X(1113)$ and $X(1114)$ are on the Euler line and are an antipodal pair on the circumcircle, their Simson lines are the asymptotes of the Jerabek rectangular circumhyperbola. These asymptotes meet in the center, $X(125)$, of the hyperbola; this center lies on the nine-point circle. The asymptotes then meet the nine-point circle again in the points $X(1312)$ and $X(1313)$. Therefore, the three points, $X(125)$, $X(1312)$, and $X(1313)$, are the vertices of a right triangle.

The theorem in Section 4 yields $X(j) = \{X(1312), X(1313)\}$-harmonic conjugate of $X(i)$ for these pairs (i, j): $(2, 858)$, $(4, 403)$, $(427, 468)$, and in a limiting sense, $(5, 30)$.

6. Euler Line and Incircle. Regarding (as usual) a, b, c as variables, the various centers and lines they determine are functions of the triple (a, b, c). Functionally speaking, the incenter does not lie on the Euler line, so that for some choices of (a, b, c), it is not surprising that the Euler has no real intersection with the incircle. The method of the previous sections nevertheless applies. Representing the points of intersection, $X(1314)$ and $X(1315)$, as X, we have

$$\cos(\angle INO) = \frac{|ON|^2 + |IN|^2 - |OI|^2}{2|ON||IN|} = \frac{|NX|^2 + |IN|^2 - r^2}{2|NX||IN|}.$$

Solving for $|NX|$ and then $|OX|$, we find, after simplifications, barycentrics for the two points:

$$X(1314) = (|OH|^2 + 2r^2 - R^2)a^2 S_A$$

$$+ (|OH|^2 + 4rR - R^2 + \sqrt{T})(S_B S_C) :: ,$$

$$X(1315) = (|OH|^2 + 2r^2 - R^2)a^2 S_A$$

$$+ (|OH|^2 + 4rR - R^2 - \sqrt{T})(S_B S_C) :: ,$$

where

$$T = 4|OH|^2(4rR - R^2) + (|OH|^2 - 3R^2 + 4r^2 + 4rR)^2.$$

The point $X(1315)$ is the one nearer to O. The Euler line meets the incircle in $2, 1$, or 0 real points according as T is positive, zero, or negative.

7. Euler Line and Brocard Circle. The Brocard circle has diameter OK, radius p, and center $U = X(182)$. As in Gallatly [3], $|OG| : |UG| = R : p$. Point L is chosen on the Euler line OH so that the line UL is perpendicular to OH, and

$$|OL| = |OG| + (p^2 - |UG|^2 - |OG|^2)/(2|OG|).$$

Now $|OK|/|OU| = 2$, so that $|OX(1316)| = 2|OL|$. This leads to

$$|OX(1316)| = (1 + p^2/|OG|^2 - p^2/R^2)|OG|.$$

We rewrite (2) as

$$\lambda(a^2 S_A, ,) + \mu(S_B S_C, ,) = \mathbb{H}(O, 3\mu/(2\lambda + \mu))(G)$$

and choose $\mu = 2(|OG|^2 R^2 - |OG|^2 p^2 + p^2 R^2)$ to find that the point of intersection other than the circumcenter is given by

$$X(1316) = (S_B^2 + S_C^2)(S_A^2 + S_B S_C) - 2a^2 S_A S_B S_C :: \ .$$

Next, we seek barycentrics for the point $X = X(187)$ of intersection of the Lemoine axis and the Brocard axis. The Lemoine axis is the radical axis of the circumcircle and the Brocard circle. Thus, $|UX|^2 - p^2 = |OX|^2 - R^2$. As $|OX| = |UX| + p$, we have $|OX| = R^2/(2p) = R^2/|OK|$, giving

$$\begin{aligned} X(187) &= a^2 R + a(2|OK|^2 - R^2)\cot\omega\cos A :: \\ &= a(\sin A - 3\cos A \tan\omega) :: \ . \end{aligned}$$

Next, we find barycentrics for $X(237)$ as the intersection of the Lemoine axis and the Euler line. Substituting for $\cos(\angle UOG)$ and simplifying yield $X(237) = a^4(S_A^2 - S_B S_C) ::$.

As noted in [4], the point $X(1316)$ is the inverse-in-circumcircle of $X(237)$, and $X(1316)$ is also the inverse-in-orthocentroidal-circle of $X(868)$. Let V denote the center, $X(381)$, of the orthocentroidal circle and $f(|OG|)$ the distance $|OX(1316)|$. The point V has distance $2|OG|$ from O, so that the distance from V to $X(1316)$ is $2|OG| - f(|OG|)$. The radius of the orthocentroidal circle is $|OG|$. Consequently, $X(868)$ has distance $|OG|^2/[2|OG| - f(|OG|)]$ from V and distance $2|OG| - |OG|^2/(2|OG| -$

$f(|OG|)$ from O. These distances lead to $X(868) = (S_A^2 - S_BS_C)(S_B - S_C)^2$ ::, as well as these points:

(Inverse-in-nine-point-circle of $X(1316)) = (S_A^2 - S_BS_C)(S_B^2 + S_C^2)$ ::
(Inverse-in-2nd-Lemoine-circle of $X(1316)) = (S_A^2 + S_BS_C)(S_B^2 - S_C^2)$ ::
(Reflection of $X(1316)$ in $X(6)$)
$$= 2S_BS_C(S_A^2 + S_BS_C) + a^2S_A(S_B^2 + S_C^2 - 4S_BS_C) ::$$
(Reflection of $X(6)$ in $X(1316)$)
$$= a^6S_A - (S_A^2 + S_BS_C)(3S_B^2 + 3S_C^2 - 2S_BS_C) ::$$

8. Lines OI and OK. Writing the circumcenter as $P = (a^2S_A, ,)$ and the incenter as $Q = (a, ,)$, we have coordinate sums $\Sigma_P = 2S^2$ and $\Sigma_Q = 2s$. Then by (2),

$$\lambda(a^2S_A, ,) + \mu(a, ,) = \mathbb{H}(O, \mu s/(S^2\lambda + \mu s))(I). \tag{6}$$

As the line OI is a diameter of the circumcircle, there are two real intersections. To find their barycentrics, we determine μ and λ so that the distances associated with the right-hand side of (6) are $\pm R$; e.g., $\mu = RS^2$ and $\lambda = (\pm|OI| - R)s$, leading to

$$(|OI| - R)sa^2S_A + RS^2a :: \ , \tag{7}$$
$$(-|OI| - R)sa^2S_A + RS^2a :: \ . \tag{8}$$

The point in (8) is the one nearer to O.

Next, we intersect the line OI with the incircle. To have the distance associated with the right-hand side of (2) equal to $|OI| \pm r$, we choose $\lambda = 1$ and $\mu = -2S(r \pm |OI|)$. The two points of intersection are then given by $a^2S_A - 4\Delta(r \pm |OI|)a$ ::. The radical axis of the circumcircle and incircle cuts the line OI in a point X satisfying $|OX|^2 - R^2 = |IX|^2 - r^2$, and using $|IX| = |OX| - |OI|$, we find the intersection of the radical axis and line OI given by

$$(2R - r)a^2S_A + 2S(r^2 - |OI|^2 - R^2)a :: .$$

Or, using Euler's formula, $|OI|^2 = R^2 - 2rR$, these barycentrics can be written as

$$(2R - r)a^2S_A + 2\Delta(r^2 + 2rR - 2R^2)a ::$$

and simplified to

$$a(a - b + c)(a + b - c)[(b + c)(a^2 + (b - c)^2) - 2a(b^2 + c^2 - bc)].$$

The line OI meets the nine-point circle in two points, one of which, X, lies between I and O. We have

$$\cos(\angle OIN) = \frac{|IN|^2 + |OI|^2 - |ON|^2}{2|IN||OI|} = \frac{|IN|^2 + |IX|^2 - |RN|^2}{2|IN||IX|}.$$

Solving for $|IX|$ and simplifying lead to these barycentrics for the two points of intersection:

$$\begin{aligned}(|OI|^2 - |IN|^2 - |ON|^2 + 2|RN|^2 \pm \sqrt{T})sa^2S_A \\ + 8\Delta^2(|ON|^2 - |RN|^2)a : : \ ,\end{aligned} \tag{9}$$

where

$$T = (|IN|^2 + |OI|^2 - |ON|^2)^2 + 4|OI|^2(|RN|^2 - |IN|^2). \tag{10}$$

Here, $|RN|$, the radius of the nine-point circle, equals $R/2$; moreover, $|ON| = 3|OG|/2$, $|IN| = R/2 - r$, and $|OI|^2 = R^2 - 2rR$. In (9), the expression containing "$+\sqrt{T}$" gives the point nearer to O. In (10), note that $|RN| \geq |IN|$, so that $T \geq 0$ for all (a, b, c), which is to say that there is always a real point of intersection. Indeed, it is easy to see that there are two distinct points of intersection unless triangle ABC is degenerate with collinear vertices.

The method leading to the barycentrics (7) and (8) applies to the points of intersection of the Brocard axis, OK, and the circumcircle.

9. Further Applications. The method using (2) extends, through the formulation of distances along selected lines, to formulations of barycentrics, hence trilinears, of many other triangle centers.

Next, recall that points (U, V, W, W') form a *harmonic range*, and W' is the $\{U, V\}$-*harmonic conjugate of* W, if

$$\frac{|UW|}{|VW|} = \frac{|UW'|}{|VW'|}.$$

For example, letting B_1 and B_2 denote the points of intersection (in Table 2) of the Brocard axis and the circumcircle, we have

$$X(187) = \{O, K\}\text{-harmonic conjugate of } X(574)$$
$$X(187) = \{B_1, B_2\}\text{-harmonic conjugate of } K.$$

The method associated with (2) can be used to obtain the following harmonic conjugacies on the line GH:

$$
\begin{aligned}
O &= \{G, H\}\text{-harmonic conjugate of } N \\
X(25) &= \{G, H\}\text{-harmonic conjugate of } X(427) \\
X(378) &= \{G, H\}\text{-harmonic conjugate of } X(403) \\
X(382) &= \{G, H\}\text{-harmonic conjugate of } X(546) \\
X(1316) &= \{G, H\}\text{-harmonic conjugate of } X(868).
\end{aligned}
$$

Closely associated with harmonic conjugates are centers of similitude of two nonconcentric circles. We begin with definitions. Suppose (U, s) and (V, t) are circles with $U \neq V$, and point P lies on (U, s) but not on line UV. The line L_P through V parallel to line UP meets (V, t) in two points: let Q be the one for which the vector $\overrightarrow{VQ}$ has the same direction as $\overrightarrow{UP}$, and let Q' be the other, so that $\overrightarrow{VQ'}$ has the same direction opposite that of $\overrightarrow{UP}$. Let $W = UV \cap PQ$ and $W' = UV \cap PQ'$. The points W and W', called the *external center of similitude* and the *internal center of similitude*, respectively, remain fixed as P varies on (U, s). Moreover, if $s < t$, then

$$
\frac{|UW|}{|VW|} = \frac{|UW'|}{|VW'|} = \frac{s}{t},
$$

so that $W' = \{U, V\}$-harmonic conjugate of W.

As noted in [5], the centers of similitude of the 2nd Lemoine circle and Parry circle are a pair of bicentric points, not triangle centers. More generally, suppose Λ is a circle with arbitrary triangle center $x : y : z$ as center and radius ρ. Then the internal center of similitude has first trilinear

$$
a(b^2 - c^2)(x + y + z)\rho T + bcSx, \tag{11}
$$

where $S = a^4 + b^4 + c^4 - b^2c^2 - c^2a^2 - a^2b^2$ and $T = b^2 + c^2 - 2a^2$. The representation (11) indicates that these centers of similitude comprise a bicentric pair.

As a final note in this section, we mention that many properties associated with points on the lines NK, OI, and OK follow from three identities involving arbitrary nonzero integers m and n:

$$
\begin{aligned}
m|NK|/n &= (m - n)\cos(B - C)\cot\omega - 2m\sin A \\
m|OI|/n &= n\cos A + m\cos B + m\cos C - m \\
m|OK|/n &= (n - m)\cos A\cot\omega + m\sin A.
\end{aligned}
$$

10. Tucker Circles. Here, we extend results given in Gallatly [3]. The Tucker circle with parameter θ' (as in [3]) has radius $r \sin\omega \csc(\omega + \theta')$ and center given by trilinears $\cos(A - \theta') : \cos(B - \theta') : \cos(C - \theta')$. The method introduced in Section 2 applies to the points of intersection of a Tucker circle and the Brocard axis. In trilinears, the results are especially attractive:

$$\begin{aligned} e\cos(A-\theta') - \cos(A+\omega) &: e\cos(B-\theta') - \cos(B+\omega) \\ &: e\cos(C-\theta') - \cos(C+\omega), \\ e\cos(A-\theta') + \cos(A+\omega) &: e\cos(B-\theta') + \cos(B+\omega) \\ &: e\cos(C-\theta') + \cos(C+\omega). \end{aligned}$$

The first of these is the one whose direction from the center of the Tucker circle is the same as the direction from O to K. Remarkably like those intersections are the internal and external centers of similitude of a Tucker circle and the Brocard circle, given, respectively, by trilinears

$$\begin{aligned} e\cos(A-\theta') + \cos(A-\omega) &: e\cos(B-\theta') + \cos(B-\omega) \\ &: e\cos(C-\theta') + \cos(C-\omega), \\ e\cos(A-\theta') - \cos(A-\omega) &: e\cos(B-\theta') - \cos(B-\omega) \\ &: e\cos(C-\theta') - \cos(C-\omega). \end{aligned}$$

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