HIROSHIMA MATH. J.
3 (1973), 1–14

Weakly Coupled Parabolic Systems with Unbounded Coefficients

Takaŝi KUSANO, Tadashi KURODA and Lu-San CHEN*)

(Received November 27, 1972)

Introduction

Consider the system of parabolic differential equations

$$(*)\qquad \sum_{i,j=1}^{n} a_{ij}^{\alpha}(x,t)\frac{\partial^2 u^\alpha}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i^{\alpha}(x,t)\frac{\partial u^\alpha}{\partial x_i}+\sum_{\beta=1}^{N} c^{\alpha\beta}(x,t)u^\beta-\frac{\partial u^\alpha}{\partial t}=0,$$
$$\alpha=1,2,\dots,N.$$

Each equation of $(*)$ contains derivatives of just one component of the unknown functions $u^1(x, t)$, $u^2(x, t)$, ..., $u^N(x, t)$, and the system $(*)$ is coupled only in the terms which are not differentiated; so that a system of this form is said to be weakly coupled [11]. Such weakly coupled parabolic systems form a class to which most of the methods and techniques employed in the study of a single parabolic equation apply with minor necessary modifications.

During the last decade weakly coupled parabolic systems, both linear and nonlinear, have been intensively investigated by several authors, remarkably by Polish mathematicians. We refer in particular to the books of Protter and Weinberger [11], Szarski [12] and Walter [13], and the relevant references quoted in them.

The purpose of this paper is to add to the theory of weakly coupled parabolic systems results concerning the asymptotic behavior for $t\to\infty$ of solutions of the Cauchy problem for the system $(*)$ with unbounded coefficients. Specifically we focus our attention on the extension of the corresponding theorems which we have recently obtained for a single parabolic equation with unbounded coefficients [4]. Our results incidentally generalize those of one of the authors for more restricted classes of weakly coupled parabolic systems [7], [8].

The main tool is the maximum principle and the use of various comparison functions constructed so as to control the behavior of the solutions under consideration. The maximum principle is proved in § 1. The asymptotic behavior of solutions of $(*)$ is studied in §§ 2 and 3; § 2 concerns the exponential decay of the solutions with unbounded initial values, while § 3 concerns the exponential growth of the positive or negative solutions with nonvanishing initial values. In § 4 it

---

*) The research of this author is supported by National Science Council.

is shown that Liouville type theorems for some weakly coupled elliptic systems can be obtained as immediate consequences of the theorems of §§ 2 and 3.

§ 1. Maximum Principle

In this section we are concerned with the weakly coupled system of parabolic inequalities

$$L^{\alpha}[u^{\alpha}]+\sum_{\beta=1}^{N} c^{\alpha\beta}(x, t)u^{\beta} \geqq 0, \qquad \alpha=1, \ldots, N, \tag{1.1}$$

where each L^{α} stands for the parabolic operator

$$L^{\alpha} \equiv \sum_{i,j=1}^{n} a_{ij}^{\alpha}(x, t)\frac{\partial^2}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i^{\alpha}(x, t)\frac{\partial}{\partial x_i}-\frac{\partial}{\partial t}.$$

It is assumed that the coefficients in (1.1) are real valued in $\bar{\Omega} \times [0, T]$, Ω being an unbounded domain in R^n with closure $\bar{\Omega}$ and boundary $\partial\Omega$, and satisfy there the following hypotheses A:

There exist constants $K_1>0$, $K_2 \geqq 0$, $K_3>0$, $\mu>0$ and λ such that

$$0 \leqq \sum_{i,j=1}^{n} a_{ij}^{\alpha}(x, t)\xi_i\xi_j \leqq K_1[\log(|x|^2+1)+1]^{-\lambda}(|x|^2+1)^{1-\mu}|\xi|^2,$$

$$|b_i^{\alpha}(x, t)| \leqq K_2(|x|^2+1)^{1/2}, \qquad i=1, \ldots, n,$$

$$c^{\alpha\beta}(x, t) \geqq 0, \qquad \alpha \neq \beta,$$

$$\sum_{\beta=1}^{N} c^{\alpha\beta}(x, t) \leqq K_3[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu},$$

for all $(x, t) \in \bar{\Omega} \times [0, T]$, all real n-vectors $\xi=(\xi_1,\ldots, \xi_n)$ and $\alpha, \beta=1, \ldots, N$. We establish the following maximum principle.

THEOREM 1. *Suppose thay hypotheses* A *hold. Let $u^{\alpha}(x, t)$, $\alpha=1,\ldots, N$, be real valued functions which are continuous in $\bar{\Omega} \times [0, T]$, differentiable once with respect to t and twice with respect to x in $\Omega \times (0, T]$, and satisfy* (1.1) *in $\Omega \times (0, T]$. Suppose that for each $\alpha=1, \ldots, N$,*

$$u^{\alpha}(x, t) \leqq 0 \text{ on } \{\partial\Omega \times [0, T]\} \cup \{\Omega \times (t=0)\}, \tag{1.2}$$

and

$$u^{\alpha}(x, t) \leqq M \exp\{k[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}\} \quad \text{in } \Omega \times (0, T], \tag{1.3}$$

where M and k are some positive constants.

Then $u^{\alpha}(x, t) \leqq 0$ in $\bar{\Omega} \times [0, T]$, $\alpha=1, \ldots, N$.

PROOF. Let $\lambda \geqq 0$. We introduce the auxiliary function

$$v(x,\ t)=\exp\{2ke^{Ht}[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}\},$$

where

$$H=8K_1(\lambda+\mu)^2(2ke+1)+4(K_1+K_2)(\lambda+\mu)n+K_3k^{-1}.$$

It is easy to verify [4] that $v(x, t)$ satisfies

$$L^{\alpha}[v]+\sum_{\beta=1}^{N}c^{\alpha\beta}v<0 \text{ in } \Omega\times(0,\ H^{-1}], \qquad \alpha=1,\ \dots,\ N. \tag{1.4}$$

Consider the system of functions

$$\begin{aligned} w^{\alpha}(x,\ t)=u^{\alpha}(x,\ t)-M\exp\{&2ke^{Ht}[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu} \\ &-k[\log(\rho^2+1)+1]^{\lambda}(\rho^2+1)^{\mu}\}, \qquad \alpha=1,\ \dots,\ N, \end{aligned}$$

in the cylinder $\bar{\Omega}_{\varrho}\times[0,\ H^{-1}]$, where $\Omega_{\varrho}=\Omega\cap(|x|<\rho)$, $\rho>0$. Since by (1.2)–(1.4)

$$L^{\alpha}[w^{\alpha}]+\sum_{\beta=1}^{N}c^{\alpha\beta}w^{\beta}>0 \qquad \text{in } \Omega\times(0,\ H^{-1}],$$

$$w^{\alpha}(x,\ t)\leqq 0 \text{ on } \{\partial\Omega_{\varrho}\times[0,\ H^{-1}]\}\cup\{\Omega_{\varrho}\times(t=0)\}, \qquad \alpha=1,\ \dots,\ N,$$

we have, by the standard maximum principle for weakly coupled parabolic systems [11] (Chapter 3, Theorem 13), $w^{\alpha}(x, t)\leqq 0$ in $\bar{\Omega}_{\varrho}\times[0, H^{-1}]$, $\alpha=1, \dots, N$.

Let (x^*, t^*) be an arbitrary fixed point of $\bar{\Omega}\times[0, H^{-1}]$. Then (x^*, t^*) is contained in $\bar{\Omega}_{\varrho}\times[0, H^{-1}]$ for all sufficiently large ρ, and we have $w^{\alpha}(x^*, t^*)\leqq 0$. Letting $\rho\to\infty$, we conclude that $u^{\alpha}(x^*, t^*)\leqq 0$, that is, $u^{\alpha}(x, t)\leqq 0$ in $\bar{\Omega}\times[0, H^{-1}]$ for each α. To show that the required inequalities hold throughout $\bar{\Omega}\times[0, T]$ it suffices to iterate the above procedure a finite number of times.

The case where $\lambda<0$ can be treated analogously.

Remarks. (i) Theorem 1 remains true if the domain Ω is replaced by the entire space R^n. In this case the condition (1.2) must be replaced by the following:

$$u^{\alpha}(x,\ 0)\leqq 0 \quad \text{for } x\in R^n, \qquad \alpha=1,\ \dots,\ N.$$

(ii) An analogue of Theorem 1 for a single parabolic inequality has recently been given by the present authors [4] (Theorem 1.1).

(iii) Theorem 1 can be generalized to give comparison theorems for weakly coupled nonlinear parabolic systems of the form

$$F_{\alpha}\left(x,\ t,\ u^1,\ \dots,\ u^N,\ \frac{\partial u^{\alpha}}{\partial x_1},\ \dots,\ \frac{\partial u^{\alpha}}{\partial x_n},\ \frac{\partial^2 u^{\alpha}}{\partial x_1^2},\ \frac{\partial^2 u^{\alpha}}{\partial x_1\partial x_2},\ \dots,\ \frac{\partial^2 u^{\alpha}}{\partial x_n^2}\right)-\frac{\partial u^{\alpha}}{\partial t}=0,$$

$$\alpha=1,\ \dots,\ N.$$

The special case where $\lambda=0$ and $\mu=1$ has been discussed in a more general setting by Besala [1].

§ 2. Decay of Solutions

In this section we shall deal with the asymptotic decay of solutions of the Cauchy problem for the weakly coupled parabolic system

$$L^\alpha[u^\alpha]+\sum_{\beta=1}^{N} c^{\alpha\beta}(x, t)u^\beta=0, \qquad \alpha=1, \dots, N, \tag{2.1}$$

$$L^\alpha \equiv \sum_{i,j=1}^{n} a_{ij}^\alpha(x, t)\frac{\partial^2}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i^\alpha(x, t)\frac{\partial}{\partial x_i}-\frac{\partial}{\partial t},$$

which is defined in the half-space $R^n \times [0, \infty)$ and satisfies there the following hypotheses B:

There exist constants $k_1>0$, $K_1>0$, $K_2 \geqq 0$, $K_3>0$, K_4, $\lambda \geqq 0$ and $\mu>0$ such that

$$k_1[\log(|x|^2+1)+1]^{-\lambda}(|x|^2+1)^{1-\mu}|\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}^\alpha(x, t)\xi_i\xi_j$$

$$\leqq K_1[\log(|x|^2+1)+1]^{-\lambda}(|x|^2+1)^{1-\mu}|\xi|^2,$$

$$|b_i^\alpha(x, t)| \leqq K_2(|x|^2+1)^{1/2}, \qquad i=1, \dots, n,$$

$$c^{\alpha\beta}(x, t) \geqq 0, \qquad \alpha \neq \beta,$$

$$\sum_{\beta=1}^{N} c^{\alpha\beta}(x, t) \leqq -K_3[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu+K_4,$$

for all $(x, t) \in R^n \times [0, \infty)$, all real n-vectors $\xi=(\xi_1, \dots, \xi_n)$ and $\alpha, \beta=1, \dots, N$.

By a solution of (2.1) we mean a system of N real valued functions $u^\alpha(x, t)$, $\alpha=1, \dots, N$, which are continuous in $R^n \times [0, \infty)$, continuously differentiable once with respect to t and twice with respect to x in $R^n \times (0, \infty)$, and satisfy the system (2.1) in $R^n \times (0, \infty)$.

The main result of this section is stated in the following theorem.

THEOREM. 2. *Suppose that hypotheses* B *hold. Let* $u^\alpha(x, t)$, $\alpha=1, \dots, N$, *be a solution of* (2.1) *with the properties*:

(a) *for any* $T>0$ *there exist positive numbers* M_T *and* k_T *such that*

$$|u^\alpha(x, t)| \leqq M_T \exp\{k_T[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu\}$$

for $(x, t) \in R^n \times (0, T]$, $\alpha=1, \dots, N$;

(b) *there exist positive numbers* M *and* k *such that*

$$|u^\alpha(x, 0)| \leqq M\exp\{k[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu\}$$

for $x \in R^n$, $\alpha = 1, \ldots, N$.

Suppose in addition that the following inequalities hold:

$$(2.2) \qquad 4K_1(\lambda+\mu)^2 k^2 + 2K_2(\lambda+\mu)nk - K_3 < 0,$$

and

$$(2.3) \qquad K_4 + 2[-k_1 n\mu + 2K_1\mu(1-\mu) + 4K_1\lambda]\gamma < 0 \qquad \text{if } 0 < \mu \leqq 1,$$

or

$$(2.3') \qquad K_4 + 2(4K_1\lambda - k_1 n\mu)\gamma - 4K_1(\lambda+\mu)^2\gamma^2 < 0 \qquad \text{if } \mu > 1,$$

where γ *is the positive root of the quadratic equation*

$$(2.4) \qquad 4K_1(\lambda+\mu)^2 y^2 + 2K_2(\lambda+\mu)ny - K_3 = 0.$$

Then $\lim_{t\to\infty} u^\alpha(x, t) = 0$, $\alpha = 1, \ldots, N$, *the convergence being of exponential order and uniform with respect to* $x \in R^n$.

PROOF. The following is an adaptation of the proof for the case of single parabolic equations [4]. We limit ourselves to the case $0 < \mu \leqq 1$; a parallel argument holds if $\mu > 1$.

Put $v^\alpha(x, t) = u^\alpha(x, t)e^{-K_4 t}$, $\alpha = 1, \ldots, N$. Then $v^\alpha(x, t)$ satisfy the system

$$L^\alpha[v^\alpha] + \sum_{\beta=1}^{n} \tilde{c}^{\alpha\beta}(x, t)v^\beta = 0 \text{ in } R^n \times (0, \infty), \qquad \alpha = 1, \ldots, N,$$

where $\tilde{c}^{\alpha\beta}(x, t) = c^{\alpha\beta}(x, t) - K_4\delta^{\alpha\beta}$, $\delta^{\alpha\beta}$ being the Kronecker symbols. It follows that

$$(2.5) \qquad \tilde{c}^{\alpha\beta}(x, t) \geqq 0, \qquad \alpha \neq \beta,$$

and

$$(2.6) \qquad \sum_{\beta=1}^{N} \tilde{c}^{\alpha\beta}(x, t) \leqq -K_3[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu.$$

We shall show that

$$(2.7) \qquad |v^\alpha(x, T_0)| \leqq M_0 \quad \text{for } x \in R^n, \qquad \alpha = 1, \ldots, N,$$

where

$$(2.8) \qquad T_0 = \frac{1}{2(\lambda+\mu)\sqrt{K_2^2 n^2 + 4K_1K_3}} \times \log \frac{K_3 k^{-1} - K_2(\lambda+\mu)n + (\lambda+\mu)\sqrt{K_2^2 n^2 + 4K_1K_3}}{K_3 k^{-1} - K_2(\lambda+\mu)n - (\lambda+\mu)\sqrt{K_2^2 n^2 + 4K_1K_3}},$$

and

$$M_0 = M \exp\left\{\frac{2K_1(\lambda+\mu)(2\lambda+2\mu+n)}{K_3k^{-1}-2K_2(\lambda+\mu)n-4K_1(\lambda+\mu)^2k}\right\}. \tag{2.9}$$

To do this, we introduce the comparison function

$$w(x, t) = M \exp\left\{k[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu\rho^{-\tau_0 t}\right.$$

$$\left.+\frac{2K_1(\lambda+\mu)(2\lambda+2\mu+n)}{\tau_0 \log\rho}(1-\rho^{-\tau_0 t})\right\},$$

where $\rho > 1$ is a parameter and

$$\tau_0 = \tau_0(\rho) = [K_3k^{-1}-2K_2(\lambda+\mu)n-4K_1(\lambda+\mu)^2k]\,(\log\rho)^{-1}.$$

It is easy to see from hypotheses B, (2.5) and (2.6) that

$$L^\alpha[w] + \sum_{\beta=1}^{N}\tilde{c}^{\alpha\beta}(x, t)w \leqq 0 \text{ in } R^n\times(0, \tau_0(\rho)^{-1}], \qquad \alpha=1, \dots, N.$$

Apply the maximum principle, Theorem 1, to the functions $w(x, t)\pm v^\alpha(x, t)$, $\alpha=1, \dots, N$. Then we conclude that $|v^\alpha(x, t)| \leqq w(x, t)$ in $R^n\times(0, \tau_0(\rho)^{-1}]$, $\alpha=1, \dots, N$. In particular,

$$|v^\alpha(x, \tau_0(\rho)^{-1})| \leqq M_1(\rho)\exp\{k\rho^{-1}[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu\}$$

for $x\in R^n$, $\alpha=1, \dots, N$, where

$$M_1(\rho) = M\exp\left\{\frac{2K_1(\lambda+\mu)(2\lambda+2\mu+n)}{\log\rho}(1-\rho^{-1})\tau_0(\rho)^{-1}\right\}.$$

Taking $t=\tau_0(\rho)^{-1}$ to be the initial time and repeating the above procedure we have

$$|v^\alpha(x, \tau_0(\rho)^{-1}+\tau_1(\rho)^{-1})| \leqq M_2(\rho)\exp\{k\rho^{-2}[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu\}$$

for $x\in R^n$, $\alpha=1, \dots, N$, where

$$\tau_1(\rho) = [K_3k^{-1}\rho-2K_2(\lambda+\mu)n-4K_1(\lambda+\mu)^2k\rho^{-1}]\,(\log\rho)^{-1}$$

and

$$M_2(\rho) = M\exp\left\{\frac{2K_1(\lambda+\mu)(2\lambda+2\mu+n)}{\log\rho}(1-\rho^{-1})(\tau_0(\rho)^{-1}+\rho^{-1}\tau_1(\rho)^{-1})\right\}.$$

Proceeding in this way we have in general

$$|v^\alpha(x, \sum_{j=0}^{p}\tau_j(\rho)^{-1})| \leqq M_{p+1}(\rho)\exp\{k\rho^{-p-1}[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu\} \tag{2.10}$$

for $x \in R^n$, $\alpha=1, \dots, N$, where

$$\tau_j(\rho)=[K_3 k^{-1}\rho^j - 2K_2(\lambda+\mu)n - 4K_1(\lambda+\mu)^2 k\rho^{-j}](\log \rho)^{-1} \tag{2.11}$$

and

$$M_{p+1}(\rho)=M\exp\left\{\frac{2K_1(\lambda+\mu)(2\lambda+2\mu+n)}{\log \rho}(1-\rho^{-1})\sum_{j=0}^{p}\rho^{-j}\tau_j(\rho)^{-1}\right\}, \tag{2.12}$$

$$p=0, 1, 2, \dots.$$

Using the inequality

$$\sum_{j=0}^{\infty}\rho^{-j}\tau_j(\rho)^{-1} \leqq \frac{1}{K_3k^{-1}-2K_2(\lambda+\mu)n-4K_1(\lambda+\mu)^2k}\,\frac{\log \rho}{1-\rho^{-1}}$$

which follows from (2.11), we see immediately from (2.12) that the sequence $\{M_\nu(\rho)\}$ is uniformly bounded by the constant M_0 defined in (2.9).

Observing that $\lim_{\varrho\to 1+0}\sum_{j=0}^{\infty}\tau_j(\rho)^{-1}=T_0$, T_0 being defined in (2.8), choose a $\rho_0>1$ such that if $1<\rho<\rho_0$

$$|v^\alpha(x, T_0)-v^\alpha(x, \sum_{j=0}^{\infty}\tau_j(\rho)^{-1})|<\varepsilon/2, \qquad \alpha=1, \dots, N,$$

where x is an arbitrary but fixed point of R^n. Fixing such a ρ find an integer P such that if $p>P$

$$|v^\alpha(x, \sum_{j=0}^{\infty}\tau_j(\rho)^{-1})-u(x, \sum_{j=0}^{p}\tau_j(\rho)^{-1})|<\varepsilon/2, \qquad \alpha=1, \dots, N.$$

For such choices of ρ and p

$$|v^\alpha(x, T_0)|<|v^\alpha(x, \sum_{j=0}^{p}\tau_j(\rho)^{-1})|+\varepsilon,$$

and taking (2.10) and the bounds $M_p(\rho)\leqq M_0$ into account,

$$|v^\alpha(x, T_0)|<M_0\exp\{k\rho^{-p-1}[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu\}+\varepsilon,$$

$\alpha=1, \dots, N$. Letting $p\to\infty$ and $\varepsilon\to 0$ we arrive at the inequality

$$|v^\alpha(x, T_0)|\leqq M_0, \qquad \alpha=1, \dots, N.$$

Since x is arbitrary we have thus established the required inequality (2.7).

In order to investigate the behavior of $v^\alpha(x, t)$ for $t>T_0$ we introduce another comparison function

$$w(x, t)=M_0\exp\{-\phi(t)[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu+\psi(t)\},$$

where

$$\phi(t)=\gamma \tanh [4K_1(\lambda+\mu)^2\gamma(t-T_0)]$$

and

$$\psi(t)=\frac{-k_1 n\mu+2K_1\mu(1-\mu)+4K_1\lambda}{2K_1(\lambda+\mu)^2}\log\cosh[4K_1(\lambda+\mu)^2\gamma(t-T_0)],$$

where γ denotes the positive root of (2.4). $W(x, t)$ is constructed so as to satisfy the differential inequalities $L^\alpha[W]+\sum_{\beta=1}^{N}\tilde{c}^{\alpha\beta}W\leqq 0$ in $R^n\times(T_0, \infty)$, $\alpha=1, \dots, N$, and the initial condition $W(x, T_0)=M_0$ on R^n. Applying the maximum principle, Theorem 1, to $W(x, t)\pm v^\alpha(x, t)$ we have $|v^\alpha(x, t)|\leqq W(x, t)$ in $R^n\times(T_0, \infty)$, $\alpha=1, \dots, N$, which amounts to

$$|v^\alpha(x, t)|\leqq M_0\{\cosh[4K_1(\lambda+\mu)^2\gamma(t-T_0)]\}^\beta\times$$
$$\times\exp\{-\gamma[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu\tanh[4K_1(\lambda+\mu)^2\gamma(t-T_0)]\}$$

for $(x, t)\in R^n\times(T_0, \infty)$, $\alpha=1, \dots, N$, where

$$\beta=\frac{-k_1 n\mu+2K_1\mu(1-\mu)+4K_1\lambda}{2K_1(\lambda+\mu)^2}.$$

This shows that, as $t\to\infty$, each $v^\alpha(x, t)$ behaves like $\exp\{2[-k_1 n\mu+2K_1\mu(1-\mu)+4K_1\lambda]\gamma t\}$. Noting that $u^\alpha(x, t)=v^\alpha(x, t)e^{K_4 t}$, we conclude that under conditions (2.2) and (2.3) (when $0<\mu\leqq 1$) each $u^\alpha(x, t)$ approaches zero exponentially as $t\to\infty$. It is obvious that the convergence is uniform with respect to $x\in R^n$. This proves the theorem.

Remarks. (i) An analogue of Theorem 2 for single parabolic equations has been proved in [4] (Theorem 2.1).

(ii) The special case of Theorem 2 in which $\lambda=0$ and $\mu=1$ has been studied by one of the authors [7] (Theorem 3).

§3. Growth of Solutions

The purpose of this section is to study the asymptotic growth of positive solutions of the weakly coupled system of parabolic inequalities

$$L^\alpha[u^\alpha]+\sum_{\beta=1}^{N}c^{\alpha\beta}(x, t)u^\beta\leqq 0, \qquad \alpha=1, \dots, N, \tag{3.1}$$

$$L^\alpha\equiv\sum_{i,j=1}^{n}a_{ij}^\alpha(x, t)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{n}b_i^\alpha(x, t)\frac{\partial}{\partial x_i}-\frac{\partial}{\partial t},$$

which is defined in the half-space $R^n\times[0, \infty)$ and satisfies there the following hypotheses C:

There exist constants $k_1>0$, $K_1>0$, $K_2\geqq 0$, $k_3>0$, $K_3>0$, $k_4\geqq 0$, $\lambda\geqq 0$ and $\mu\in(0, 1]$ such that

$$k_1[\log(|x|^2+1)+1]^{-\lambda}(|x|^2+1)^{1-\mu}|\xi|^2\leqq \sum_{i,j=1}^{n} a_{ij}^{\alpha}(x, t)\xi_i\xi_j$$

$$\leqq K_1[\log(|x|^2+1)+1]^{-\lambda}(|x|^2+1)^{1-\mu}|\xi|^2,$$

$$|b_i^{\alpha}(x, t)|\leqq K_2(|x|^2+1)^{1/2}, \qquad i=1, \dots, n,$$

$$c^{\alpha\beta}(x, t)\geqq 0, \qquad \alpha\neq\beta,$$

$$-k_3[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}+k_4\leqq \sum_{\beta=1}^{N} c^{\alpha\beta}(x, t)$$

$$\leqq K_3[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}$$

for all $(x, t)\in R^n\times[0, \infty)$, all real n-vectors $\xi=(\xi_1, \dots, \xi_n)$ and $\alpha, \beta=1, \dots, N$. We shall prove the following.

THEOREM 3. *Suppose that hypotheses* C *hold. Let* $u^{\alpha}(x, t)$, $\alpha=1, \dots, N$, *be real valued functions which are continuous in* $R^n\times[0, \infty)$, *continuously differentiable once with respect to* t *and twice with respect to* x *in* $R^n\times(0, \infty)$, *and satisfy the system* (3.1) *in* $R^n\times(0, \infty)$. *Let in addition the following conditions be satisfied:*

(a) *for each* $T>0$ *there exist positive numbers* M_T *and* k_T *such that*

$$u^{\alpha}(x, t)\geqq -M_T\exp\{k_T[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}\}$$

for $(x, t)\in R^n\times(0, T]$, $\alpha=1, \dots, N$;

(b) $u^{\alpha}(x, 0)\geqq 0$ *and* $u^{\alpha}(x, 0)\not\equiv 0$ *for* $x\in R^n$, $\alpha=1, \dots, N$.

Suppose that

$$k_4-2K_1(\lambda+\mu)n\gamma-4k_1\mu^2\gamma^2>0, \tag{3.2}$$

where γ *is the positive root of the quadratic equation in* y

$$4k_1\mu^2y^2-2(2\lambda K_1+K_2n)(\lambda+\mu)y-k_3=0. \tag{3.3}$$

Then $\lim_{t\to\infty} u^{\alpha}(x, t)=\infty$, $\alpha=1, \dots, N$, *the divergence being of exponential order and uniform on every compact of* R^n.

PROOF. Observe first that $u^{\alpha}(x, t)>0$ in $R^n\times(0, \infty)$, $\alpha=1, \dots, N$, according to the maximum principle, Theorem 1, and the strong maximum principle for weakly coupled parabolic systems [11]. It can be shown as in [8] that for each $t_0>0$ there exist positive numbers M and k such that

$$u^{\alpha}(x, t_0)\geqq M\exp\{-k[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}\}$$

for $x \in R^n$, $\alpha=1, \ldots, N$.

We introduce the comparison function $v(x, t)$ defined by

$$v(x, t)=M \exp\{-\phi(t)[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}+\psi(t)\},$$

where

$$\phi(t)=\frac{\gamma(\delta+k)e^{4\mu^2 k_1(\gamma+\delta)(t-t_0)}+\delta(k-\gamma)}{(\delta+k)e^{4\mu^2 k_1(\gamma+\delta)(t-t_0)}-(k-\gamma)},$$

and

$$\begin{aligned}\psi(t)=&(k_4-4k_1\mu^2\gamma^2)(t-t_0)\\ &-\frac{K_1(\lambda+\mu)}{2k_1\mu^2}\log\frac{(\delta+k)e^{4\mu^2 k_1(\gamma+\delta)(t-t_0)}-(k-\gamma)}{(\gamma+\delta)e^{4\mu^2 k_1\delta(t-t_0)}}\\ &-(\gamma-\delta)\log\frac{(\delta+k)e^{4\mu^2 k_1(\gamma+\delta)(t-t_0)}-(k-\gamma)}{e^{4\mu^2 k_1(\gamma+\delta)(t-t_0)}}\\ &+\frac{(\gamma+\delta)(k-\gamma)}{(\delta+k)e^{4\mu^2 k_1(\gamma+\delta)(t-t_0)}-(k-\gamma)}-(\delta-\gamma)\log(\gamma+\delta)-(k-\gamma).\end{aligned}$$

Here δ denotes the absolute value of the negative root of (3.3). It is not difficult to verify that $L^{\alpha}[v]+\sum_{\beta=1}^{N}c^{\alpha\beta}(x, t)v \geqq 0$ in $R^n \times (t_0, \infty)$ and $v(x, t_0)=M\exp\{-k[\log([x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}\}$ on R^n [4]. Now apply the maximum principle, Theorem 1, to the functions $v(x, t)-u^{\alpha}(x, t)$, $\alpha=1, \ldots, N$. Then it follows that $u^{\alpha}(x, t) \geqq v(x, t)$ in $R^n \times (t_0, \infty)$. Since $\phi(t)$ is bounded for $t>t_0$, the limiting behavior of $v(x, t)$ as $t \to \infty$ is determined by the factor $\exp\{\psi(t)\}$, which behaves just like $\exp\{(k_4-4k_1\mu^2\gamma^2-2K_1 n(\lambda+\mu)\gamma)t\}$ for large t. Consequently, each $u^{\alpha}(x, t)$ tends exponentially to infinity as $t \to \infty$; obviously the divergence is uniform on every compact x-set. Thus the proof is complete.

REMARKS. (i) An analogue of Theorem 3 for a single parabolic inequality has been proved in [4] (Theorem 3.1).

(ii) The special case of Theorem 3 in which $\lambda=0$, and $\mu=1$ has been obtained by one of the authors [8] (Theorem 2).

(iii) Let, in hypotheses C, the condition "$c^{\alpha\beta}(x, t) \geqq 0$, $\alpha \neq \beta$" be replaced by the more stringent "$c^{\alpha\beta}(x, t)>0$, $\alpha \neq \beta$". Then the conclusion of Theorem 3 remains true under the weaker initial condition:

(b′) There exists an α such that $u^{\alpha}(x, 0) \geqq 0$ and $u^{\alpha}(x, 0) \not\equiv 0$ for $x \in R^n$.

§4. Weakly Coupled Elliptic Systems

Inspired by Eidel'man and Porper [5], we present here Liouville type theorems for some weakly coupled elliptic systems with unbounded coefficients.

We begin by stating a Liouville theorem for the system

$$\sum_{i,j=1}^{n} a_{ij}^{\alpha}(x)\frac{\partial^2 u^\alpha}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i^\alpha(x)\frac{\partial u^\alpha}{\partial x_i}+\sum_{\beta=1}^{N} c^{\alpha\beta}(x)u^\beta=0, \qquad \alpha=1, \ldots, N, \tag{4.1}$$

whose coefficients are defined and real valued in R^n and satisfy there following hypotheses:

There exist constants $k_1>0$, $K_1>0$, $K_2\geqq 0$, $k_3>0$, $K_3>0$, $k_4\geqq 0$, $\lambda\geqq 0$ and $\mu\in(0, 1]$ such that

$$k_1[\log(|x|^2+1)+1]^{-\lambda}(|x|^2+1)^{1-\mu}|\xi|^2\leqq \sum_{i,j=1}^{n} a_{ij}^\alpha(x)\xi_i\xi_j$$

$$\leqq K_1[\log(|x|^2+1)+1]^{-\lambda}(|x|^2+1)^{1-\mu}|\xi|^2,$$

$$|b_i^\alpha(x)|\leqq K_2(|x|^2+1)^{1/2}, \qquad i=1, \ldots, n,$$

$$c^{\alpha\beta}(x)>0, \qquad \alpha\neq\beta,$$

$$-k_3[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}+k_4\leqq \sum_{\beta=1}^{N} c^{\alpha\beta}(x)$$

$$\leqq K_3[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}$$

for all $x\in R^n$, all real n-vectors $\xi=(\xi_1, \ldots, \xi_n)$ and α, $\beta=1, \ldots, N$.

THEOREM 4. *Suppose that the above hypotheses hold. Suppose in addition that*

$$k_4-2K_1(\lambda+\mu)n\gamma-4k_1\mu^2\gamma^2>0, \tag{4.2}$$

where γ is the positive root of the quadratic equation

$$4k_1\mu^2y^2-2(2\lambda K_1+K_2 n)(\lambda+\mu)y-k_3=0.$$

Let $u^\alpha(x)$, $\alpha=1, \ldots, N$, be a solution of (4.1) *in R^n with the properties:*

(i) *there exist positive constants M and k such that*

$$|u^\alpha(x)|\leqq M\exp\{k[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}\}$$

for $x\in R^n$, $\alpha=1, \ldots, N$;

(ii) *$u^\alpha(x)\geqq 0$, or $u^\alpha(x)\leqq 0$, for $x\in R^n$,* $\quad\alpha=1, \ldots, N$.

Then $u^\alpha(x)\equiv 0$ for $x\in R^n$, $\alpha=1, \ldots, N$.

PROOF. We observe that $u^{\alpha}(x)$, $\alpha=1, \ldots, N$, is the unique solution to the following Cauchy problem

$$(4.3)\qquad \sum_{i,j=1}^{n} a_{ij}^{\alpha}(x)\frac{\partial^2 v^{\alpha}}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i^{\alpha}(x)\frac{\partial v^{\alpha}}{\partial x_i}+\sum_{\beta=1}^{N} a^{\alpha\beta}(x)v^{\beta}-\frac{\partial v^{\alpha}}{\partial t}=0$$

$$\text{for } (x, t)\in R^n\times(0, \infty).$$

$$v^{\alpha}(x, 0)=u^{\alpha}(x) \quad \text{for } x\in R^n, \qquad \alpha=1, \ldots, N.$$

Let $u^{\alpha}(x)\geqq 0$ for $x\in R^n$, $\alpha=1,\ldots, N$. If $u^{\alpha}(x)\not\equiv 0$ for some α, then it follows from Theorem 3 (Remark (iii)) that $\lim_{t\to\infty} v^{\alpha}(x, t)\equiv u^{\alpha}(x)=\infty$ for $x\in R^n$, $\alpha=1, \ldots, N$. This is obviously a contradiction. The case where $u^{\alpha}(x)\leqq 0$ for $x\in R^n$ can be treated similarly.

EXAMPLE 1. That condition (4.2) is sharp is illustrated by the elliptic system

$$\Delta u^1-(|x|^2-n+1)u^1+u^2=0,$$

$$\Delta u^2+u^1-(|x|^2-n+1)u^2=0,$$

which has a solution $u^1(x)=u^2(x)=e^{-|x|^2/2}$, where Δ is the n-dimensional Laplace operator. For this system, $\lambda=0$, $\mu=1$, $k_1=K_1=1$, $K_2=0$, $k_3=1$, $k_4=n+1$, $K_3=n$ and $\gamma=1/2$. But condition (4.2) is violated: $k_4-2K_1(\lambda+\mu)n\gamma-4k_1\mu^2\gamma^2=0$.

Our final theorem concerns the complex valued entire solutions of the elliptic system containing a complex parameter ω:

$$(4.4)\qquad \sum_{i,j=1}^{n} a_{ij}^{\alpha}(x)\frac{\partial^2 u^{\alpha}}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i^{\alpha}(x)\frac{\partial u^{\alpha}}{\partial x_i}+\sum_{\beta=1}^{N} c^{\alpha\beta}(x)u^{\beta}=\omega u^{\alpha}, \quad \alpha=1, \ldots, N.$$

Assume that there exist constants $k_1>0$, $K_1>0$, $K_2\geqq 0$, $K_3>0$, K_4, $\lambda\geqq 0$ and $\mu>0$ such that

$$k_1[\log(|x|^2+1)+1]^{-\lambda}(|x|^2+1)^{1-\mu}|\xi|^2\leqq \sum_{i,j=1}^{n} a_{ij}^{\alpha}(x)\xi_i\xi_j$$

$$\leqq K_1[\log(|x|^2+1)+1]^{-\lambda}(|x|^2+1)^{1-\mu}|\xi|^2,$$

$$|b_i^{\alpha}(x)|\leqq K_2(|x|^2+1)^{1/2}, \qquad i=1, \ldots, n.$$

$$c^{\alpha\beta}(x)\geqq 0, \qquad \alpha\neq\beta,$$

$$\sum_{\beta=1}^{N} c^{\beta\alpha}(x)\leqq -K_3[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}+K_4,$$

for all $x\in R^n$, all real n-vectors $\xi=(\xi_1, \ldots, \xi_n)$ and α, $\beta=1, \ldots, N$. Assume furthermore that

$$K_4+2[-k_1n\mu+2K_1\mu(1-\mu)+4K_1\lambda]\gamma<0 \qquad \text{if } 0<\mu\leqq 1, \tag{4.5}$$

or

$$K_4+2(4K_1\lambda-k_1n\mu)\gamma-4K_1(\lambda+\mu)^2\gamma^2<0 \qquad \text{if } \mu>1, \tag{4.5$'$}$$

where γ is the positive root of the quadratic equation

$$4K_1(\lambda+\mu)^2y^2+2K_2(\lambda+\mu)ny-K_3=0.$$

THEOREM 5. *Suppose the above hypotheses hold. Let $u^\alpha(x)$, $\alpha=1, \dots, N$, be a complex valued solution of the system* (4.4) *in R^n with the property that*

$$|\mathrm{Re}\, u^\alpha(x)|\leqq M\exp\{k[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu\}, \tag{4.6}$$

$$|\mathrm{Im}\, u^\alpha(x)|\leqq M'\exp\{k'[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu\}$$

for $x\in R^n$, $\alpha=1, \dots, N$, where M, M', k, k' are positive constants. If $\mathrm{Re}\,\omega\leqq 0$ *and $k<\gamma$, then $u^\alpha(x)\equiv 0$ for $x\in R^n$, $\alpha=1, \dots, N$.*

PROOF. Putting $\omega=\sigma_1+\sqrt{-1}\,\sigma_2$, and $u^\alpha(x)=u_1^\alpha(x)+\sqrt{-1}\,u_2^\alpha(x)$, $\alpha=1, \dots, N$, we form the functions

$$v^\alpha(x, t)=e^{-\sigma_1 t}[u_1^\alpha(x)\cos\sigma_2 t+u_2^\alpha(x)\sin\sigma_2 t], \qquad \alpha=1, \dots, N.$$

It is easy to see [9] that $v^\alpha(x, t)$ satisfy the parabolic system (4.3) in $R^n\times(0, \infty)$ and the initial condition $v^\alpha(x, 0)=u_1^\alpha(x)$ on R^n, $\alpha=1, \dots, N$. According to Theorem 2, $\lim_{t\to\infty} v^\alpha(x, t)=0$ for $x\in R^n$, $\alpha=1, \dots, N$. Since $\sigma_1\leqq 0$ by hypothesis, all the functions $u_1^\alpha(x)$, $u_2^\alpha(x)$ must vanish identically in R^n, thereby completing the proof of the theorem.

EXAMPLE 2. Consider the elliptic system

$$\Delta u^1-(|x|^2+n+1)u^1+u^2=0, \tag{4.7}$$

$$\Delta u^2+u^1-(|x|^2+n+1)u^2=0.$$

For this system, $\lambda=0$, $\mu=1$, $k_1=K_1=1$, $K_2=0$, $K_3=1$, $K_4=1-n$, and $\gamma=1/2$. Hence, by Theorem 5, the solution of (4.7) satisfying

$$|u^1(x)|\leqq Me^{k|x|^2} \quad\text{and}\quad |u^2(x)|\leqq Me^{k|x|^2} \quad\text{with}\quad k<1/2, \tag{4.8}$$

vanishes identically in R^n. Since $u^1(x)=u^2(x)=e^{|x|^2/2}$ is a solution of (4.7), it is not possible to replace k by 1/2 in (4.8). This shows that the growth restriction (4.6) with $k<\gamma$ cannot be relaxed.

REMARK. Among numerous papers dealing with Liouville theorems for elliptic equations we refer in particular to the papers by Besala [2], Besala and Ugowski [3], Eidel'man and Porper [5], Krzyżański [6], and Oddson [10].

References

[1] P. Besala, On solutions of Fourier's first problem for a system of non-linear parabolic equations in an unbounded domain, Ann. Polon. Math. **13** (1963), 247–265.

[2] P. Besala, On solutions of non-linear second order elliptic equations defined in unbounded domains, Atti Sem. Mat. Fis. Univ. Modena **13** (1964), 74–86.

[3] P. Besala and H. Ugowski, Some uniqueness theorems for solutions of parabolic and elliptic partial differential equations in unbounded regions, Colloq. Math. **20** (1969), 127–141.

[4] L.-S. Chen, T. Kuroda and T. Kusano, Some parabolic equations with unbounded coefficients, Funkc. Ekvac., **16** (1973), 1–28.

[5] S. D. Eidel'man and F. O. Porper, On the behavior of solutions of second order parabolic equations with dissipation, Differencial'nye Uravneija **7** (1971), 1684–1695. (Russian)

[6] M. Krzyżański, Sur le problème de Dirichlet pour l'équation linéaire du type elliptique dans un domaine non borné, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. Ser. VIII **4** (1948), 408–416.

[7] T. Kusano, Asymptotic behavior of solutions of parabolic differential equations with unbounded coefficients, J. Sci. Hiroshima Univ. Ser. A–I **33** (1969), 151–159.

[8] T. Kusano, On the growth of solutions of parabolic differential inequalities with unbounded coefficients, Funkc. Ekvac. **13** (1970) 45–50.

[9] V. G. Miŝin, On the spectrum of an elliptic equation, Mat. Zametki **11** (1972), 577–583. (Russian)

[10] J. K. Oddson, Phragmèn-Lindelöf and comparison theorems for elliptic equations with mixed boundary conditions, Arch. Rational Mech. Anal. **26** (1967), 316–334.

[11] M. H. Protter and H. F. Weinberger, Maximum Principles in Differential Equations, Prentice-Hall, Englewood Cliffs, N. J. 1967.

[12] J. Szarski, Differential Inequalities, Monografie Matematyczne, Tom 43, Warszawa 1965.

[13] W. Walter, Differential and Integral Inequalities, Springer, Berlin 1970.

Department of Mathematics
Hiroshima University, Hiroshima, Japan,
Mathematical Institute
Tôhoku University, Sendai, Japan
and
Department of Mathematics
National Central University, Chung-Li, Taiwan