HIROSHIMA MATH. J.
20 (1990), 213–222

On Morrey spaces of measures: basic properties and potential estimates

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(Received May 19, 1989)

1. Introduction

This paper deals with various estimates for the Riesz potentials and the heat kernel acting on the spaces of Radon measures μ on R^n whose total variation $|\mu|$ satisfies

$$|\mu|(B(z, r)) \leqq Cr^{n(1-1/p)} \tag{1.1}$$

for some $1 \leqq p \leqq \infty$. Here $B(z, r)$ denotes the open ball with radius r centered at $z \in R^n$ and the constant C is independent of r and z. The spaces of measures of the form $\mu = f\,dx$, with $f \in L^1_{\mathrm{loc}}$ and dx the Lebesgue measure, satisfying (1.1) are called Morrey spaces and have been extensively studied in connection with the regularity theory for weak solutions of nonlinear elliptic equations (see [1, 2, 5]). However, those measures are considered mostly on bounded domains since the main interest is in the local behavior of the densities.

In this paper we consider the measures satisfying (1.1) on the whole space R^n and discuss the boundedness of the Riesz potentials and the heat kernel acting on such measures. Our results are stated in the same way as in the well-known case of L^p spaces and seem to be more or less known to mathematical publicity. For example, Peetre [6] states a more general result than ours without proof; however, it seems to be not so easy to guess a proof of his result by reading the other parts of [6]. For this reason we give in this paper the detailed proofs of our results for later use.

The author encountered the spaces of measures satisfying (1.1) during the study of viscous vortex flow in three-dimensional space [3]. Although some parts of the results in this paper are already included in [3], we present here the full version of the results for the reader's convenience. In Section 2 we define the Morrey spaces of measures and investigate their elementary structures. In particular, we shall show that the norms of the Morrey spaces possess several properties in common with the norms of the Lebesgue spaces L^p and the Lorentz-Marcinkiewicz spaces L^p_w. It should be noticed that, contrary to the case of L^p and L^p_w spaces, the interpolation property of the Morrey spaces with respect to various bounded linear operators still remains an open problem. Section 3 is the main part of this paper, in which we discuss the boundedness of the Riesz potentials and the heat kernel acting on the Morrey spaces. The estimates given there possess the same form as in the case of those potentials acting on L^p spaces.

The author is grateful to Professors S. Oharu and F-Y. Maeda for their interest in this work and for helpful discussions. The present work was initiated while the author was visiting the University of Paderborn in Federal Republic of Germany as an Alexander von Humboldt research fellow. The support and the warm hospitality of the Alexander von Humboldt Foundation and of the University of Paderborn are gratefully acknowledged here.

2. Definition and basic properties of Morrey spaces

We employ the standard notation: $L^p = L^p(R^n)$ denotes the usual Lebesgue space and $L^p_w = L^p_w(R^n)$ the Lorentz-Marcinkiewicz space of measurable functions f such that $\lambda(\alpha) = \text{meas}\,\{x; |f(x)| > \alpha\}$, $\alpha > 0$, satisfies

$$\|f\|^*_{p,w} \equiv \sup_{\alpha>0} \alpha\lambda(\alpha)^{1/p} < \infty .$$

DEFINITION 2.1. A Radon measure μ on R^n belongs to the Morrey space $\mathcal{M}^p$, $1 \leqq p \leqq \infty$, if there exists a constant C such that the total variation $|\mu|$ satisfies

$$|\mu|(B(z, r)) \leqq Cr^{n(1-1/p)}$$

for all $z \in R^n$ and $r > 0$, where $B(z, r)$ is the open ball with radius r and center z. We set $M^p = L^1_{\text{loc}} \cap \mathcal{M}^p$.

PROPOSITION 2.2. *$\mathcal{M}^p$ is a Banach space with norm*

$$\|\mu\|_p = \sup_{z\in R^n, r>0} r^{-n(1-1/p)}|\mu|(B(z, r))$$

and M^p is a closed subspace of $\mathcal{M}^p$.

PROOF. Let μ_m be an arbitrary Cauchy sequence; for each $\varepsilon > 0$ there is an N so that

$$\|\mu_m - \mu_l\|_p < \varepsilon \qquad \text{for all} \quad m, l \geqq N , \tag{2.1}$$

and, moreover, there is a $C > 0$ with

$$\|\mu_m\|_p \leqq C \qquad \text{for all} \quad m . \tag{2.2}$$

By (2.2), μ_m is uniformly bounded in total variation on each fixed open ball; so we can extract a subsequence $\mu_{m'}$ which converges weakly to some measure μ on each open ball. Since (2.2) implies

$$|\mu|(B(z, r)) \leqq \liminf_{m' \to \infty} |\mu_{m'}|(B(z, r)) \leqq Cr^{n(1-1/p)},$$

it follows that $\mu \in \mathscr{M}^p$ and

$$\|\mu\|_p \leqq \liminf_{m' \to \infty} \|\mu_{m'}\|_p .$$

Applying the same argument to $\mu_{m'} - \mu_l$ yields, by (2.1),

$$\|\mu - \mu_l\|_p \leqq \liminf_{m' \to \infty} \|\mu_{m'} - \mu_l\|_p \leqq \varepsilon \qquad \text{for} \quad l \geqq N .$$

This shows the first assertion. The second assertion is easily verified if we note that this time the functions μ_m converge in L^1_{loc} topology.

PROPOSITION 2.3. (i) *$\mathscr{M}^1$ is the set of finite measures; and $M^1 = L^1$.*
(ii) *$\mathscr{M}^\infty = M^\infty = L^\infty$ with equivalent norms.*

PROOF. Statement (i) is obvious from the definition. Let $\mu \in \mathscr{M}^\infty$; then $|\mu|(B) \leqq C\|\mu\|_\infty |B|$, where $|B|$ is Lebesgue measure of open balls B and C depends only on n. From this we easily see that $|\mu|(E) \leqq C\|\mu\|_\infty |E|$ for all Borel sets E and hence μ is absolutely continuous with respect to Lebesgue measure. By Lebesgue's theorem on differentiation, the density f of μ is bounded in absolute value by $C\|\mu\|_\infty$ almost everywhere, and we get $\mathscr{M}^\infty \subset L^\infty$ with the continuous injection. The reverse inclusion is obvious. This proves (ii).

PROPOSITION 2.4. (i) *$L^p \subset L^p_w \subset M^p$, $1 < p < \infty$, with continuous injections.*
(ii) *$\mathscr{M}^p(R^{n-1}) \subset \mathscr{M}^{np/(n-1)}(R^n)$, $1 \leqq p \leqq \infty$, under the map: $\mu \to \mu \times dx_n$.*

PROOF. (i) The first inclusion is well known. To show the second, we have only to show that $f \in L^p_w$ if and only if there is a constant C so that

$$\int_E |f|\, dx \leqq C|E|^{1-1/p} \qquad \text{for all Borel sets } E . \tag{2.3}$$

Assume first that $f \in L^p_w$ and let $\lambda_E(\alpha)$ be the distribution function of $f|_E$. Since we may assume that $|E|$ is finite, $\lambda_E(\alpha) \leqq \min(\lambda(\alpha), |E|) \leqq \min(C\alpha^{-p}, |E|)$. Thus, denoting $\beta = (C/|E|)^{1/p}$, we obtain

$$\int_E |f|\, dx = \int_0^\infty \lambda_E(\alpha)\, d\alpha = \int_0^\beta \lambda_E(\alpha)\, d\alpha + \int_\beta^\infty \lambda_E(\alpha)\, d\alpha$$

$$\leqq |E|\beta + C\int_\beta^\infty \alpha^{-p}\, d\alpha = pC|E|^{1-1/p}/(p-1) .$$

Conversely, suppose f satisfies (2.3). Setting $E = B(0, r) \cap \{x; |f(x)| > \alpha\}$, applying Tschebyscheff's inequality and then letting $r \to \infty$ yields $\alpha\lambda(\alpha)^{1/p} \leqq C$. This proves (i). For (ii), direct calculation gives

$$\int_{B(z,r)} |\mu|(dx')\, dx_n \leqq \int_{z_n-r}^{z_n+r} \int_{B'(z',r)} |\mu|(dx')\, dx_n$$
$$\leq C\|\mu\|_p r^{1+(n-1)(1-1/p)} = C\|\mu\|_p r^{n(1-(n-1)/np)},$$

where $z = (z', z_n)$, and B' stands for open balls in R^{n-1}. This proves (ii).

PROPOSITION 2.5. *Let $\mu_\lambda(E) = \lambda^n \mu(E/\lambda)$ for $\lambda > 0$. Then*

$$\|\mu_\lambda\|_p = \lambda^{n/p}\|\mu\|_p .$$

PROOF. That $\|\mu_\lambda\|_p \leqq \lambda^{n/p}\|\mu\|_p$ is directly verified. This in turn implies that $\|\mu\|_p = \|(\mu_\lambda)_{1/\lambda}\|_p \leqq \lambda^{-n/p}\|\mu_\lambda\|_p$. This completes the proof.

PROPOSITION 2.6. *Let $p_0 \neq p_1$, $0 \leqq \theta \leqq 1$, and $1/p = (1-\theta)/p_0 + \theta/p_1$. Then $\mathscr{M}^{p_0} \cap \mathscr{M}^{p_1} \subset \mathscr{M}^p$* and $\|\mu\|_p \leqq \|\mu\|_{p_0}^{1-\theta}\|\mu\|_{p_1}^{\theta}$.

PROOF. In fact, we have

$$|\mu|(B(z, r)) = [|\mu|(B(z, r))]^{1-\theta}[|\mu|(B(z, r))]^{\theta}$$
$$\leqq [\|\mu\|_{p_0} r^{n(1-1/p_0)}]^{1-\theta}[\|\mu\|_{p_1} r^{n(1-1/p_1)}]^{\theta} = \|\mu\|_{p_0}^{1-\theta}\|\mu\|_{p_1}^{\theta} r^{n(1-1/p)} .$$

PROPOSITION 2.7. *If $\mu \in \mathscr{M}^p$ and $\nu \in \mathscr{M}^1$, then the convolution $\mu * \nu$ lies in $\mathscr{M}^p$ and satisfies the estimate $\|\mu * \nu\|_p \leqq \|\mu\|_p \|\nu\|_1$.*

PROOF. Denoting by $\chi_{z,r}(x)$ the indicator function of $B(z, r)$, we have

$$|\mu * \nu|(B(z, r)) \leqq \int |\nu|(dy) \int \chi_{z,r}(x+y)|\mu|(dx) = \int |\mu|(B(z-y, r))|\nu|(dy)$$
$$\leqq \|\nu\|_1 \|\mu\|_p r^{n(1-1/p)} .$$

This proves the result.

PROPOSITION 2.8. *If $\mu \in \mathscr{M}^p$, then $\int (1+|x|)^{-n-1}|\mu|(dx)$ is finite; in particular, μ is a tempered distribution.*

PROOF. Indeed, direct calculation gives

$$\int (1+|x|)^{-n-1}|\mu|(dx) \leqq |\mu|(B(0, 1)) + \int_{|x| \geqq 1} |x|^{-n-1}|\mu|(dx) .$$

The last term is estimated as

$$= \int_0^1 |\mu|[\{x; |x|^{-n-1} > s\}]\, ds = \int_0^1 |\mu|(B(0, s^{-1/(n+1)}))\, ds$$

$$\leqq \|\mu\|_p \int_0^1 s^{-n/(n+1)p'}\, ds = C\|\mu\|_p, \qquad 1/p' = 1 - 1/p.$$

This completes the proof.

3. Potential estimates

In this section we denote $\Lambda = (-\Delta)^{1/2}$; so $\Lambda^{-\alpha}$, $0 < \alpha < n$, is the Riesz potential of order α. We begin by establishing the Sobolev type estimates.

PROPOSITION 3.1. (i) *Let* $0 \leqq \alpha < n/p$. *If* $\Lambda^\alpha f \in \mathcal{M}^p$, *then* $f \in M^q$ *with* $1/q = 1/p - \alpha/n$ *and the estimate*

$$\|f\|_q \leqq C\|\Lambda^\alpha f\|_p \tag{3.1}$$

holds with a constant C independent of f.

(ii) *If* $\Lambda^\alpha f \in \mathcal{M}^p \cap \mathcal{M}^q$ *and* $n/p < \alpha < n/q$, then $f \in L^\infty$ *and we have*

$$\|f\|_\infty \leqq C\|\Lambda^\alpha f\|_p^{1-\theta}\|\Lambda^\alpha f\|_q^\theta, \qquad \theta = (\alpha - n/p)/(n/q - n/p). \tag{3.2}$$

PROOF. We set $\mu = \Lambda^\alpha f$ so that $f = \Lambda^{-\alpha}\mu$. For fixed $A > 0$, we decompose the kernel function $c_{n,\alpha}|x|^{\alpha-n}$ as

$$K_1(x) = \begin{cases} c_{n,\alpha}|x|^{\alpha-n} & (|x| \leqq A) \\ 0 & \text{otherwise} \end{cases}; \qquad K_2(x) = c_{n,\alpha}|x|^{\alpha-n} - K_1(x).$$

Then we have

$$|f(x)| \leqq K_1 * |\mu| + K_2 * |\mu| \equiv I_1 + I_2$$

and

$$I_2(x) \leqq C\int_{|x-y|>A} |x-y|^{\alpha-n}|\mu|(dy) = C\int_0^{A^{\alpha-n}} |\mu|[|x-y|^{\alpha-n} > s]\, ds$$

$$= C\int_0^{A^{\alpha-n}} |\mu|(B(x, s^{-1/(n-\alpha)})\, ds \leqq C\|\mu\|_p \int_0^{A^{\alpha-n}} s^{-n/(n-\alpha)p'}\, ds$$

$$= C\|\mu\|_p A^{\alpha-n/p}, \qquad p' = p/(p-1).$$

Hence

$$\int_{B(z,r)} I_2(x)\, dx \leqq C\|\mu\|_p A^{\alpha-n/p} r^n. \tag{3.3}$$

On the other hand, denoting by $\chi_{z,r}$ the indicator function of $B(z, r)$,

$$\int_{B(z,r)} I_1(x)\,dx \leqq \int \chi_{z,r}(x) \int K_1(x-y)|\mu|(dy) = \int |\mu|(dy) \int \chi_{z,r}(x+y) K_1(x)\,dx$$

$$= \int K_1(x)|\mu|(B(z-x, r))\,dx \leqq C\|\mu\|_p r^{n(1-1/p)} A^\alpha .$$

Combining this with (3.3) yields

$$\int_{B(z,r)} |f(x)|\,dx \leqq C\|\mu\|_p (A^{\alpha-n/p} r^n + A^\alpha r^{n(1-1/p)}) .$$

Taking the minimum with respect to $A > 0$ gives (3.1). We next prove (3.2). This time, we estimate I_1 and I_2 above as

$$I_2(x) \leqq C\|\mu\|_q A^{\alpha-n/q}; \qquad I_1(x) \leqq C\|\mu\|_p A^{\alpha-n/p} .$$

Adding these and taking the minimum with respect to $A > 0$ yields (3.2).

PROPOSITION 3.2. *Let* $e^{t\Delta}\mu(x) = \int E_t(x-y)\mu(dy)$, $E_t(x) = (4\pi t)^{-n/2} \times \exp(-|x|^2/4t)$.

(i) $\|e^{t\Delta}\mu\|_p \leqq \|\mu\|_p$ *for all p with* $1 \leqq p \leqq \infty$,

(ii) $\|\nabla^k e^{t\Delta}\mu\|_q \leqq Ct^{-k/2-(n/p-n/q)/2}\|\mu\|_p$ *whenever* $1 \leqq p \leqq q \leqq \infty$, *where* ∇^k *stands for the k-th derivatives with respect to x.*

PROOF. (i) Using the identity $\int E_t(x)\,dx = 1$, we have

$$\int_{B(z,r)} |e^{t\Delta}\mu|\,dx \leqq \int E_t(x)|\mu|(B(z-x, r))\,dx \leqq \|\mu\|_p r^{n(1-1/p)} .$$

(ii) Using the well-known estimate

$$|\nabla^k E_t(x)| \leqq C_1 t^{-(k+n)/2} \exp[-C_2|x|^2/t]$$

we see that

$$\|\nabla^k e^{t\Delta}\mu\|_p \leqq Ct^{-k/2}\|\mu\|_p ; \qquad \|\nabla^k e^{t\Delta}\mu\|_\infty \leqq Ct^{-k/2-n/2p}\|\mu\|_p , \qquad 1 \leqq p \leqq \infty .$$

For $p < q < \infty$, Proposition 2.6 gives

$$\|\nabla^k e^{t\Delta}\mu\|_q \leqq \|\nabla^k e^{t\Delta}\mu\|_p^{p/q} \|\nabla^k e^{t\Delta}\mu\|_\infty^{1-p/q} \leqq Ct^{-k/2-(n/p-n/q)/2}\|\mu\|_p .$$

The proof is complete.

The following corresponds to Gagliardo-Nirenberg inequality in L^p case.

PROPOSITION 3.3. *If $n/p < \alpha$, $f \in M^p$ and $\Lambda^\alpha f \in \mathcal{M}^p$, then $f \in L^\infty$ and*

$$\|f\|_\infty \leqq C\|f\|_p^{1-n/\alpha p}\|\Lambda^\alpha f\|_p^{n/\alpha p}$$

with a constant C independent of f.

PROOF. First observe that under our assumption the operator

$$(1-\Delta)^{-\alpha/2} = \Gamma(\alpha/2)^{-1}\int_0^\infty t^{\alpha/2-1}e^{-t(1-\Delta)}\,dt\,,$$

where $\Gamma(\beta)$ is the gamma function, is bounded from $\mathcal{M}^p$ to L^∞. Indeed, the foregoing result gives the estimate

$$\|(1-\Delta)^{-\alpha/2}\mu\|_\infty \leqq C\int_0^\infty t^{\alpha/2-1}\|e^{-t(1-\Delta)}\mu\|_\infty\,dt \leqq C\|\mu\|_p\int_0^\infty t^{\alpha/2-n/2p-1}e^{-t}\,dt$$
$$= C\|\mu\|_p\,.$$

Since $(1-\Delta)^{\alpha/2} = \nu_1 * + \nu_2 * \Lambda^\alpha$ (see [7]) for some $\nu_i \in \mathcal{M}^1$, $i = 1, 2$, we obtain

$$\|f\|_\infty \leqq C\|(1-\Delta)^{\alpha/2}f\|_p \leqq C(\|f\|_p + \|\Lambda^\alpha f\|_p)\,.$$

Here we have used Proposition 2.7. Substituting $f_\lambda(x) = f(x/\lambda)$, $\lambda > 0$, and applying Proposition 2.5, we obtain

$$\|f\|_\infty \leqq C(\lambda^{n/p}\|f\|_p + \lambda^{n/p-\alpha}\|\Lambda^\alpha f\|_p)\,.$$

Taking the minimum with respect to $\lambda > 0$ gives the desired result.

We finally establish Morrey and John-Nirenberg type estimates ([1, 2, 4, 5]).

DEFINITION. A measurable function f is said to be in BMO if

$$[f]_{BMO} \equiv \sup_{z\in R^n, r>0} r^{-n}\int_{B(z,r)}|f-\bar{f}_r(z)|\,dx < \infty\,,$$

where $\bar{f}_r(z)$ is the average: $\bar{f}_r(z) = |B(z,r)|^{-1}\int_{B(z,r)} f\,dx.$

To estimate the Hölder seminorm

$$[f]_\beta = \sup_{y\in R^n}\|f(\cdot + y) - f(\cdot)\|_\infty/|y|^\beta\,, \qquad 0 < \beta \leqq 1\,,$$

we use the following result of Campanato ([1], [2]).

LEMMA 3.4. *A function f is uniformly Hölder continuous on R^n with exponent β if and only if there is a constant $C > 0$ so that, for all $z \in R^n$ and $r > 0$,*

$$\int_{B(z,r)}|f-\bar{f}_r(z)|\,dx \leqq Cr^{n+\beta}\,.$$

The infimum of C on the right-hand side is equivalent to $[f]_\beta$.

Our result is now stated in the following way:

PROPOSITION 3.5. (i) *If $\Lambda^\alpha f \in \mathcal{M}^p$, $\alpha = n/p$, then $f \in BMO$ and we have*

$$[f]_{BMO} \leqq C\|\Lambda^\alpha f\|_p$$

with a constant C independent of f.

(ii) *If $\Lambda^\alpha f \in \mathcal{M}^p$ and $1 > \alpha - n/p > 0$, then f is uniformly Hölder continuous with exponent $\alpha - n/p$ and we have*

$$[f]_{\alpha - n/p} \leqq C\|\Lambda^\alpha f\|_p$$

with a constant C independent of f.

We begin the proof by establishing the following

LEMMA 3.6. *Let $0 < \alpha < 1$ and let $R_\alpha(x) = c_{n,\alpha}|x|^{\alpha-n}$ be the kernel of $\Lambda^{-\alpha}$. Then there is a constant C so that*

$$\int |R_\alpha(x+y) - R_\alpha(x)|\, dx \leqq C|y|^\alpha \qquad \text{for all} \quad y \in R^n .$$

PROOF. We write the above integral as

$$\int |R_\alpha(x+y) - R_\alpha(x)|\, dx = \int_{|x| \leqq 2|y|} + \int_{|x| \geqq 2|y|} \equiv J_1 + J_2 .$$

We easily see that

$$\begin{aligned} J_1 &\leqq \int_{|x| \leqq 2|y|} (|R_\alpha(x+y)| + |R_\alpha(x)|)\, dx \leqq \int_{|x| \leqq 3|y|} |R_\alpha(x)|\, dx \\ &= C|y|^\alpha . \end{aligned}$$

To estimate J_2 observe that since $|x| \geqq 2|y|$ implies $|x + \theta y| \geqq |x| - |y| \geqq |x|/2$ for $0 \leqq \theta \leqq 1$, we obtain

$$|R_\alpha(x+y) - R_\alpha(x)| \leqq |y| \int_0^1 |\nabla R_\alpha|(x + \theta y)\, d\theta \leqq C|y||x|^{\alpha-n-1} \quad \text{for} \quad |x| \geqq 2|y| .$$

Integrating this yields $J_2 \leqq C|y|^\alpha$, and this proves Lemma 3.6.

PROOF OF PROPOSITION 3.5. Let $\mu = \Lambda^\alpha f$ so that $f = \Lambda^{-\alpha}\mu$ and $\mu \in \mathcal{M}^p$. We first assume that $0 < \alpha < 1$. Since

$$|B(0,r)||f(x) - \bar{f}_r(z)| = \left|\int_{B(z,r)} (f(x) - f(y))\, dy\right| \leqq \int_{B(z,r)} |f(x) - f(y)|\, dy ,$$

we obtain

$$|B(0,r)|\int_{B(z,r)}|f-\bar{f}_r(z)|\,dx \leqq \iint_{B(z,r)\times B(z,r)}|f(x)-f(y)|\,dx\,dy$$

$$\leqq \int_{|\eta|\leqq 2r}d\eta\int_{B(z,r)}|f(y+\eta)-f(y)|\,dy\,.$$

Using Lemma 3.6, we see that

$$\int_{B(z,r)}|f(y+\eta)-f(y)|\,dy \leqq \int\chi_{z,r}(y)\,dy\int|R_\alpha(y+\eta-\zeta)-R_\alpha(y-\zeta)|\,|\mu|(d\zeta)$$

$$=\int|R_\alpha(y+\eta)-R_\alpha(y)|\,|\mu|(B(z-y,r))\,dy$$

$$\leqq C|\eta|^\alpha\|\mu\|_p r^{n(1-1/p)}$$

and hence

$$|B(0,r)|\int_{B(z,r)}|f-\bar{f}_r(z)|\,dx \leqq C\|\mu\|_p r^n r^{n(1-1/p+\alpha/n)}\,.$$

This proves assertions (i) and (ii) in case $0<\alpha<1$. The case $\alpha\geqq 1$ is proved by applying Proposition 3.1 and thereby reducing the problem to the case $0<\alpha<1$. The proof is complete.

Proposition 3.5 (ii) does not include the case $\alpha-n/p=1$. To treat this case, we have to modify the definition of seminorm $[f]_1$ to the form

$$[f]_1=\sup_{y\in R^n}\|f(\cdot+y)+f(\cdot-y)-2f(\cdot)\|_\infty/|y|\,.$$

Using this seminorm, we can show the following

PROPOSITION 3.7. *If $\Lambda^\alpha f\in\mathcal{M}^p$ with $\alpha-n/p=1$ and $f\in M^q$ for some q, then*

$$[f]_1\leqq C\|\Lambda^\alpha f\|_p$$

with a constant C independent of f.

PROOF. Consider the Poisson semigroup [7]

$$e^{-t\Lambda}=\pi^{-1/2}\int_0^\infty s^{-1/2}e^{-s}e^{t^2\Delta/4s}\,ds\,.$$

Then Proposition 3.2 yields the estimates

$$\|e^{-tA}\mu\|_p \leqq \|\mu\|_p \quad (1 \leqq p \leqq \infty);$$

$$\|A^{\beta}e^{-tA}\mu\|_q \leqq Ct^{-\beta-(n/p-n/q)}\|\mu\|_p \quad (1 \leqq p \leqq q \leqq \infty).$$

For f satisfying the assumption, we consider the regularization $f^{\varepsilon} = e^{\varepsilon A}f$, which is in L^{∞} and satisfies

$$\|A^2 e^{-tA}f^{\varepsilon}\|_{\infty} \leqq Ct^{-2+\alpha-n/p}\|A^{\alpha}f^{\varepsilon}\|_p = Ct^{-1}\|A^{\alpha}f^{\varepsilon}\|_p \leqq Ct^{-1}\|A^{\alpha}f\|_p .$$

Hence one can apply the results of [7, Chap. V, Sect. 4] to obtain

$$\|f^{\varepsilon}\|_{\infty} + [f^{\varepsilon}]_1 \leqq C(\|f^{\varepsilon}\|_{\infty} + \|A^{\alpha}f\|_p)$$

with C independent of f and $\varepsilon > 0$. Inserting $(f^{\varepsilon})_{\lambda}(x) = (f^{\varepsilon})(x/\lambda)$, $\lambda > 0$, and applying Proposition 2.5 now gives

$$\|f^{\varepsilon}\|_{\infty} + \lambda^{-1}[f^{\varepsilon}]_1 \leqq C(\|f^{\varepsilon}\|_{\infty} + \lambda^{-1}\|A^{\alpha}f\|_p).$$

Multiplying both sides by λ and then letting $\lambda \to 0$, we obtain

$$[f^{\varepsilon}]_1 \leqq C\|A^{\alpha}f\|_p ,$$

where C is independent of ε. Since f^{ε} is bounded in L^1_{loc}, a simple limiting argument yields the desired assertion. This completes the proof.

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