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Fibred Riemannian spaces with contact structure

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Introduction

Theory of foliated Riemannian manifolds has been studied by many authors in the name of fibred spaces ([3]) or foliated manifolds ([8], [14]) by use of Riemannian submersions ([7]). K. Ogiue ([5], [6]) studied relations between an almost contact structure and an almost complex structure induced in the base space of fibering. On the other hand, A. Morimoto ([4]) defined an almost complex structure in the product of two almost contact spaces. Y. Tashiro and the present author ([12]) have recently induced an almost complex structure in the total space of a fibred Riemannian space, the base space and each fibre of which are almost contact, and investigated relations among their structures.

The purpose of this paper is to study fibred spaces with almost contact metric structure induced from the base space with almost complex structure and each fibre with almost contact metric structure of general dimensions. A typical example of these is the Hopf fibering π: $S^{4n+3}(1) \to QP(n)$ with totally geodesic fibre S^3 (cf. [2], [11]).

In §1, we shall summarize fundamental properties and known results of the fibred Riemannian space. We shall induce, in §§2 and 3, an almost contact metric structure on the total space by use of the almost complex structure on the base space and almost contact metric structure on each fibre and discuss relations of them. §4 is devoted to the study of space form and we shall prove that the base space is locally Euclidean if the total space is Sasakian space form with conformal fibres. An example having this property will be given. In the last section, we shall investigate relations between the integrability of the almost complex structure and the normality of the induced almost contact metric structure on the total space by use of Nijenhuis tensors.

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§1. Fibred Riemannian spaces

Let $\{\tilde{M}, M, \tilde{g}, \pi\}$ be a fibred Riemannian space, that is, $\{\tilde{M}, \tilde{g}\}$ is an m-dimensional total space with Riemannian metric $\tilde{g}$, M an n-dimensional base space, π: $\tilde{M} \to M$ the projection with maximum rank n. The fibre passing through a point

$\tilde{P}\in\tilde{M}$ is denoted by $\bar{M}(\tilde{P})$ or generally $\bar{M}$ and the metric tensor $\tilde{g}$ is projectable.

Manifolds, geometric objects and mappings we deal with are supposed to be of class C^∞ and to be connected. Throughout this paper the range of indices are as follows:

$$h, i, j, k, l = 1, 2, \dots, m,$$
$$a, b, c, d, e = 1, 2, \dots, n,$$
$$\alpha, \beta, \gamma, \delta, \varepsilon = n+1, \dots, m,$$
$$A, B, C, D, E = 1, 2, \dots, m,$$
$$P, Q, R, S = 1, 2, \dots, m, m+1,$$
$$\sharp = m+1.$$

The summation convention on repeated indices will be used with respect to their own ranges.

We take a coordinate neighborhoods $(\tilde{U}, z^h)$ in $\tilde{M}$ and (U, x^a) in M such that $\pi(\tilde{U}) = U$, where z^h and x^a are coordinates in $\tilde{U}$ and U respectively. Then the projection π: $\tilde{M}\to M$ can be expressed by equations

$$x^a = x^a(z^h) \tag{1.1}$$

which are differentiable functions of coordinates z^h in $\tilde{U}$ with Jacobian $(\partial x^a/\partial z^i)$ of maximum rank n. Take a fibre $\bar{M}$ such that $\bar{M}\cap\tilde{U}\neq\phi$. Then there are in $\bar{M}\cap\tilde{U}$ local coordinates y^α, and (x^a, y^α) form a coordinate system in $\tilde{U}$.

If we put

$$E_i{}^a = \partial x^a/\partial z^i \quad \text{and} \quad C^h{}_\alpha = \partial z^h/\partial y^\alpha, \tag{1.2}$$

then $E_i{}^a$ are components of a local covector field E^a defined in $\tilde{U}$ for each fixed index a, and $C^h{}_\alpha$ are those of a vector field C_α for each fixed index α. The vector fields C_α form a natural frame tangent to $\bar{M}(\tilde{P})$ and

$$E_i{}^a C^i{}_\beta = 0. \tag{1.3}$$

We put $\tilde{g} = (\tilde{g}_{ji})$ in $(\tilde{U}, z^h)$ and $(\tilde{g}^{ji}) = (\tilde{g}_{ji})^{-1}$. The components of the induced metric tensor $\bar{g}$ of the fibre $\bar{M}$ are then given by

$$\bar{g}_{\gamma\beta} = \tilde{g}_{ji} C^j{}_\gamma C^i{}_\beta. \tag{1.4}$$

If we put

$$g_{cb} = \tilde{g}_{ji} E^j{}_c E^i{}_b \qquad \text{and} \quad (g^{ba}) = (g_{ba})^{-1}, \tag{1.5}$$

then $g = (g_{cb})$ is the projection of $\tilde{g}$ and the metric tensor of the base space M. We obtain

$$E^h{}_a = \tilde{g}^{hi} g_{ab} E_i{}^b,$$

and we put

$$C_i{}^\alpha = \tilde{g}_{ih} \bar{g}^{\alpha\beta} C^h{}_\beta.$$

The frame (E^a, C^α) is dual to (E_b, C_β). We write (E_B) for (E_b, C_β) and (E^A) for (E^a, C^α), if necessary.

Denoting by $\mathscr{L}_\beta$ the Lie derivation with respect to the vector field C_β, we have

$$\mathscr{L}_\beta E^a = 0, \quad \mathscr{L}_\beta C_\alpha = 0, \quad \mathscr{L}_\beta E_b = -P_{b\beta}{}^\alpha C_\alpha, \quad \mathscr{L}_\beta C^\alpha = P_{c\beta}{}^\alpha E^c, \tag{1.6}$$

where $P_{c\beta}{}^\alpha$ are local functions in $\tilde{U}$

Denote by $\tilde{\nabla}$ the Riemannian connection of the total space $\{\tilde{M}, \tilde{g}\}$. It is known that $\tilde{\nabla}$ is projectable and the projection ∇ is the Riemannian connection of the base space $\{M, g\}$, components of which are given by

$$\Gamma^a_{cb} = (1/2) g^{ae}(\partial_c g_{be} + \partial_b g_{ce} - \partial_e g_{bc}).$$

We have the following equations (see [3], [12]):

$$\begin{aligned} \tilde{\nabla}_j E^h{}_b &= \Gamma^a_{c\,b} E_j{}^c E^h{}_a - L_{cb}{}^\alpha E_j{}^c C^h{}_\alpha + L_b{}^a{}_\gamma C_j{}^\gamma E^h{}_a - h_\gamma{}^\alpha{}_b C_j{}^\gamma C^h{}_\alpha, \\ \tilde{\nabla}_j C^h{}_\beta &= L_c{}^a{}_\beta E_j{}^c E^h{}_a - (h_\beta{}^\alpha{}_c - P_{c\beta}{}^\alpha) E_j{}^c C^h{}_\alpha + h_{\gamma\beta}{}^a C_j{}^\gamma E^h{}_a + \bar{\Gamma}^\alpha_{\gamma\beta} C_j{}^\gamma C^h{}_\alpha, \end{aligned} \tag{1.7}$$

$$\begin{aligned} \tilde{\nabla}_j E_i{}^a &= -\Gamma^a_{cb} E_j{}^c E_i{}^b - L_c{}^a{}_\beta (E_j{}^c C_i{}^\beta + C_j{}^\beta E_i{}^c) - h_{\gamma\beta}{}^a C_j{}^\gamma C_i{}^\beta, \\ \tilde{\nabla}_j C_i{}^\alpha &= L_{cb}{}^\alpha E_j{}^c E_i{}^b + (h_\beta{}^\alpha{}_c - P_{c\beta}{}^\alpha) E_j{}^c C_i{}^\beta + h_\gamma{}^\alpha{}_b C_j{}^\gamma E_i{}^b - \bar{\Gamma}^\alpha_{\gamma\beta} C_j{}^\gamma C_i{}^\beta, \end{aligned} \tag{1.8}$$

where $L_{cb}{}^\alpha$, $h_{\gamma\beta}{}^a$ and $\bar{\Gamma}^\alpha_{\gamma\beta}$ are local functions in $\tilde{U}$ and

$$L_c{}^a{}_\beta = L_{cb}{}^\alpha g^{ba} \bar{g}_{\alpha\beta}, \quad h_\gamma{}^\alpha{}_b = h_{\gamma\beta}{}^a \bar{g}^{\beta\alpha} g_{ba}.$$

Putting $\tilde{\nabla}_c = E^j{}_c \tilde{\nabla}_j$ and $\tilde{\nabla}_\gamma = C^j{}_\gamma \tilde{\nabla}_j$, we obtain

$$\tilde{\nabla}_\gamma E_b = L_b{}^a{}_\gamma E_a - h_\gamma{}^\alpha{}_b C_\alpha, \quad \tilde{\nabla}_\gamma C_\beta = h_{\gamma\beta}{}^a E_a + \bar{\Gamma}^\alpha_{\gamma\beta} C_\alpha \tag{1.9}$$

by use of (1.7). Hence $h_{\gamma\beta}{}^a$ are components of the second fundamental tensor with respect to the normal vector E_a of each fibre $\bar{M}$, $\bar{\Gamma}^\alpha_{\gamma\beta}$ are coefficients of the Riemannian connection $\bar{\nabla}$ of the induced metric $\bar{g}$ in $\bar{M}$ and $L_{cb}{}^\alpha$ are coefficients of the normal connection of $\bar{M}$. Therefore we see that

$$h_{\gamma\beta}{}^a = h_{\beta\gamma}{}^a, \tag{1.10}$$

$$\bar{\Gamma}^\alpha_{\gamma\beta} = (1/2)\bar{g}^{\alpha\varepsilon}(\partial_\gamma \bar{g}_{\beta\varepsilon} + \partial_\beta \bar{g}_{\gamma\varepsilon} - \partial_\varepsilon \bar{g}_{\gamma\beta}) \tag{1.11}$$

and

$$L_{cb}{}^{\alpha} + L_{bc}{}^{\alpha} = 0. \tag{1.12}$$

Denoting by $\mathscr{L}_c$ the Lie derivation with respect to E_c, we have

$$\begin{aligned} &\mathscr{L}_c E_b = 2L_{bc}{}^{\alpha} C_\alpha, \qquad \mathscr{L}_c C_\beta = P_{c\beta}{}^{\alpha} C_\alpha, \\ &\mathscr{L}_c E^a = 0, \qquad \mathscr{L}_c C^\alpha = 2L_{cb}{}^{\alpha} E^b - P_{c\beta}{}^{\alpha} C^\beta. \end{aligned} \tag{1.13}$$

§2. Almost contact structure in a fibred Riemannian space

We consider a fibred Riemannian space $\tilde{M}$ such that the base space M is an almost complex space and that each fibre $\bar{M}$ is an almost contact space. Denoting the almost complex structure of M and its lift in the total space $\tilde{M}$ by J which is independent of the fibre and the almost contact structure of each fibre $\bar{M}$ by $(\bar{\phi}, \bar{\xi}, \bar{\eta})$ which is in general dependent on points of the base space M. The structure $(\bar{\phi}, \bar{\xi}, \bar{\eta})$ satisfies the equations

$$\bar{\phi}^2 = -I + \bar{\eta} \otimes \bar{\xi}, \quad \bar{\phi}(\bar{\xi}) = 0, \quad \bar{\eta} \otimes \bar{\phi} = 0, \quad \bar{\eta}(\bar{\xi}) = 1.$$

If we define

$$\begin{aligned} &\tilde{\phi} = J_{ba} E^b \otimes E^a + \bar{\phi}_{\beta\alpha} C^\beta \otimes C^\alpha, \\ &\tilde{\eta} = (0, \bar{\eta}_\alpha), \quad \tilde{\xi} = {}^t(0, \bar{\xi}^\alpha), \end{aligned} \tag{2.1}$$

${}^tA$ being the transposed matrix of A, then we can easily see that

$$\tilde{\phi}^2 = -I + \tilde{\eta} \otimes \tilde{\xi}, \quad \tilde{\phi}(\tilde{\xi}) = 0, \quad \tilde{\eta} \otimes \tilde{\phi} = 0, \quad \tilde{\eta}(\tilde{\xi}) = 1,$$

where I is the identity map of degree m. Therefore $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ is an almost contact structure on the fibred Riemannian space $\tilde{M}$, which will be called the *induced* one. We state the following:

PROPOSITION 2.1. *Let M be an almost complex space and each fibre $\bar{M}$ is an almost contact space. Then the fibred Riemannian space $\tilde{M}$ admits an almost contact structure.*

Moreover, if M is an almost Hermitian manifold with the almost Hermitian metric g and$(\bar{M}, \bar{g})$ is an almost contact metric manifold, then the Riemannian metric $\tilde{g}$ on the fibred Riemannian space is defined by

$$\tilde{g} = g_{ba} E^b \otimes E^a + \bar{g}_{\beta\alpha} C^\beta \otimes C^\alpha, \tag{2.2}$$

and $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ becomes an almost contact metric structure.

By means of (1.7), (1.8) and (2.1), we can derive the equations

$$(2.3)\qquad (\tilde{\nabla}_c\tilde{\Phi})E_b=(\nabla_cJ_b{}^e)E_e+(L_{cb}{}^\beta\bar{\Phi}_\beta{}^\alpha-L_{ca}{}^\alpha J_b{}^a)C_\alpha,$$

$$(2.4)\qquad (\tilde{\nabla}_c\tilde{\Phi})C_\alpha=(L_c{}^b{}_\gamma\bar{\Phi}_\alpha{}^\gamma-L_c{}^a{}_\alpha J_a{}^b)E_b+({}^*\nabla_c\bar{\Phi}_\alpha{}^\gamma)C_\gamma,$$

$$(2.5)\qquad (\tilde{\nabla}_\gamma\tilde{\Phi})E_b=({}^{**}\nabla_\gamma J_b{}^a)E_a+(h_\gamma{}^\lambda{}_b\bar{\Phi}_\lambda{}^\mu-h_\lambda{}^\mu{}_aJ_b{}^a)C_\mu,$$

$$(2.6)\qquad (\tilde{\nabla}_\gamma\tilde{\Phi})C_\alpha=(h_{\gamma\beta}{}^b\bar{\Phi}_\alpha{}^\beta-h_{\gamma\alpha}{}^aJ_a{}^b)E_b+(\bar{\nabla}_\gamma\bar{\Phi}_\alpha{}^\beta)C_\beta,$$

$$(2.7)\qquad \tilde{\nabla}_c\tilde{\xi}=(L_c{}^b{}_\alpha\xi^\alpha)E_b+({}^*\nabla_c\xi^\alpha)C_\alpha,$$

$$(2.8)\qquad \tilde{\nabla}_\gamma\tilde{\xi}=(h_{\gamma\alpha}{}^b\xi^\alpha)E_b+(\bar{\nabla}_\gamma\xi^\mu)C_\mu,$$

$$(2.9)\qquad \tilde{\nabla}_c\tilde{\eta}=(L_{cb}{}^\alpha\bar{\eta}_\alpha)E^b+({}^*\nabla_c\bar{\eta}_\alpha)C^\alpha,$$

$$(2.10)\qquad \tilde{\nabla}_\gamma\tilde{\eta}=(h_\gamma{}^\alpha{}_b\bar{\eta}_\alpha)E^b+(\bar{\nabla}_\gamma\bar{\eta}_\mu)C^\mu,$$

where we have put

$$(2.11)\qquad \nabla_cJ_b{}^e=\partial_cJ_b{}^e+\Gamma_{cd}^eJ_b{}^d-\Gamma_{cb}^aJ_a{}^e,$$

$$(2.12)\qquad {}^*\nabla_c\bar{\Phi}_\alpha{}^\gamma=\partial_c\bar{\Phi}_\alpha{}^\gamma+Q_{c\beta}{}^\gamma\bar{\Phi}_\alpha{}^\beta-Q_{c\alpha}{}^\beta\bar{\Phi}_\beta{}^\gamma,$$

$$(2.13)\qquad {}^*\nabla_c\xi^\alpha=\partial_c\xi^\alpha+Q_{c\beta}{}^\alpha\xi^\beta,$$

$$(2.14)\qquad {}^*\nabla_c\bar{\eta}_\beta=\partial_c\bar{\eta}_\beta-Q_{c\beta}{}^\alpha\bar{\eta}_\alpha,$$

$$(2.15)\qquad {}^{**}\nabla_\gamma J_b{}^a=\partial_\gamma J_b{}^a+L_d{}^a{}_\gamma J_b{}^d-L_b{}^d{}_\gamma J_d{}^a,$$

$$(2.16)\qquad \bar{\nabla}_\gamma\bar{\Phi}_\alpha{}^\beta=\partial_\gamma\bar{\Phi}_\alpha{}^\beta+\bar{\Gamma}_{\gamma\mu}^\beta\bar{\Phi}_\alpha{}^\mu-\bar{\Gamma}_{\gamma\alpha}^\mu\bar{\Phi}_\mu{}^\beta,$$

$$(2.17)\qquad \bar{\nabla}_\gamma\xi^\mu=\partial_\gamma\xi^\mu+\bar{\Gamma}_{\gamma\alpha}^\mu\xi^\alpha,$$

$$(2.18)\qquad \bar{\nabla}_\gamma\bar{\eta}_\beta=\partial_\gamma\bar{\eta}_\beta-\bar{\Gamma}_{\gamma\beta}^\alpha\bar{\eta}_\alpha,$$

$$(2.19)\qquad Q_{c\beta}{}^\gamma=P_{c\beta}{}^\gamma-h_\beta{}^\gamma{}_c.$$

An almost contact metric structure is said to be *contact* if

$$d\tilde{\eta}(\tilde{X},\tilde{Y})=\tilde{g}(\tilde{\Phi}\tilde{X},\tilde{Y})$$

for any vector fields $\tilde{X}$ and $\tilde{Y}$ in $\tilde{M}$ ([10]).

If we put $\tilde{\theta}(\tilde{X},\tilde{Y})=\tilde{g}(\tilde{\Phi}\tilde{X},\tilde{Y})$, $\bar{\theta}(\bar{X},\bar{Y})=\bar{g}(\bar{\Phi}\bar{X},\bar{Y})$ and $\Omega(X,Y)=g(JX,Y)$ for vector fields X, Y in M and $\bar{X}$, $\bar{Y}$ in $\bar{M}$, then we get

$$(2.20)\qquad (d\tilde{\theta})^H=d\Omega,$$

$$(2.21)\qquad (d\tilde{\theta})E_c\otimes E_b\otimes C_\alpha=2L_{cb}{}^\beta\bar{\Phi}_{\beta\alpha}.$$

$$(2.22)\qquad (d\tilde{\theta})E_c\otimes C_\alpha\otimes E_b=2{}^{**}\nabla_\alpha J_{cb},$$

$$(2.23)\qquad (d\tilde{\theta})E_c\otimes C_\beta\otimes C_\alpha={}^*\nabla_c\bar{\Phi}_{\beta\alpha}+h_{\beta\gamma c}\bar{\Phi}_\alpha{}^\gamma+h_\alpha{}^\gamma{}_c\bar{\Phi}_{\gamma\beta},$$

$$(2.24)\qquad (d\tilde{\theta})C_\gamma\otimes E_b\otimes E_a=2L_{ba}{}^\beta\bar{\Phi}_{\beta\gamma},$$

(2.25) $$(d\tilde{\theta})^V = d\bar{\theta},$$

(2.26) $$(d\tilde{\eta})E_c \otimes E_b = 2L_{cb}{}^{\alpha}\bar{\eta}_{\alpha},$$

(2.27) $$(d\tilde{\eta})E_c \otimes C_{\alpha} = {}^*\nabla_c\bar{\eta}_{\alpha} - h_{\alpha}{}^{\gamma}{}_c\bar{\eta}_{\gamma},$$

(2.28) $$(d\tilde{\eta})^V = d\bar{\eta}$$

by use of (2.1) and (2.3)$\sim$(2.10), where A^H and A^V are horizontal part and vertical part of a form A respectively. If the induced structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is contact, then we obtain $d\bar{\theta}=0$, $(d\tilde{\eta})E_c \otimes E_b = 2J_{cb}$, $(d\tilde{\eta})E_c \otimes C_{\alpha}=0$ and $(d\tilde{\eta})C_{\gamma} \otimes C_{\mu} = \bar{\phi}_{\gamma\mu}$. Therefore we can state

PROPOSITION 2.2. *A necessary and sufficient condition for the induced almost contact metric structure on $\tilde{M}$ to be contact is that the base space M is almost kaehlerian, each fibre $\bar{M}$ is contact, and the equations ${}^*\nabla_b\bar{\eta}_{\gamma} = h_{\gamma}{}^{\alpha}{}_b\bar{\eta}_{\alpha}$ and $L_{cb}{}^{\alpha}\bar{\eta}_{\alpha} = J_{cb}$ hold.*

If the induced almost contact metric structure on $\tilde{M}$ is K-contact, that is, $\tilde{\theta}(\tilde{X}, \tilde{Y}) = (\tilde{\nabla}_{\tilde{X}}\tilde{\eta})\tilde{Y}$, then we obtain

(2.29) $$\tilde{\phi}E_c = (L_{cb}{}^{\alpha}\bar{\eta}_{\alpha})E^b + ({}^*\nabla_c\bar{\eta}_{\alpha})C^{\alpha},$$

(2.30) $$\tilde{\phi}C_{\gamma} = (h_{\gamma}{}^{\alpha}{}_b\bar{\eta}_{\alpha})E^b + (\bar{\nabla}_{\gamma}\bar{\eta}_{\mu})C^{\mu},$$

and state the following with the aid of (2.1).

PROPOSITION 2.3. *The fibred almost contact space $\tilde{M}$ is a K-contact manifold if and only if the following conditions are satisfied:*

(1) *M is almost Kaehlerian,*

(2) *each fibre $\bar{M}$ is a K-contact manifold,*

(3) ${}^*\nabla_c\bar{\eta}_{\alpha} = 0,$

(4) $h_{\gamma}{}^{\alpha}{}_b\bar{\eta}_{\alpha} = 0,$ *and*

(5) $L_{cb}{}^{\alpha}\bar{\eta}_{\alpha} = J_{cb}.$

§3. Fibred Sasakian manifold

By means of (2.3)$\sim$(2.10), components of the covariant derivation $\tilde{\nabla}\tilde{\phi}$ and $\tilde{\nabla}\tilde{\xi}$ are given by

(3.1) $$(\tilde{\nabla}\tilde{\theta})E_c \otimes E_b \otimes E_a = \nabla_c J_{ba},$$

(3.2) $$(\tilde{\nabla}\tilde{\theta})E_c \otimes E_b \otimes C_{\alpha} = L_{cb}{}^{\beta}\bar{\phi}_{\beta\alpha} - L_{ca\alpha}J_b{}^a,$$

(3.3) $$(\tilde{\nabla}\tilde{\theta})E_c\otimes C_\alpha\otimes E_b = L_{cb\gamma}\bar{\phi}_\alpha{}^\gamma - L_c{}^a{}_\alpha J_{ab},$$

(3.4) $$(\tilde{\nabla}\tilde{\theta})E_c\otimes C_\alpha\otimes C_\gamma = {}^*\nabla_c\bar{\phi}_{\alpha\gamma},$$

(3.5) $$(\tilde{\nabla}\tilde{\theta})C_\gamma\otimes E_b\otimes E_a = L_{da\gamma}J_b{}^d - L_b{}^d{}_\gamma J_{da},$$

(3.6) $$(\tilde{\nabla}\tilde{\theta})C_\gamma\otimes E_b\otimes C_\mu = h_\gamma{}^\lambda{}_b\bar{\phi}_{\lambda\mu} - h_{\gamma\mu a}J_b{}^a,$$

(3.7) $$(\tilde{\nabla}\tilde{\theta})C_\gamma\otimes C_\alpha\otimes E_b = h_{\gamma\beta b}\bar{\phi}_\alpha{}^\beta - h_{\gamma\alpha}{}^a J_{ab},$$

(3.8) $$(\tilde{\nabla}\tilde{\theta})C_\gamma\otimes C_\alpha\otimes C_\beta = \bar{\nabla}_\gamma\bar{\phi}_{\alpha\beta}.$$

(3.9) $$(\tilde{\nabla}\tilde{\xi})E_c\otimes E^b = L_c{}^b{}_\alpha\bar{\xi}^\alpha,$$

(3.10) $$(\tilde{\nabla}\tilde{\xi})E_c\otimes C^\alpha = {}^*\nabla_c\bar{\xi}^\alpha,$$

(3.11) $$(\tilde{\nabla}\tilde{\xi})C_\gamma\otimes E^b = h_{\gamma\alpha}{}^b\bar{\xi}^\alpha,$$

(3.12) $$(\tilde{\nabla}\tilde{\xi})C_\gamma\otimes C^\mu = \bar{\nabla}_\gamma\bar{\xi}^\mu.$$

In this section, we assume that the induced almost contact structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on $\tilde{M}$ is Sasakian and call such an $\tilde{M}$ a *fibred Sasakian space.*

It is well known ([1, p. 73]) that *an almost contact metic structure* $(\tilde{\phi},\tilde{\xi},\tilde{\eta},\tilde{g})$ *is Sasakian if and only if the equation*

(3.13) $$(\tilde{\nabla}_{\tilde{X}}\tilde{\phi})\tilde{Y} = -\tilde{g}(\tilde{X}, \tilde{Y})\tilde{\xi} + \tilde{\eta}(\tilde{Y})\tilde{X}$$

holds forarbitrary vector fields $\tilde{X}$ *and* $\tilde{Y}$ *on* $\tilde{M}$. Then it follows from (3.1)$\sim$(3.12) and (3.13) that we have the equations

(3.14) $$\nabla_c J_{ba} = 0,$$

(3.15) $$L_{cb}{}^\beta\bar{\phi}_{\beta\alpha} - L_{ca\alpha}J_b{}^a = -\bar{\eta}_\alpha g_{cb},$$

(3.16) $${}^*\nabla_c\bar{\phi}_\alpha{}^\gamma = 0,$$

(3.17) $$L_d{}^a{}_\gamma J_b{}^d - L_b{}^d{}_\gamma J_d{}^a = 0,$$

(3.18) $$h_\gamma{}^\lambda{}_b\bar{\phi}_\lambda{}^\mu - h_\gamma{}^\mu{}_a J_b{}^a = 0,$$

(3.19) $$\bar{\nabla}_\gamma\bar{\phi}_\alpha{}^\beta = \bar{\eta}_\alpha\delta_\gamma^\beta - \bar{\xi}^\beta\bar{g}_{\gamma\alpha},$$

(3.20) $$L_{cb\alpha}\bar{\xi}^\alpha = J_{cb},$$

(3.21) $${}^*\nabla_c\bar{\xi}^\alpha = 0,$$

(3.22) $$h_{\gamma\alpha}{}^b\bar{\xi}^\alpha = 0,$$

(3.23) $$\bar{\nabla}_\gamma\bar{\xi}^\mu = \bar{\phi}_\gamma{}^\mu.$$

Contracting (3.18) with respect to γ and μ, we get $h_{\alpha\beta}{}^a\bar{g}^{\alpha\beta} = 0$, that is,each fibre $\bar{M}$ is a minimal submanifold of $\tilde{M}$.

On the other hand, if we take the skew-symmetric part of (3.15), then we obtain

$$L_{cb}{}^{\beta}\bar{\Phi}_{\beta}{}^{\gamma}=0 \tag{3.24}$$

by use of (3.17) and the skew-symmetry of L. Hence we can easily get

$$L_{cb}{}^{\gamma}=J_{cb}\xi^{\gamma} \tag{3.25}$$

by use of (3.20) and (3.24), and see that the structure tensor L does not vanish everywhere. Moreover, the relations (3.15) and (3.17) are fulfilled by the equation (3.25). Therefore we can state the following

THEOREM 3.1. *Let (J, g) be an almost Hermitian manifold on M and $\bar{M}$ be an almost contact metric manifold. Then the induced almost contact metric structure on $\tilde{M}$ is Sasakian if and only if the structure (J, g) is Kaehlerian, $\bar{M}$ is Sasakian and the equations* (3.16), (3.18) *and* (3.25) *hold. In this case, each fibre $\bar{M}$ is a minimal submanifold of $\tilde{M}$.*

A necessary and sufficient condition for $\tilde{M}$ to have isometric fibres (resp. conformal fibres) is $h_{\gamma\beta}{}^{a}=0$ (resp. $h_{\gamma\beta}{}^{a}=\bar{g}_{\gamma\beta}A^{a}$, where $A=A^{a}E_{a}$ is the mean curvature vector along each fibre in $\tilde{M}$).

COROLLARY 3.2. *If a fibred Sasakian space $\tilde{M}$ has conformal fibres, then $\tilde{M}$ has isometric and totally geodesic fibres.*

PROOF. It is easily seen that the conditions $h_{\gamma\beta}{}^{a}=\bar{g}_{\gamma\beta}A^{a}$ and $\bar{g}^{\gamma\beta}h_{\gamma\beta}{}^{a}=0$ imply $h_{\gamma\beta}{}^{a}=0$.

Moreover, considering the equations (3.21) and (3.25), we directly have the following:

COROLLARY 3.3. *In a fibred Sasakian spoce $\tilde{M}$, the structure tensor L is parallel on M in the sense of $*\nabla_{d}L_{cb}{}^{\alpha}=0$.*

§4. Fibred Riemannian manifold with space form

In this section, first of all, we recall curvature properties of the fibred Riemannian space. The curvature tensor $\tilde{K}$ of $\tilde{M}$ is defined by

$$\tilde{K}(\tilde{X}, \tilde{Y})\tilde{Z}=\tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z}-\tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z}-\tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z} \tag{4.1}$$

for any vector fields $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ in $\tilde{M}$. If we put

$$\tilde{K}(E_{D}, E_{C})E_{B}=\tilde{K}_{DCB}{}^{a}E_{a}+\tilde{K}_{DCB}{}^{\alpha}C_{\alpha},$$

then $\tilde{K}_{DCB}{}^{A}$ are components of the curvature tensor $\tilde{K}$ with respect to the frame $\{E_{a},$

$C_\alpha\}$. Denoting by $\tilde{K}_{kji}{}^h$ components of $\tilde{K}$ in $(\tilde{U}, z^h)$, we have the relations

$$\tilde{K}_{DCB}{}^A = \tilde{K}_{kji}{}^h E^k{}_D E^j{}_C E^i{}_B E_h{}^A. \tag{4.2}$$

Taking account of (1.7), (4.1) and (4.2), we get the following structure equations of the fibred Riemannian space $\tilde{M}$ ([12]);

$$\tilde{K}_{dcb}{}^a = K_{dcb}{}^a - L_d{}^a{}_\varepsilon L_{cb}{}^\varepsilon + L_c{}^a{}_\varepsilon L_{db}{}^\varepsilon + 2L_{dc}{}^\varepsilon L_b{}^a{}_\varepsilon, \tag{4.3}$$

$$\tilde{K}_{dcb}{}^\alpha = -{}^*\nabla_d L_{cb}{}^\alpha + {}^*\nabla_c L_{db}{}^\alpha - 2L_{dc}{}^\varepsilon h_\varepsilon{}^\alpha{}_b, \tag{4.4}$$

$$\begin{aligned}\tilde{K}_{dc\beta}{}^\alpha = {}& {}^*\nabla_c h_\beta{}^\alpha{}_d - {}^*\nabla_d h_\beta{}^\alpha{}_c + 2{}^{**}\nabla_\beta L_{dc}{}^\alpha + L_{de}{}^\alpha L_c{}^e{}_\beta \\ & - L_{ce}{}^\alpha L_d{}^e{}_\beta - h_\varepsilon{}^\alpha{}_d h_\beta{}^\varepsilon{}_c + h_\varepsilon{}^\alpha{}_c h_\beta{}^\varepsilon{}_d,\end{aligned} \tag{4.5}$$

$$\tilde{K}_{d\gamma b}{}^a = {}^*\nabla_d L_b{}^a{}_\gamma - L_d{}^a{}_\varepsilon h_\gamma{}^\varepsilon{}_b + L_{db}{}^\varepsilon h_{\gamma\varepsilon}{}^a - L_b{}^a{}_\varepsilon h_\gamma{}^\varepsilon{}_d, \tag{4.6}$$

$$\tilde{K}_{d\gamma b}{}^\alpha = -{}^*\nabla_d h_\gamma{}^\alpha{}_b + {}^{**}\nabla_\gamma L_{db}{}^\alpha + L_d{}^e{}_\gamma L_{eb}{}^\alpha + h_\gamma{}^\varepsilon{}_d h_\varepsilon{}^\alpha{}_b, \tag{4.7}$$

$$\tilde{K}_{\delta\gamma b}{}^a = L_{\delta\gamma b}{}^a + h_\delta{}^\varepsilon{}_b h_{\gamma\varepsilon}{}^a - h_\gamma{}^\varepsilon{}_b h_{\delta\varepsilon}{}^a, \tag{4.8}$$

$$\tilde{K}_{\delta\gamma\beta}{}^a = {}^{**}\nabla_\delta h_{\gamma\beta}{}^a - {}^{**}\nabla_\gamma h_{\delta\beta}{}^a, \tag{4.9}$$

$$\tilde{K}_{\delta\gamma\beta}{}^\alpha = \bar{K}_{\delta\gamma\beta}{}^\alpha + h_{\delta\beta}{}^e h_\gamma{}^\alpha{}_e - h_{\gamma\beta}{}^e h_\delta{}^\alpha{}_e, \tag{4.10}$$

where we have put

$$K_{dcb}{}^a = \partial_d \Gamma^a_{cb} - \partial_c \Gamma^a_{db} + \Gamma^a_{de}\Gamma^e_{cb} - \Gamma^a_{ce}\Gamma^e_{db}, \tag{4.11}$$

$${}^*\nabla_d L_{cb}{}^\alpha = \partial_d L_{cb}{}^\alpha - \Gamma^e_{dc} L_{eb}{}^\alpha - \Gamma^e_{db} L_{ce}{}^\alpha + Q_{d\varepsilon}{}^\alpha L_{cb}{}^\varepsilon, \tag{4.12}$$

$${}^*\nabla_d L_c{}^a{}_\beta = \partial_d L_c{}^a{}_\beta + \Gamma^a_{de} L_c{}^e{}_\beta - \Gamma^e_{dc} L_e{}^a{}_\beta - Q_{d\beta}{}^\varepsilon L_c{}^a{}_\varepsilon. \tag{4.13}$$

$${}^*\nabla_d h_{\gamma\beta}{}^a = \partial_d h_{\gamma\beta}{}^a + \Gamma^a_{de} h_{\gamma\beta}{}^e - Q_{d\gamma}{}^\varepsilon h_{\varepsilon\beta}{}^a - Q_{d\beta}{}^\varepsilon h_{\gamma\varepsilon}{}^a, \tag{4.14}$$

$${}^*\nabla_d h_\beta{}^\alpha{}_b = \partial_d h_\beta{}^\alpha{}_b - \Gamma^e_{db} h_\beta{}^\alpha{}_e + Q_{d\varepsilon}{}^\alpha h_\beta{}^\varepsilon{}_b - Q_{d\beta}{}^\varepsilon h_\varepsilon{}^\alpha{}_b, \tag{4.15}$$

$${}^{**}\nabla_\delta L_{cb}{}^\alpha = \partial_\delta L_{cb}{}^\alpha + \bar{\Gamma}^\alpha_{\delta\varepsilon} L_{cb}{}^\varepsilon - L_c{}^e{}_\delta L_{eb}{}^\alpha - L_b{}^e{}_\delta L_{ce}{}^\alpha, \tag{4.16}$$

$${}^{**}\nabla_\delta L_b{}^a{}_\beta = \partial_\delta L_b{}^a{}_\beta - \bar{\Gamma}^\varepsilon_{\delta\beta} L_b{}^a{}_\varepsilon + L_e{}^a{}_\delta L_b{}^e{}_\beta - L_b{}^e{}_\delta L_e{}^a{}_\beta, \tag{4.17}$$

$${}^{**}\nabla_\delta h_{\gamma\beta}{}^a = \partial_\delta h_{\gamma\beta}{}^a - \bar{\Gamma}^\varepsilon_{\delta\gamma} h_{\varepsilon\beta}{}^a - \bar{\Gamma}^\varepsilon_{\delta\beta} h_{\gamma\varepsilon}{}^a + L_e{}^a{}_\delta h_{\gamma\beta}{}^e, \tag{4.18}$$

$${}^{**}\nabla_\delta h_\beta{}^\alpha{}_b = \partial_\delta h_\beta{}^\alpha{}_b + \bar{\Gamma}^\alpha_{\delta\varepsilon} h_\beta{}^\varepsilon{}_b - \bar{\Gamma}^\varepsilon_{\delta\beta} h_\varepsilon{}^\alpha{}_b - L_b{}^e{}_\delta h_\beta{}^\alpha{}_e, \tag{4.19}$$

$$L_{\delta\gamma b}{}^a = \partial_\delta L_b{}^a{}_\gamma - \partial_\gamma L_b{}^a{}_\delta + L_e{}^a{}_\delta L_b{}^e{}_\gamma - L_e{}^a{}_\gamma L_b{}^e{}_\delta, \tag{4.20}$$

$$\bar{K}_{\delta\gamma\beta}{}^\alpha = \partial_\delta \bar{\Gamma}^\alpha_{\gamma\beta} - \partial_\gamma \bar{\Gamma}^\alpha_{\delta\beta} + \bar{\Gamma}^\alpha_{\delta\varepsilon}\bar{\Gamma}^\varepsilon_{\gamma\beta} - \bar{\Gamma}^\alpha_{\gamma\varepsilon}\bar{\Gamma}^\varepsilon_{\delta\beta}. \tag{4.21}$$

Now we assume that the induced Sasakian structure on $\tilde{M}$ is of constant $\tilde{\phi}$-sectional curvature $\tilde{k}$. Then the curvature tensor $\tilde{K}$ has the form [5];

$$\begin{aligned}\tilde{K}(\tilde{X}, \tilde{Y})\tilde{Z} = {}& \{(\tilde{k}+3)/4\}\{g(\tilde{Y}, \tilde{Z})\tilde{X} - \tilde{g}(\tilde{X}, \tilde{Z})\tilde{Y}\} \\ & + \{(\tilde{k}-1)/4\}\{\tilde{\eta}(\tilde{X})\tilde{\eta}(\tilde{Z})\tilde{Y} - \tilde{\eta}(\tilde{Y})\tilde{\eta}(\tilde{Z})\tilde{X} + \tilde{g}(\tilde{X}, \tilde{Z})\tilde{\eta}(\tilde{Y})\tilde{\xi}\end{aligned} \tag{4.22}$$

$$-\tilde{g}(\tilde{Y},\tilde{Z})\tilde{\eta}(\tilde{X})\xi+\tilde{\theta}(\tilde{Y},\tilde{Z})\tilde{\phi}\tilde{X}-\tilde{\theta}(\tilde{X},\tilde{Z})\tilde{\phi}\tilde{Y}$$
$$-2\tilde{\theta}(\tilde{X},\tilde{Y})\tilde{\phi}\tilde{Z}\}.$$

Therefore the equations (4.3)$\sim$(4.10) give rise to

$$K_{dcb}{}^{a}=\{(\tilde{k}+3)/4\}(g_{cb}\delta_d^a-g_{db}\delta_c^a)+\{(\tilde{k}-1)/4\}(J_{cb}J_d{}^a-J_{db}J_c{}^a$$
$$-2J_{dc}J_b{}^a)+L_d{}^a{}_\varepsilon L_{cb}{}^\varepsilon-L_c{}^a{}_\varepsilon L_{db}{}^\varepsilon-2L_{dc}{}^\varepsilon L_b{}^a{}_\varepsilon, \tag{4.23}$$

$$*\nabla_c L_{db}{}^\alpha-*\nabla_d L_{cb}{}^\alpha=2L_{dc}{}^\varepsilon h_\varepsilon{}^\alpha{}_b, \tag{4.24}$$

$$*\nabla_c h_\beta{}^\alpha{}_d-*\nabla_d h_\beta{}^\alpha{}_c+2**\nabla_\beta L_{dc}{}^\alpha+L_{de}{}^\alpha L_c{}^e{}_\beta-L_{ce}{}^\alpha L_d{}^e{}_\beta$$
$$-h_\varepsilon{}^\alpha{}_d h_\beta{}^\varepsilon{}_c+h_\varepsilon{}^\alpha{}_c h_\beta{}^\varepsilon{}_d=\{-(\tilde{k}-1)/2\}J_{dc}\tilde{\phi}_\beta{}^\alpha, \tag{4.25}$$

$$*\nabla_d L_b{}^a{}_\gamma=L_d{}^a{}_\varepsilon h_\gamma{}^\varepsilon{}_b-L_{db}{}^\varepsilon h_{\gamma\varepsilon}{}^a+L_b{}^a{}_\varepsilon h_\gamma{}^\varepsilon{}_d, \tag{4.26}$$

$$**\nabla_\gamma L_{db}{}^\alpha-*\nabla_d h_\gamma{}^\alpha{}_b+L_d{}^e{}_\gamma L_{eb}{}^\alpha+h_\gamma{}^\varepsilon{}_d h_\varepsilon{}^\alpha{}_b$$
$$=-\{(\tilde{k}+3)/4\}g_{db}\delta_\gamma^\alpha+\{(\tilde{k}-1)/4\}(g_{db}\bar{\eta}_\gamma\xi^\alpha-J_{db}\tilde{\phi}_\gamma{}^\alpha), \tag{4.27}$$

$$L_{\delta\gamma b}{}^a=h_\gamma{}^\varepsilon{}_b h_{\delta\varepsilon}{}^a-h_\delta{}^\varepsilon{}_b h_{\gamma\varepsilon}{}^a-\{(\tilde{k}-1)/2\}\tilde{\phi}_{\delta\gamma}J_b{}^a, \tag{4.28}$$

$$**\nabla_\delta h_{\gamma\beta}{}^a-**\nabla_\gamma h_{\delta\beta}{}^a=0, \tag{4.29}$$

$$\bar{K}_{\delta\gamma\beta}{}^\alpha=\{(\tilde{k}+3)/4\}\ (\bar{g}_{\gamma\beta}\delta_\delta^\alpha-\bar{g}_{\delta\beta}\delta_\gamma^\alpha)-\{(\tilde{k}-1)/4\}\ (\bar{\eta}_\gamma\bar{\eta}_\beta\delta_\delta^\alpha-\bar{\eta}_\delta\bar{\eta}_\beta\delta_\gamma^\alpha$$
$$+\bar{g}_{\gamma\beta}\bar{\eta}_\delta\xi^\alpha-\bar{g}_{\delta\beta}\bar{\eta}_\gamma\xi^\alpha-\bar{\phi}_{\gamma\beta}\bar{\phi}_\delta{}^\alpha+\bar{\phi}_{\delta\beta}\bar{\phi}_\gamma{}^\alpha$$
$$+2\bar{\phi}_{\delta\gamma}\bar{\phi}_\beta{}^\alpha)+h_{\gamma\beta}{}^e h_\delta{}^\alpha{}_e-h_{\delta\beta}{}^e h_\gamma{}^\alpha{}_e. \tag{4.30}$$

Denoting by k and $\bar{k}$ the scalar curvature of M and $\bar{M}$ respectively, we have

$$k=n(n+2)\ (\tilde{k}+3)/4, \tag{4.31}$$

$$\bar{k}=(s-1)\ \{s(\tilde{k}+3)+\tilde{k}-1\}/4-\|h_{\alpha\beta}{}^a\|^2, \tag{4.32}$$

by means of (3.25), where $s=\dim\bar{M}$. Combining the equations (3.25) and (4.23), we can see that

$$K_{dcb}{}^a=(\tilde{k}+3)\ (g_{cb}\delta_d^a-g_{db}\delta_c^a+J_d{}^aJ_{cb}-J_c{}^aJ_{db}-2J_{dc}J_b{}^a)/4. \tag{4.33}$$

Hence we have

THEOREM 4.1. *If the induced Sasakian structure on $\tilde{M}$ is of constant $\tilde{\phi}$-sectinal curvature $\tilde{k}$, then the manifold M with Kaehler structure (J, g) is of constant holomorphic sectional curvature $\tilde{k}+3$.*

Next, consider the case where $\tilde{M}$ is a fibred space with conformal fibres. Then, by Corollary 3.2, each fibre $\bar{M}$ is totally geodesic and the second fundamental form h of $\bar{M}$ vanishes identically. Hence the equation (4.25) reduces to

$$4J_{dc}\bar{\Phi}_{\beta}{}^{\alpha} = -(\tilde{k}-1)J_{dc}\bar{\Phi}_{\beta}{}^{\alpha} \tag{4.34}$$

and we have $\tilde{k} = -3$. Thus the equations (4.22), (4.28), (4.30), (4.32) and (4.33) are reduced to

$$\begin{aligned} \tilde{K}(\tilde{X},\tilde{Y})\tilde{Z} = & -\tilde{\eta}(\tilde{X})\tilde{\eta}(\tilde{Z})\tilde{Y} + \tilde{\eta}(\tilde{Y})\tilde{\eta}(\tilde{Z})\tilde{X} - \tilde{g}(\tilde{X}, \tilde{Z})\tilde{\eta}(\tilde{Y})\tilde{\xi} \\ & + \tilde{g}(\tilde{Y}, \tilde{Z})\tilde{\eta}(\tilde{X})\tilde{\xi} - \tilde{\theta}(\tilde{Y}, \tilde{Z})\tilde{\Phi}\tilde{X} + \tilde{\theta}(\tilde{X}, \tilde{Z})\tilde{\Phi}\tilde{Y} \\ & + \tilde{\theta}(\tilde{X}, \tilde{Y})\tilde{\Phi}\tilde{Z}, \end{aligned} \tag{4.35}$$

$$L_{\delta\gamma b}{}^{a} = 2\bar{\Phi}_{\delta\gamma}J_{b}{}^{a}, \tag{4.36}$$

$$\begin{aligned} \bar{K}_{\delta\gamma\beta}{}^{\alpha} = & \bar{\eta}_{\gamma}\bar{\eta}_{\beta}\delta_{\delta}^{\alpha} - \bar{\eta}_{\delta}\bar{\eta}_{\beta}\delta_{\gamma}^{\alpha} + \bar{g}_{\gamma\beta}\bar{\eta}_{\delta}\bar{\xi}^{\alpha} - \bar{g}_{\delta\beta}\bar{\eta}_{\gamma}\bar{\xi}^{\alpha} - \bar{\Phi}_{\gamma\beta}\bar{\Phi}_{\delta}{}^{\alpha} \\ & + \bar{\Phi}_{\delta\beta}\bar{\Phi}_{\gamma}{}^{\alpha} + 2\bar{\Phi}_{\delta\gamma}\bar{\Phi}_{\beta}{}^{\alpha}, \end{aligned} \tag{4.37}$$

$$\bar{k} = -(s-1), \tag{4.38}$$

$$K_{dcb}{}^{a} = 0 \tag{4.39}$$

respectively. However, by use of Theorem 3.1, the equation (4.36) is valid if the space $\tilde{M}$ is Sasakian. Thus we have the following

THEOREM 4.2. *Let $\tilde{M}$ be a fibred Sasakian space with conformal fibres. If $\tilde{M}$ is a space of constant $\tilde{\Phi}$-sectional curvature $\tilde{k}$, then*

(1) *the total space is a Sasakian space form with $\tilde{k} = -3$,*

(2) *the base space M is a locally Euclidean space, and*

(3) *the fibre $\bar{M}$ is a Sasakian space form with constant $\bar{\Phi}$-sectional curvature -3.*

Conversely, if the base space M is a locally Euclidean space and each fibre $\bar{M}$ is a Sasakian space form with constant $\bar{\Phi}$-sectional curvature -3, then $\tilde{M}$ is a Sasakian space form with constant $\tilde{\Phi}$-sectional curvature -3.

EXAMPLE. Euclidean plane E^2 with coordinates (x_1, x_2) and flat metric has an almost complex structure

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now, we consider the Euclidean space E^3 with Cartesian coordinates (y_1, y_2, z) and define $\bar{\Phi}$, $\bar{\xi}$, $\bar{\eta}$, $\bar{g}$ by

$$\bar{\phi}=\frac{1}{4}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\bar{\xi}={}^t(0, 0, 2),$$

$$\bar{\eta}=(-y_2, 0, 1)/2,$$

$$\bar{g}=\frac{1}{4}\begin{pmatrix} 1+y_2{}^2 & 0 & -y_2 \\ 0 & 1 & 0 \\ -y_2 & 0 & 1 \end{pmatrix},$$

$$\bar{g}^{-1}=4\begin{pmatrix} 1 & 0 & y_2 \\ 0 & 1 & 0 \\ y_2 & 0 & 1+y_2{}^2 \end{pmatrix}.$$

Then E^3 with $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ becomes a Sasakian space form with constant $\bar{\phi}$-sectional curvature -3 (cf. Sasaki [9]), and we denote it by $E^3(-3)$.

Next we consider a symmetric tensor field in the 5-dimensional Euclidean space E^5 with Cartesian coordinates (x_1, x_2, y_1, y_2, z) define $\tilde{g}$ by

$$\tilde{g}=\frac{1}{4}\begin{pmatrix} 1+x_2{}^2 & 0 & x_2y_2 & 0 & -x_2 \\ 0 & 1 & 0 & 0 & 0 \\ x_2y_2 & 0 & 1+y_2{}^2 & 0 & -y_2 \\ 0 & 0 & 0 & 1 & 0 \\ -x_2 & 0 & -y_2 & 0 & 1 \end{pmatrix}.$$

Then $\tilde{g}$ is a positive definite Riemannian metric. The inverse matrix of $\tilde{g}$ is given by

$$\tilde{g}^{-1}=4\begin{pmatrix} 1 & 0 & 0 & 0 & x_2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & y_2 \\ 0 & 0 & 0 & 1 & 0 \\ x_2 & 0 & y_2 & 0 & 1+x_2{}^2+y_2{}^2 \end{pmatrix}.$$

Then E^5 becomes a Sasakian space form $E^5(-3)$ with constant $\tilde{\phi}$-sectional curvature -3 by taking the following tensors as its structure tensor;

$$\tilde{\phi}=\frac{1}{4}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{\xi}={}^t(0, 0, 0, 0, 2),$$

$$\tilde{\eta}=(-x_2, 0, -y_2, 0, 1)/2.$$

The vector fields

$$C_1=(0, 0, 1, 0, 0),$$

$$C_2=(0, 0, 0, 1, 0),$$

$$C_3=(0, 0, 0, 0, 1),$$

$$E_1={}^t(2, 0, 0, 0, 2x_2) \text{ and}$$

$$E_2={}^t(0, 2, 0, 0, 0)$$

form a frame field in $E^5(-3)$ and the Euclidean space $\{E^5(-3), \tilde{g}\}$ becomes a fibred Riemannian space having E^2 as the base space and $E^3(-3)$ as the fibre. It is well known that the Sasakian space form $E^3(-3)$ is a totally geodesic submanifold of E^5 (-3). A similar argument can be applied to the case of general dimensions and we see that a Sasakian space form $E^m(-3)$ has a fibred structure with Euclidean base space E^n and Sasakian space form $E^{m-n}(-3)$ as fibre.

§5. Integrability and normality

In this section, we study relations between the normality of the almost contact metric structures on $\tilde{M}$ and $\bar{M}$ and the integrability of the almost complex structure on M. Now consider the product manifold $\tilde{M}\times E^1$, E^1 being a 1-dimensional Euclidean space. If we define on $\tilde{M}\times E^1$ a tensor field F of type (1, 1) with local components $F_P{}^Q$ by

$$(5.1)\qquad (F_P{}^Q)=\begin{pmatrix} J_b{}^a & 0 & 0 \\ 0 & \bar{\phi}_\beta{}^\alpha & -\bar{\xi}^\alpha \\ 0 & \bar{\eta}_\beta & 0 \end{pmatrix}$$

in each $\{\tilde{U}\times E^1, x^P\}$, then we can see that $F^2=-1$ on $\tilde{M}\times E^1$, that is, $\{\tilde{M}\times E^1, F\}$ becomes an almost complex manifold. Hence we can calculate the components of Nijenhuis tensors $\tilde{N}_{PQ}{}^R$ of F as follows:

(5.2) $$\tilde{N}_{cb}{}^a = J_c{}^e(\partial_e J_b{}^a - \partial_b J_e{}^a) - J_b{}^e(\partial_e J_c{}^a - \partial_c J_e{}^a),$$

(5.3) $$\tilde{N}_{cb}{}^\alpha = 0,$$

(5.4) $$\tilde{N}_{cb}{}^\sharp = 0,$$

(5.5) $$\tilde{N}_{c\beta}{}^a = 0,$$

(5.6) $$\tilde{N}_{c\beta}{}^\alpha = J_c{}^e(\partial_e \bar{\Phi}_\beta{}^\alpha) + \bar{\Phi}_\beta{}^\gamma(\partial_c \bar{\Phi}_\gamma{}^\alpha),$$

(5.7) $$\tilde{N}_{c\beta}{}^\sharp = J_c{}^e(\partial_e \bar{\eta}_\beta) + \bar{\Phi}_\beta{}^\alpha(\partial_c \bar{\eta}_\alpha),$$

(5.8) $$\tilde{N}_{\gamma\beta}{}^a = 0,$$

(5.9) $$\tilde{N}_{\gamma\beta}{}^\alpha = \bar{\Phi}_\gamma{}^\lambda(\partial_\lambda \bar{\Phi}_\beta{}^\alpha - \partial_\beta \bar{\Phi}_\lambda{}^\alpha) - \bar{\Phi}_\beta{}^\lambda(\partial_\lambda \bar{\Phi}_\gamma{}^\alpha - \partial_\gamma \bar{\Phi}_\lambda{}^\alpha)$$
$$+ \bar{\eta}_\gamma(\partial_\beta \bar{\xi}^\alpha) - \bar{\eta}_\beta(\partial_\gamma \bar{\xi}^\alpha),$$

(5.10) $$\tilde{N}_{\gamma\beta}{}^\sharp = \bar{\Phi}_\gamma{}^\lambda(\partial_\lambda \bar{\eta}_\beta - \partial_\beta \bar{\eta}_\lambda) - \phi_\beta{}^\lambda(\partial_\lambda \bar{\eta}_\gamma - \partial_\gamma \bar{\eta}_\lambda),$$

(5.11) $$\tilde{N}_{\sharp a}{}^b = 0,$$

(5.12) $$\tilde{N}_{\sharp a}{}^\beta = J_a{}^d(\partial_d \bar{\xi}^\beta) + \bar{\xi}^\gamma(\partial_a \bar{\Phi}_\gamma{}^\beta),$$

(5.13) $$\tilde{N}_{\sharp a}{}^\sharp = \bar{\xi}^\alpha(\partial_a \bar{\eta}_\alpha),$$

(5.14) $$\tilde{N}_{\sharp\beta}{}^a = 0,$$

(5.15) $$\tilde{N}_{\sharp\beta}{}^\alpha = -\bar{\xi}^\gamma(\partial_\gamma \bar{\Phi}_\beta{}^\alpha - \partial_\beta \bar{\Phi}_\gamma{}^\alpha) + \bar{\Phi}_\beta{}^\gamma(\partial_\gamma \bar{\xi}^\alpha),$$

(5.16) $$\tilde{N}_{\sharp\beta}{}^\sharp = -\bar{\xi}^\gamma \partial_\gamma \bar{\eta}_\beta.$$

It follows from these equations that the components $\tilde{N}_{cb}{}^a$ coincide with those of the Nijenhuis tensor

(5.17) $$N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y]$$

of J on M, where X and Y are vector fields on M, and the components $\tilde{N}_{\gamma\beta}{}^\alpha$, $\tilde{N}_{\gamma\beta}{}^\sharp$, $\tilde{N}_{\sharp\beta}{}^\alpha$ and $\tilde{N}_{\sharp\beta}{}^\sharp$ coincide with the $\bar{N}_{\gamma\beta}{}^\alpha$, $\bar{N}_{\gamma\beta}{}^\sharp$, $\bar{N}_{\sharp\beta}{}^\alpha$ and $\bar{N}_{\sharp\beta}{}^\sharp$ respectively which are components of Nijenhuis tensor of $\bar{M}\times E^1$ with an almost complex structure

(5.18) $$\bar{F} = \begin{pmatrix} \bar{\Phi}_\beta{}^\alpha & -\bar{\xi}^\alpha \\ \bar{\eta}_\beta & 0 \end{pmatrix}.$$

Moreover, by virtue of a formula (see [13], p. 191)

$$\tilde{N}_{PQ}{}^{R}F_{S}{}^{Q}+\tilde{N}_{PS}{}^{Q}F_{Q}{}^{R}=0,$$

it is well known that *if the components $\tilde{N}_{\gamma\beta}{}^{\alpha}$ vanish identically on $\bar{M}\times E^{1}$, then the other components $\tilde{N}_{\gamma\beta}{}^{\#}$, $\tilde{N}_{\#\beta}{}^{\#}$ and $\tilde{N}_{\#\beta}{}^{\gamma}$ vanish.* An almost contact structure on $\tilde{M}$ is said to be *normal* if the almost complex structure F on $\tilde{M}\times E^{1}$ is integrable ([9], [10]), equivalently if $N_{CB}{}^{A}$ vanish identically. Consequently, we have

THEOREM 5.1. *If the induced almost contact structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ on $\tilde{M}$ is normal, then the almost complex structure J on M is integrable and the almost contact structure $(\bar{\phi}, \bar{\xi}, \bar{\eta})$ on $\bar{M}$ is normal.*

Moreover, if the almost contact structure $(\bar{\phi}, \bar{\xi}, \bar{\eta})$ on $\bar{M}$ is independent of the base space, then all the components of $\tilde{N}_{PQ}{}^{R}$ vanish identically under the assumptions that the almost complex structure J on M is integrable and that the almost contact structure on $\bar{M}$ is normal. Finally we get

THEOREM 5.2. *If the almost contact structure $(\bar{\phi}, \bar{\xi}, \bar{\eta})$ on $\bar{M}$ is independent of the base space, then a necessary and sufficient condition in order that the induced almost contact structure on $\tilde{M}$ is normal is that J on M is integrable and $(\bar{\phi}, \bar{\xi}, \bar{\eta})$ on $\bar{M}$ is normal.*

REMARK. *If the almost contact structure $(\bar{\phi}, \bar{\xi}, \bar{\eta})$ on $\bar{M}$ in the condition of Theorem* 5.2 *is replaced by contact structure on $\bar{M}$, then we can see that the same result is valid but the fibred Riemannian space $\tilde{M}$ is never locally trivial because the integrability tensor $L=(L_{cb}{}^{\alpha})$ does not vanish by means of the equation* (3.25).

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