HIROSHIMA MATH. J.
**19** (1989), 575–585

# Intersections of subideals of Lie algebras

Dedicated to the memory of Professor Sigeaki Tôgô

Masanobu HONDA

(Received January 4, 1989)

## Introduction

The author [5], [6] and Nomura [11] have investigated the class $\mathfrak{L}^{\infty}$ of Lie algebras in which the join of any collection of subideals is always a subideal. On the other hand, concerning the class $\mathfrak{L}_{\infty}$ of Lie algebras in which the intersection of any collection of subideals is always a subideal, very little is known except the fact that $\mathfrak{M} \leq \mathfrak{L}_{\infty}$ ([5, Lemma 3.2]), where $\mathfrak{M}$ is the class of Lie algebras having an upper bound for the steps of all subideals. The purpose of this paper is to present further results concerning the class $\mathfrak{L}_{\infty}$ and investigate related classes.

In Section 2 we shall first prove that in any Lie algebra the intersection of any collection of descendant (resp. weakly descendant, serial, weakly serial) subalgebras is always descendant (resp. weakly descendant, serial, weakly serial) (Theorem 2.2). We shall secondly characterize the class $\mathfrak{L}_{\infty}$ as the class of Lie algebras in which every descendant subalgebra is a subideal (Theorem 2.3).

The group-theoretic analogue of the class $\mathfrak{M}$ is usually denoted by $\mathfrak{B}$. Robinson [12] has proved that if a group $G$ has a normal subgroup $N$ such that $N$ has a composition series of finite length and $G/N$ is in the class $\mathfrak{B}$, then $G$ is in the class $\mathfrak{B}$. In Section 3 we shall prove that if a Lie algebra $L$ has an ideal $I$ such that $I$ has a composition series of finite length and $L/I$ is in the class $\mathfrak{L}_{\infty}$ (resp. $\mathfrak{L}_{\infty}$(asc), $\mathfrak{M}$, $\mathfrak{D}$(asc, si)), then $L$ is in the class $\mathfrak{L}_{\infty}$ (resp. $\mathfrak{L}_{\infty}$(asc), $\mathfrak{M}$, $\mathfrak{D}$(asc, si)) (Theorem 3.4), where $\mathfrak{L}_{\infty}$(asc) is the class of Lie algebras in which the intersection of any collection of ascendant subalgebras is always ascendant, and $\mathfrak{D}$(asc, si) is the class of Lie algebras in which every ascendant subalgebra is a subideal.

In Section 4 we shall first prove that if a Lie algebra having an abelian ideal of codimension 1 is in the class $\mathfrak{L}_{\infty}$, then it must be in the class $\mathfrak{M}$ (Proposition 4.2). Secondly we shall present a sufficient condition for Lie algebras in the class $\mathfrak{L}_{\infty}$ to be nilpotent.

## 1.

Throughout this paper we always consider not necessarily finite-dimensional Lie algebras over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified. Notation and terminology are mainly based on [2]. For the sake of convenience we explain some terms which we use here. Any notation not explained here may be found in [2].

Let $L$ be a Lie algebra over $\mathfrak{k}$ and $n$ be an integer $\geqq 0$. $H \leq L$ (resp. $H \lhd L$, $H \lhd^n L$, $H \operatorname{si} L$) we mean that $H$ is a subalgebra (resp. an ideal, an $n$-step subideal, a subideal) of $L$. If $H \operatorname{si} L$, then there is the smallest integer $m$ with respect to $H \lhd^m L$, which we denote by $\operatorname{si}(L: H)$ as in [5]. $H$ is a weak subideal of $L$, denoted by $H \operatorname{wsi} L$, if there exist an integer $n \geqq 0$ and a chain $\{H_i: 0 \leqq i \leqq n\}$ of subspaces of $L$ such that

(a) $H_0 = H$ and $H_n = L$,

(b) $[H_{i+1}, H] \subseteq H_i$ $(0 \leqq i < n)$.

$H$ is an ascendant subalgebra (resp. a weakly ascendant subalgebra) of $L$, denoted by $H \operatorname{asc} L$ (resp. $H \operatorname{wasc} L$), if there exist an ordinal $\sigma$ and an ascending chain $\{H_\alpha: \alpha \leqq \sigma\}$ of subalgebras (resp. subspaces) of $L$ such that

(a) $H_0 = H$ and $H_\sigma = L$,

(b) $H_\alpha \lhd H_{\alpha+1}$ (resp. $[H_{\alpha+1}, H] \subseteq H_\alpha$) for all ordinals $\alpha < \sigma$,

(c) $H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha$ for all limit ordinals $\lambda \leqq \sigma$.

Then the ascending chain $\{H_\alpha: \alpha \leqq \sigma\}$ is said to be an ascending series (resp. a weakly ascending series) from $H$ to $L$. $H$ is a descendant subalgebra (resp. a weakly descendant subalgebra) of $L$, denoted by $H \operatorname{dsc} L$ (resp. $H \operatorname{wdsc} L$), if there exist an ordinal $\sigma$ and a descending chain $\{H_\alpha: \alpha \leqq \sigma\}$ of subalgebras (resp. subspaces) of $L$ such that

(a) $H_0 = L$ and $H_\sigma = H$,

(b) $H_{\alpha+1} \lhd H_\alpha$ (resp. $[H_\alpha, H] \subseteq H_{\alpha+1}$) for all ordinals $\alpha < \sigma$,

(c) $H_\lambda = \bigcap_{\alpha<\lambda} H_\alpha$ for all limit ordinals $\lambda \leqq \sigma$.

Then the descending chain $\{H_\alpha: \alpha \leqq \sigma\}$ is said to be a descending series (resp. a weakly descending series) from $H$ to $L$. $H$ is a serial subalgebra (resp. a weakly serial subalgebra) of $L$, denoted by $H \operatorname{ser} L$ (resp. $H \operatorname{wser} L$), if there exist a totally ordered set $\Sigma$ and a family $\{\Lambda_\sigma, V_\sigma: \sigma \in \Sigma\}$ of subalgebras (resp. subspaces) of $L$ such that

(a) $H \subseteq V_\sigma \subseteq \Lambda_\sigma$ for all $\sigma \in \Sigma$,

(b) $\Lambda_\tau \subseteq V_\sigma$ if $\tau < \sigma$,

(c) $L \setminus H = \bigcup_{\sigma \in \Sigma} (\Lambda_\sigma \setminus V_\sigma)$,

(d) $V_\sigma \lhd \Lambda_\sigma$ (resp. $[\Lambda_\sigma, H] \subseteq V_\sigma$) for all $\sigma \in \Sigma$.

Then the family $\{\Lambda_\sigma, V_\sigma: \sigma \in \Sigma\}$ is said to be a series (resp. a weak series) from $H$ to $L$.

Let $H \leq L$. The ideal closure series (resp. the weak closure series) of $H$ in $L$, which we denote by $\{H^{L,\alpha}: \alpha \geqq 0\}$ (resp. $\{H_{L,\alpha}: \alpha \geqq 0\}$) as in [5], is defined inductively as follows:

(a) $H^{L,0} = L$ (resp. $H_{L,0} = L$);

(b) $H^{L,\alpha+1} = \sum_{n \geqq 0} [H,_n H^{L,\alpha}]$ (resp. $H_{L,\alpha+1} = [H_{L,\alpha}, H] + H$) for each ordinal $\alpha$;

(c) $H^{L,\lambda} = \bigcap_{\alpha<\lambda} H^{L,\alpha}$ (resp. $H_{L,\lambda} = \bigcap_{\alpha<\lambda} H_{L,\alpha}$) for each limit ordinal $\lambda$. Then it is easy to see that $H \operatorname{dsc} L$ (resp. $H \operatorname{wdsc} L$) if and only if $H^{L,\sigma} = H$ (resp. $H_{L,\sigma} = H$) for some ordinal $\sigma$.

A class $\mathfrak{X}$ is a collection of Lie algebras together with their isomorphic copies and 0-dimensional Lie algebras. Lie algebras in a class $\mathfrak{X}$ are called $\mathfrak{X}$-algebras. $\mathfrak{A}$ (resp. $\mathfrak{F}$, $\mathfrak{F}_n$, $\mathfrak{G}$, $\mathfrak{N}$, $\mathfrak{N}_n$, $\mathrm{R}\mathfrak{N}$, $\mathfrak{S}$, $\mathfrak{Z}$) is the class of Lie algebras which are abelian (resp. finite-dimensional, finite-dimensional of dimension $\leqq n$, finitely generated, nilpotent, nilpotent of class $\leqq n$, residually nilpotent, simple, hypercentral). $\mathfrak{B}$ (resp. $\mathfrak{Gr}$) is the class of Lie algebras $L$ such that $\langle x \rangle \operatorname{si} L$ (resp. $\langle x \rangle \operatorname{asc} L$) for all $x \in L$. $\mathfrak{B}$-algebras (resp. $\mathfrak{Gr}$-algebras) are called Baer (resp. Gruenberg) algebras. Similarly we use $\tilde{\mathfrak{G}}\mathfrak{r}$, as in [8], to denote the class of Lie algebras $L$ such that $\langle x \rangle \operatorname{dsc} L$ for all $x \in L$. Furthermore, we use $\grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$, as in [8], to denote the class of Lie algebras $L$ such that $L^{\alpha} = \{0\}$ for some ordinal $\alpha$. Evidently $\mathrm{R}\mathfrak{N} \leq \grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$. Let $\mathfrak{X}_1$, $\mathfrak{X}_2$, $\mathfrak{X}_3$ and $\mathfrak{X}$ be classes of Lie algebras and $n$ be an integer $> 0$. Then the classes $\mathfrak{X}_1\mathfrak{X}_2$, $\mathfrak{X}_1\mathfrak{X}_2\mathfrak{X}_3$, $\mathfrak{X}^n$, $\mathrm{E}\mathfrak{X}$, $\mathrm{E}(\lhd)\mathfrak{X}$ and $\acute{\mathrm{E}}(\lhd)\mathfrak{X}$ are defined as follows:

$L \in \mathfrak{X}_1\mathfrak{X}_2$ iff $L$ has an ideal $I$ such that $I \in \mathfrak{X}_1$ and $L/I \in \mathfrak{X}_2$;

$L \in \mathfrak{X}_1\mathfrak{X}_2\mathfrak{X}_3$ iff $L \in (\mathfrak{X}_1\mathfrak{X}_2)\mathfrak{X}_3$;

$L \in \mathfrak{X}^n$ iff there exists an ascending series $\{L_i: i \leqq n\}$ from $\{0\}$ to $L$ such that $L_{i+1}/L_i \in \mathfrak{X}$ $(0 \leqq i < n)$;

$L \in \mathrm{E}\mathfrak{X}$ iff $L \in \mathfrak{X}^n$ for some integer $n > 0$;

$L \in \mathrm{E}(\lhd)\mathfrak{X}$ (resp. $\acute{\mathrm{E}}(\lhd)\mathfrak{X}$) iff there exist an integer $n \geqq 0$ (resp. an ordinal $\sigma$) and an ascending series $\{L_i: i \leqq n\}$ (resp. $\{L_\alpha: \alpha \leqq \sigma\}$), consisting of ideals of $L$, from $\{0\}$ to $L$ such that each factor $L_{i+1}/L_i \in \mathfrak{X}$ (resp. $L_{\alpha+1}/L_\alpha \in \mathfrak{X}$).

In particular, $\mathrm{E}\mathfrak{A}$ is the class of soluble Lie algebras. The following five classes of Lie algebras, introduced in [5] and [7], will be mainly studied in this paper.

$L \in \mathfrak{A}_1$ iff either $L \in \mathfrak{A}$ or $L \in \mathfrak{A}^2$ with $\dim(L/L^2) = 1$.

$L \in \mathfrak{L}_\infty$ (resp. $\mathfrak{L}_\infty(\mathrm{asc})$) iff $H_\alpha \operatorname{si} L$ (resp. $H_\alpha \operatorname{asc} L$) $(\alpha \in A)$ implies $\bigcap_{\alpha\in A} H_\alpha \operatorname{si} L$ (resp. $\bigcap_{\alpha\in A} H_\alpha \operatorname{asc} L$).

$L \in \mathfrak{M}_n$ iff $\operatorname{si}(L: H) \leqq n$ for all subideals $H$ of $L$

$L \in \mathfrak{M}$ iff $L \in \mathfrak{M}_n$ for some integer $n \geqq 0$.

Let $\Delta_i$ be any of the relations $\leq$, si, asc, dsc $(i = 1, 2)$. Then we introduce the new class $\mathfrak{D}(\Delta_1, \Delta_2)$ of Lie algebras as follows:

$L \in \mathfrak{D}(\Delta_1, \Delta_2)$ iff $H \Delta_1 L$ always implies $H \Delta_2 L$.

In particular, $\mathfrak{D}(\leq, \mathrm{si})$ is usually denoted by $\mathfrak{D}$. We also abbreviate $\mathfrak{D}(\leq, \mathrm{asc})$ to $\mathfrak{D}(\mathrm{asc})$. In [7] $\mathfrak{D}(\mathrm{asc}, \mathrm{si})$ is denoted by $\mathfrak{M}(\mathrm{asc})$. The classes $\mathfrak{D}(\leq, \mathrm{dsc})$, $\mathfrak{D}(\mathrm{asc}, \mathrm{dsc})$ and $\mathfrak{D}(\mathrm{dsc}, \mathrm{asc})$ will not concern us in this paper.

## 2.

In group theory, Hartley [3] has proved that in any group the intersection of any collection of serial subgroups is always serial. In this section we shall first prove the similar results in Lie theory and present a generalization of [4, Proposition 2.6]. We shall secondly characterize $\mathfrak{L}_\infty$-algebras as Lie algebras in which every descendant subalgebra is a subideal.

We begin with

LEMMA 2.1. *Let $L$ be a Lie algebra and $\Delta$ be any of the relations* si, wsi, asc, wasc, dsc, wdsc, ser, wser. *Let $H_\alpha \Delta L (\alpha \in A)$ and set $K = \bigcap_{\alpha \in A} H_\alpha$. Then there exist an ordinal $\sigma$ and a descending chain $\{K_\beta: \beta \leqq \sigma\}$ of subalgebras of $L$ satisfying the following conditions*:

(a) $K_0 = L$ *and* $K_\sigma = K$;

(b) $K_{\beta+1} \Delta K_\beta$ *for all ordinals* $\beta < \sigma$;

(c) $K_\lambda = \bigcap_{\beta<\lambda} K_\beta$ *for all limit ordinals* $\lambda \leqq \sigma$.

*In particular, if* $|A| < \infty$ *then* $\sigma < \omega$ *and so* $K \Delta L$.

PROOF. Let the elements of $A$ be well-ordered as $A = \{\alpha: \alpha < \sigma\}$ for some ordinal $\sigma$. Then we can construct a descending chain $\{K_\beta: \beta \leqq \sigma\}$ as follows: $K_0 = L$, $K_\beta = \bigcap_{\alpha<\beta} H_\alpha$ $(0 < \beta \leqq \sigma)$. It is easy to see that for any ordinal $\beta < \sigma$, $K_{\beta+1} = K_\beta \cap H_\beta \Delta K_\beta$. Therefore $\{K_\beta: \beta \leqq \sigma\}$ is a required chain.

We have the first main result of this section, generalizing [4, Proposition 2.6], in the following

THEOREM 2.2. *Let $L$ be a Lie algebra and $\Delta$ be any of the relations* dsc, wdsc, ser, wser. *If $H_\alpha \Delta L (\alpha \in A)$, then $\bigcap_{\alpha \in A} H_\alpha \Delta L$.*

PROOF. If $\Delta$ means dsc or wdsc, then the result is immediately deduced from Lemma 2.1. Assume that $\Delta$ means ser (resp. wser). Set $K = \bigcap_{\alpha \in A} H_\alpha$. Then by Lemma 2.1 we can easily see that there are a reversely well-ordered set $\Sigma$ and a family $\{\Lambda_\sigma, V_\sigma: \sigma \in \Sigma\}$ of subalgebras of $L$ containing $K$ such that

(a) $V_\sigma$ ser $\Lambda_\sigma$ (resp. $V_\sigma$ wser $\Lambda_\sigma$) for all $\sigma \in \Sigma$,

(b) $\Lambda_\tau \leq V_\sigma$ if $\tau < \sigma$,

(c) $L \backslash K = \bigcup_{\sigma \in \Sigma} (\Lambda_\sigma \backslash V_\sigma)$.

For each $\sigma \in \Sigma$, there are a totally ordered set $\Sigma_\sigma$ and a series (resp. a weak series) $\{\Lambda_{\sigma,\tau}, V_{\sigma,\tau}: \tau \in \Sigma_\sigma\}$ from $V_\sigma$ to $\Lambda_\sigma$. Set $\Sigma^* = \{(\sigma, \tau): \sigma \in \Sigma, \tau \in \Sigma_\sigma\}$. Then we can define a total ordering on $\Sigma^*$ as follows: $(\sigma, \tau) < (\sigma', \tau')$ if $\sigma < \sigma'$ or if $\sigma = \sigma'$ and $\tau < \tau'$. For each $(\sigma, \tau) \in \Sigma^*$, set $\Lambda_{(\sigma,\tau)} = \Lambda_{\sigma,\tau}$ and $V_{(\sigma,\tau)} = V_{\sigma,\tau}$. Then it is not hard to show that $\{\Lambda_{(\sigma,\tau)}, V_{(\sigma,\tau)}: (\sigma, \tau) \in \Sigma^*\}$ is a series (resp. a weak series) from $K$ to $L$. Thus we have $K \operatorname{ser} L$ (resp. $K \operatorname{wser} L$).

REMARK. Theorem 2.2 is not true for any of the relations si, wsi, asc, wasc. In fact, the Lie algebra $L$ constructed in [6, p. 354, Example] has a descending chain $\{H_n: n < \omega\}$ of subideals such that $\bigcap_{n<\omega} H_n$ is self-idealizing in $L$. Then $\bigcap_{n<\omega} H_n$ is not weakly ascendant in $L$.

We can now prove the second main result of this section.

THEOREM 2.3. $\mathfrak{L}_\infty = \mathfrak{D}(\mathrm{dsc}, \mathrm{si})$.

PROOF. By Lemma 2.1 we clearly have $\mathfrak{D}(\mathrm{dsc}, \mathrm{si}) \leq \mathfrak{L}_\infty$. Let $L \in \mathfrak{L}_\infty$ and $H \operatorname{dsc} L$. Then there is a descending series $\{H_\alpha : \alpha \leqq \sigma\}$ from $H$ to $L$. By transfinite induction on $\alpha$ we can show that $H_\alpha \operatorname{si} L$ for each ordinal $\alpha \leqq \sigma$. Hence $L \in \mathfrak{D}(\mathrm{dsc}, \mathrm{si})$ and therefore $\mathfrak{L}_\infty \leq \mathfrak{D}(\mathrm{dsc}, \mathrm{si})$.

REMARK. The proof of Theorem 2.3 carries over in group theory without difficulties. Thus the group-theoretic analogue of Theorem 2.3 is also true.

Using [8, Theorem 2.9(2)] we have $\mathrm{R}\mathfrak{N} \leq \grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}} \leq \tilde{\mathfrak{G}}\mathfrak{r}$. By making use of this result and Theorem 2.3, we obtain the following corollary.

COROLLARY 2.4. $\mathfrak{L}_\infty \cap \grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}} \leq \mathfrak{L}_\infty \cap \tilde{\mathfrak{G}}\mathfrak{r} = \mathfrak{L}_\infty \cap \mathfrak{B}$.

## 3.

In group theory, a group $G$ is said to be a $\mathfrak{B}$-group if there exists an upper bound for the defects of all subnormal subgroups of $G$. Robinson [12, Lemma 1] has proved that every extension of a group having a composition series of finite length by a $\mathfrak{B}$-group is also a $\mathfrak{B}$-group. In this section we shall establish the similar results concerning the classes $\mathfrak{L}_\infty = \mathfrak{D}(\mathrm{dsc}, \mathrm{si})$, $\mathfrak{L}_\infty(\mathrm{asc})$, $\mathfrak{M}$ and $\mathfrak{D}(\mathrm{asc}, \mathrm{si})$ of Lie algebras.

A composition series of finite length for a Lie algebra $L$ is a finite ascending series $\{L_i : i \leqq n\}$ from $\{0\}$ to $L$ with each $L_{i+1}/L_i \in \mathfrak{S}$. The class $\mathrm{E}\mathfrak{S}$ is the class of Lie algebras having composition series of finite length. By [2, Proposition 1.7.5] we have $\mathrm{E}\mathfrak{S} = \text{Min-si} \cap \text{Max-si}$. By making use of [1, Theorem 4.7] and [2, Theorem 8.2.3], we can easily see that $\text{Min-si} \leq \acute{\mathrm{E}}(\lhd)(\mathfrak{F} \cup \mathfrak{S})$. It follows that

$$\text{Min-si} \cap \text{Max-si} \leq \acute{\mathrm{E}}(\lhd)(\mathfrak{F} \cup \mathfrak{S}) \cap \text{Max-si} \leq \mathrm{E}(\lhd)(\mathfrak{F} \cup \mathfrak{S}) \leq \mathrm{E}(\mathfrak{F} \cup \mathfrak{S}).$$

By induction on $n$ we can show that $\mathfrak{F}_n \leq \mathrm{E}\mathfrak{S}$ $(n = 1, 2, \cdots)$. This implies that $\mathrm{E}(\mathfrak{F} \cup \mathfrak{S}) = \mathrm{E}\mathfrak{S}$. Therefore we have

LEMMA 3.1. $\text{Min-si} \cap \text{Max-si} = \mathrm{E}(\lhd)(\mathfrak{F} \cup \mathfrak{S}) = \mathrm{E}(\mathfrak{F} \cup \mathfrak{S}) = \mathrm{E}\mathfrak{S}$.

Now we need the following two lemmas.

LEMMA 3.2. *Let $L$ be a Lie algebra and $n$ be a positive integer. Then $L \in \mathfrak{S}^n$ if and only if whenever $\{L_i : i \leqq m\}$ is a strictly ascending series from $\{0\}$ to $L$ with $m < \omega$, then $m \leqq n$.*

PROOF. Let $\mathfrak{X}_n$ denote the class of Lie algebras $L$ such that whenever $\{L_i$: $i$

$\leqq m\}$ is a strictly ascending series from $\{0\}$ to $L$ with $m < \omega$, then $m \leqq n$. By induction on $n$ we first show that $\mathfrak{S}^n \leq \mathfrak{X}_n$ $(n = 1, 2, \cdots)$. It is trivial for $n = 1$. Let $n \geqq 1$ and $L \in \mathfrak{S}^{n+1}$, and let $\{L_i: i \leqq m\}$ be a strictly ascending series from $\{0\}$ to $L$ with $m < \omega$. By inductive hypothesis $L$ has an ideal $I$ such that $I \in \mathfrak{X}_n$ and $L/I \in \mathfrak{S}$. Let $\psi$ denote the natural map $L \to L/I$. Then $\{\psi(L_i): i \leqq m\}$ (resp. $\{L_i \cap I: i \leqq m\}$) is an ascending series from $\{0\}$ to $L/I$ (resp. $I$). Let $r$ be the smallest integer with respect to $\psi(L_r) = \psi(L)$. In order to show that $m \leqq n + 1$, we may assume that $r > 0$. Let $r \leqq i < m$. Since $L_i + I = L$, by the modular law $L_{i+1} = L_i + (L_{i+1} \cap I)$. It follows that $L_i \cap I < L_{i+1} \cap I$, since $L_i \neq L_{i+1}$. By the minimality of $r$, $L_{r-1} \leqq I$. Hence we have

$$\{0\} = L_0 \cap I < \cdots < L_{r-1} \cap I \leq L_r \cap I < \cdots < L_m \cap I = I.$$

Since $I \in \mathfrak{X}_n$, we have $m \leqq n + 1$. Therefore we obtain $\mathfrak{S}^{n+1} \leq \mathfrak{X}_{n+1}$. This completes the induction. Conversely we show that $\mathfrak{X}_n \leq \mathfrak{S}^n$. Let $L \in \mathfrak{X}_n \backslash \{0\}$ and let $\mathscr{S}$ denote the collection of strictly ascending series from $\{0\}$ to $L$ of finite length. Since $L \in \mathfrak{X}_n$, $\mathscr{S}$ has an element $\{L_i: i \leqq m\}$ of maximal length. By the maximality of $m$ we have $L_{i+1}/L_i \in \mathfrak{S}$ $(0 \leqq i < m)$. Hence $L \in \mathfrak{S}^m \leq \mathfrak{S}^n$, since $m \leqq n$. Thus we obtain $\mathfrak{X}_n \leq \mathfrak{S}^n$.

LEMMA 3.3. *Let $L$ be a Lie algebra and $I$ be an ideal of $L$ such that $I \in$ Min-si $\cap$ Max-si. Then there exists a positive integer $n$ such that for any ascendant subalgebra $H$ of $L$, $H$ si $H + I$ with* si$(H + I: H) \leqq n$.

PROOF. By Lemma 3.1 there exists a positive integer $n$ such that $I \in \mathfrak{S}^n$. Since $H \operatorname{asc} L$, $H \operatorname{asc} H + I$. Let $\{H_\alpha: \alpha \leqq \sigma\}$ be a strictly ascending series from $H$ to $H + I$. For any $\alpha < \sigma$, since $H + (H_\alpha \cap I) = H_\alpha \neq H_{\alpha+1} = H + (H_{\alpha+1} \cap I)$, we have $H_\alpha \cap I \neq H_{\alpha+1} \cap I$. Hence $\{H_\alpha \cap I: \alpha \leqq \sigma\}$ is a strictly ascending series from $H \cap I$ to $I$. Owing to [14, Theorem], we have $I \in \mathfrak{D}$(asc, si) $\cap$ Max-si $\leq$ Max-asc. Thus $\sigma$ must be a finite ordinal. Since $\{0\} \leq H_0 \cap I < H_1 \cap I < \cdots < H_\sigma \cap I = I$, by Lemma 3.2 we have $\sigma \leqq n$. Therefore $H$ si $H + I$ and si$(H + I: H) \leqq n$.

We now set about proving the main result of this section.

THEOREM 3.4. *Let $\mathfrak{X}$ be any of the classes $\mathfrak{L}_\infty$, $\mathfrak{L}_\infty$(asc), $\mathfrak{M}$ and $\mathfrak{D}$(asc, si). Then* (Min-si $\cap$ Max-si)$\mathfrak{X} = \mathfrak{X}$.

PROOF. Let $L \in$ (Min-si $\cap$ Max-si)$\mathfrak{X}$. Then $L$ has an ideal $I$ such that $I \in$ Min-si $\cap$ Max-si and $L/I \in \mathfrak{X}$. By Lemma 3.3 there exists a positive integer $n$ such that for any $H \operatorname{asc} L$, $H$ si $H + I$ with si$(H + I: H) \leqq n$. Then for each of the following three cases, we show that $L \in \mathfrak{X}$.

Case 1. $\mathfrak{X} = \mathfrak{M}$; then $L/I \in \mathfrak{M}_m$ for some integer $m > 0$. Let $H$ si $L$. Then $H \lhd^n H + I$. Since $(H + I)/I$ si $L/I$, we have $H + I \lhd^m L$. It follows that $H \lhd^{m+n} L$. Therefore we have $L \in \mathfrak{M}_{m+n} \leq \mathfrak{M}$.

Case 2. $\mathfrak{X} = \mathfrak{D}(\mathrm{asc}, \mathrm{si})$; for any ascendant subalgebra $H$ of $L$, we have $H \lhd^n H + I \,\mathrm{si}\, L$, since $(H + I)/I \,\mathrm{asc}\, L/I \in \mathfrak{D}(\mathrm{asc}, \mathrm{si})$. Thus $L \in \mathfrak{D}(\mathrm{asc}, \mathrm{si})$.

Case 3. $\mathfrak{X} = \mathfrak{L}_\infty$ (resp. $\mathfrak{L}_\infty(\mathrm{asc})$); then we set $\Delta = \mathrm{si}$ (resp. asc). Let $H_\alpha \Delta L$ and $K_\alpha = H_\alpha + I$ $(\alpha \in A)$. For each $\alpha \in A$, since $H_\alpha \,\mathrm{si}\, K_\alpha$ with $\mathrm{si}(K_\alpha\colon H_\alpha) \leqq n$, we have

$$H_\alpha = H_\alpha^{K_\alpha, n} \lhd H_\alpha^{K_\alpha, n-1} \lhd \cdots \lhd H_\alpha^{K_\alpha, 0} = K_\alpha.$$

Therefore we have

$$\bigcap\nolimits_{\alpha \in A} H_\alpha = \bigcap\nolimits_{\alpha \in A} H_\alpha^{K_\alpha, n} \lhd \bigcap\nolimits_{\alpha \in A} H_\alpha^{K_\alpha, n-1} \lhd \cdots \lhd \bigcap\nolimits_{\alpha \in A} H_\alpha^{K_\alpha, 0} = \bigcap\nolimits_{\alpha \in A} K_\alpha.$$

Since for each $\alpha \in A$ $K_\alpha/I \,\Delta\, L/I \in \mathfrak{L}_\infty$ (resp. $\mathfrak{L}_\infty(\mathrm{asc})$), $\bigcap_{\alpha \in A} K_\alpha \,\Delta\, L$. Hence $\bigcap_{\alpha \in A} H_\alpha \,\mathrm{si} \bigcap_{\alpha \in A} K_\alpha \,\Delta\, L$ and therefore $\bigcap_{\alpha \in A} H_\alpha \,\Delta\, L$. Thus we have $L \in \mathfrak{L}_\infty$ (resp. $\mathfrak{L}_\infty(\mathrm{asc})$).

By [2, Lemma 1.3.7] we have $\mathfrak{N} \leq \mathfrak{M}$. On the other hand, for a Lie algebra $L$ it is well known that if $L = \zeta_\sigma(L)$ for some ordinal $\sigma$, then for any subalgebra $H$ of $L$, $\{H + \zeta_\alpha(L)\colon \alpha \leqq \sigma\}$ is an ascending series from $H$ to $L$. It follows that $\mathfrak{Z} \leq \mathfrak{D}(\mathrm{asc}) \leq \mathfrak{L}_\infty(\mathrm{asc})$. By making use of these results, [5, Lemma 3.2] and Theorems 2.3 and 3.4, we obtain

Corollary 3.5. (1) $\mathfrak{F}\mathfrak{N} \leq (\text{Min-si} \cap \text{Max-si})\mathfrak{N}$
$\leq (\text{Min-si} \cap \text{Max-si})\mathfrak{M} = \mathfrak{M}$
$\leq (\text{Min-si} \cap \text{Max-si})\mathfrak{L}_\infty = \mathfrak{L}_\infty = \mathfrak{D}(\mathrm{dsc}, \mathrm{si})$.

(2) $\mathfrak{F}\mathfrak{Z} \leq (\text{Min-si} \cap \text{Max-si})\mathfrak{D}(\mathrm{asc})$
$\leq (\text{Min-si} \cap \text{Max-si})\mathfrak{L}_\infty(\mathrm{asc}) = \mathfrak{L}_\infty(\mathrm{asc})$.

Remark. As stated in Corollary 3.5 we have $\mathfrak{F}\mathfrak{N} \leq \mathfrak{M} \leq \mathfrak{L}_\infty$ and $\mathfrak{F}\mathfrak{Z} \leq \mathfrak{L}_\infty(\mathrm{asc})$. However, we should note that $\mathfrak{N}\mathfrak{F} \not\leq \mathfrak{L}_\infty$ and $\mathfrak{Z}\mathfrak{F} \not\leq \mathfrak{L}_\infty(\mathrm{asc})$. In fact, by [6, p. 354, Example] and the proof of [7, Theorem 5.1(3)] we have $\mathfrak{A}\mathfrak{F}_1 \not\leq \mathfrak{L}_\infty$ and $\mathfrak{A}_1 \not\leq \mathfrak{L}_\infty(\mathrm{asc})$, respectively. Furthermore, the latter fact is in contrast to the fact that $\mathfrak{A}_1 \leq \mathfrak{M}$ ([5, Theorem 2.10]).

In Corollary 3.5, if the ground field $\mathfrak{k}$ is of characteristic $p > 0$, then by [5, Remark to Lemma 3.2] we have $\mathfrak{M} < \mathfrak{L}_\infty$. Moreover, the following example shows that even if the ground field $\mathfrak{k}$ is of arbitrary characteristic, then $(\text{Min-si} \cap \text{Max-si})\mathfrak{N} < \mathfrak{M}$ and $(\text{Min-si} \cap \text{Max-si})\mathfrak{D}(\mathrm{asc}) < \mathfrak{L}_\infty(\mathrm{asc})$.

Example 3.6. Let $A$ be an abelian Lie algebra over $\mathfrak{k}$ with basis $\{a_i\colon i \in \boldsymbol{Z}\}$. For a derivation $x$ of $A$, $x \in T$ if its image $Ax$ is of finite-dimensional and if the restriction of $x$ to $Ax$ has trace zero in the uaual sense. Then by [13, Lemma 4.1] $T$ is an infinite-dimensional simple Lie algebra. We constuct the split extension $L = A \dotplus T$ of $A$ by $T$, which is one of the Lie algebras constructed in [6, p. 355, Example]. Then evidently

$$\{H\colon H \leq A \text{ or } H = L\} \subseteq \{H\colon H \lhd^2 L\} \subseteq \{H\colon H \,\mathrm{asc}\, L\}.$$

Let $H \operatorname{asc} L$. Levič [10] has proved that every simple Lie algebra has no non-trivial ascendant subalgebras. By using this result, we can see that $H \leq A$ or $H + A = L$. Assume that $H + A = L$. For each $i \in \boldsymbol{Z}$, we define a derivation $x_i$ of $A$ by $a_j x_i = \delta_{ij} a_i - \delta_{i+1j} a_{i+1}$ $(j \in \boldsymbol{Z})$. Clearly $x_i \in T$. Since $L = H + A$, $x_i = h + a$ for some $h \in H$ and some $a \in A$. By using [7, Lemma 2.1], we can find an integer $n > 0$ such that $[a_{i},{}_n h] \in H$. Since $A$ is an abelian ideal of $L$, $[a_{i},{}_n h] = [a_{i},{}_n x_i] = a_i$. It follows that $a_i \in H$ for all $i \in \boldsymbol{Z}$. Hence $L = H + A = H$ and therefore

$$\{H\colon H \operatorname{asc} L\} = \{H\colon H \leq A \text{ or } H = L\} = \{H\colon H \lhd^2 L\}.$$

Thus we have $L \in \mathfrak{M} \cap \mathfrak{D}(\text{asc, si}) \leq \mathfrak{L}_\infty(\text{asc})$. Next we prove that $L$ is not in the class $(\text{Min-si} \cap \text{Max-si})\mathfrak{D}(\text{asc})$. Assume, to the contrary, that $L$ has an ideal $I$ such that $I \in \text{Min-si} \cap \text{Max-si}$ and $L/I \in \mathfrak{D}(\text{asc})$. Obviously $L$ is not in the class $\text{Min-si} \cup \text{Max-si}$. Hence $I \leq A$. Since $A/I \lhd L/I \in \mathfrak{D}(\text{asc}) = \text{Q}\mathfrak{D}(\text{asc})$, we have $T \cong (L/I)/(A/I) \in \mathfrak{D}(\text{asc})$. But $T$ has no non-trivial ascendant subalgebras. Thus $T \in \mathfrak{F}_1$, a contradiction.

Finally we show that there is no inclusion between the class $\mathfrak{L}_\infty(\text{asc})$ and the class $\mathfrak{D}(\text{asc, si})$, and neither between the class $\mathfrak{L}_\infty = \mathfrak{D}(\text{dsc, si})$ and the class $\mathfrak{D}(\text{asc, si})$, in the following

PROPOSITION 3.7. (1) $\mathfrak{L}_\infty(\text{asc}) \not\leq \mathfrak{D}(\text{asc, si})$ *and* $\mathfrak{L}_\infty \not\leq \mathfrak{D}(\text{asc, si})$.

(2) *Assume that the ground field* $\mathfrak{k}$ *is of characteristic zero. Then* $\mathfrak{D}(\text{asc, si}) \not\leq \mathfrak{L}_\infty(\text{asc})$ *and* $\mathfrak{D}(\text{asc, si}) \not\leq \mathfrak{L}_\infty$.

PROOF. (1) Let $X$ be an abelian Lie algebra over $\mathfrak{k}$ with basis $\{x_i\colon i = 0, 1, \cdots\}$ and $\sigma$ be a derivation of $X$ such that $x_0\sigma = 0$ and $x_i\sigma = x_{i-1}$ $(i \geqq 1)$. Form the split extension $L = X \dotplus \langle\sigma\rangle$ of $X$ by $\langle\sigma\rangle$. Then it is well known (cf. [2, p.119]) that $L \in \mathfrak{Z} \setminus \mathfrak{B}$. Thus we have $L \in \mathfrak{D}(\text{asc}) \setminus \mathfrak{D}(\text{asc, si})$. Since $L^2 = [X, \sigma] = X$, $L^2$ is of codimension 1 and so $L \in \mathfrak{A}_1$. By [5, Lemma 2.9] we have $L \in \mathfrak{L}_\infty$.

(2) Let $W$ be a Lie algebra over $\mathfrak{k}$ with basis $\{w_i\colon i = 1, 2, \cdots\}$ and multiplication $[w_i, w_j] = (i - j)\, w_{i+j}$. Then by [2, Theorem 8.7.1] and [9, Theorem] we have $W \in \text{R}\mathfrak{N} \cap \text{Max}$. By using induction on $n$, we can easily see that $[w_2,{}_n w_1] = (n!) w_{n+2}$ $(n = 0, 1, \cdots)$. Hence $\langle w_1\rangle$ is not a subideal of $W$ and therefore $W$ is not in the class $\mathfrak{B}$. It follows from Corollary 2.4 that $W$ is not in the class $\mathfrak{L}_\infty$. It is clear that $\text{Max} \leq \mathfrak{D}(\text{asc, si})$. Therefore we obtain $W \in \mathfrak{D}(\text{asc, si}) \setminus \mathfrak{L}_\infty(\text{asc})$.

## 4.

As stated in Section 3 the inclusions $\mathfrak{N} \leq \mathfrak{M} \leq \mathfrak{L}_\infty$ hold. In this section we shall present sufficient conditions for a Lie algebra in the class $\mathfrak{L}_\infty$ to be in the class $\mathfrak{M}$ and, furthermore, to be nilpotent.

We need the following lemma.

LEMMA 4.1. *Let $n$ be a positive integer and let $L\in\mathfrak{L}_\infty\cap\mathfrak{A}\mathfrak{N}_n$. Then:*

(1) *To each subalgebra $H$ of $L$, there corresponds an integer $m\geqq 0$ such that $[L^{n+1},_m H]+H=[L^{n+1},_{m+1} H]+H$.*

(2) *If $L$ has an abelian ideal $A$ and a nilpotent subalgebra $N$ of class $\leqq n$ such that $L=A+N$ and $A\cap N=\{0\}$, then $L^{m+n+1}=L^{m+n+2}$ for some integer $m\geqq 0$.*

PROOF. (1) Let $H\leq L$ and $M=H+L^{n+1}$. Since $L^{n+1}$ is an abelian ideal of $L$, as in the proof of [4, Lemma 4.1] we can easily see that $H^{M,\alpha}=H_{M,\alpha}$ for all ordinals $\alpha$. It is clear that $M=H+L^{n+1}\lhd H+L^n\lhd\cdots\lhd H+L=L$. Hence we have $M\in\mathfrak{L}_\infty$. Therefore by [5, Proposition 3.1(3)] there exists an integer $m=m(H)\geqq 0$ such that $H^{M,m}=H^{M,m+1}$. It follows that $H_{M,m}=H_{M,m+1}$. Moreover, for any integer $k\geqq 0$, $H_{M,k}=[L^{n+1},_k H]+H$. Thus we have the result.

(2) By (1) there exists an integer $m\geqq 0$ such that $[L^{n+1},_m N]+N=[L^{n+1},_{m+1} N]+N$. Since $L^{n+1}\leq A\in\mathfrak{A}$, for any integer $k\geqq 0$, $[L^{n+1},_k N]=[L^{n+1},_k A+N]=L^{k+n+1}$. Hence $L^{m+n+1}+N=L^{m+n+2}+N$ and therefore $L^{m+n+1}=L^{m+n+2}+(L^{m+n+1}\cap N)=L^{m+n+2}$.

As stated in [6, p.354, Example], Lie algebras in the class $\mathfrak{A}\mathfrak{F}_1$ need not be in the class $\mathfrak{L}_\infty$. However, the following proposition shows that if a Lie algebra in the class $\mathfrak{A}\mathfrak{F}_1$ is also in the class $\mathfrak{L}_\infty$, then it must be in the class $\mathfrak{M}$.

PROPOSITION 4.2. $\mathfrak{L}_\infty\cap\mathfrak{A}\mathfrak{F}_1=\mathfrak{M}\cap\mathfrak{A}\mathfrak{F}_1$.

PROOF. Let $L\in\mathfrak{L}_\infty\cap\mathfrak{A}\mathfrak{F}_1$. Then $L$ has an abelian ideal $A$ and an element $x$ such that $L=A+\langle x\rangle$. We may suppose that $A\cap\langle x\rangle=\{0\}$. By Lemma 4.1(2) $L^{m+2}=L^{m+3}$ for some integer $m\geqq 0$. It follows that $L^\omega=L^{m+2}$. By induction on $k$ we can easily verify that $L^{k+1}=[A,_k x]$ $(k=1,2,\cdots)$. Let $H$ si $L$ with $s=\mathrm{si}(L:H)$. We show that $s\leq m+2$. It is clear that $H\leq A$ or $H+A=L$. If $H\leq A$ then $s\leqq 2\leqq m+2$. Assume that $H+A=L$. Then $x=h+a$ for some $h\in H$ and some $a\in A$. Since $H\lhd^s L$, $L^{s+1}=[A,_s x]=[A,_s h]\leq H$. Hence $L^{m+2}=L^\omega\leq L^{s+1}\leq H$. It follows that $H=H+L^{m+2}\lhd H+L^{m+1}\lhd\cdots\lhd H+L=L$. Therefore we have $s\leqq m+1<m+2$, so that $L\in\mathfrak{M}_{m+2}$. Thus we obtain $\mathfrak{L}_\infty\cap\mathfrak{A}\mathfrak{F}_1\leq\mathfrak{M}\cap\mathfrak{A}\mathfrak{F}_1$.

Finally we consider a sufficient condition for a Lie algebra in the class $\mathfrak{L}_\infty$ to be nilpotent.

As stated in Remark to Corollary 3.5, Lie algebras in the class $\mathfrak{N}\mathfrak{G}$ need not be in the class $\mathfrak{L}_\infty$. Moreover, Lie algebras in the class $\mathfrak{L}_\infty\cap\mathfrak{N}\mathfrak{G}$ need not be nilpotent (see Remark (2) to Lemma 4.3). However, we have

LEMMA 4.3. $\mathfrak{L}_\infty \cap \mathfrak{N}\mathfrak{G} \cap \acute{\mathfrak{G}}\mathrm{r} = \mathfrak{N}$.

PROOF. Since by Corollary 2.4 $\mathfrak{L}_\infty \cap \acute{\mathfrak{G}}\mathrm{r} \leq \mathfrak{B}$, it is enough to show that $\mathfrak{B} \cap \mathfrak{N}\mathfrak{G} \leq \mathfrak{N}$. Let $L \in \mathfrak{B} \cap \mathfrak{N}\mathfrak{G}$. Then $L$ has a nilpotent ideal $N$ such that $L/N$ is finitely generated. There is a finitely generated subalgebra $X$ of $L$ such that $L = X + N$. By [2, Theorem 7.1.5(b), (c)] $X$ is a nilpotent subideal of $L$. It follows from [2, Theorem 2.2.13] that $L = X + N \in \mathfrak{N}$. Thus we have $\mathfrak{B} \cap \mathfrak{N}\mathfrak{G} \leq \mathfrak{N}$.

REMARK. (1) If we remove the class $\mathfrak{L}_\infty$ from the equation stated in Lemma 4.3, then it becomes a failure. In fact, the Lie algebra constructed in [6, p.354, Example] is not a Gruenberg algebra, but in the class $\mathfrak{A}\mathfrak{F}_1 \cap \mathrm{R}\mathfrak{N}$. It follows from [8, Theorem 2.9(2)] that $\mathfrak{N}\mathfrak{G} \cap \acute{\mathfrak{G}}\mathrm{r} > \mathfrak{N}$.

(2) If we replace the class $\acute{\mathfrak{G}}\mathrm{r}$ with the class $\mathfrak{G}\mathrm{r}$ in the equation stated in Lemma 4.3, then it becomes a failure. In fact, the Lie algebra constructed in the proof of Proposition 3.7(1) is not a Baer algebra, but in the class $\mathfrak{L}_\infty \cap \mathfrak{A}\mathfrak{F}_1 \cap \mathfrak{Z}$. This implies that $\mathfrak{L}_\infty \cap \mathfrak{N}\mathfrak{G} \cap \mathfrak{G}\mathrm{r} > \mathfrak{N}$.

LEMMA 4.4. $\mathfrak{M} \cap \acute{\mathfrak{G}}\mathrm{r} = \mathfrak{M} \cap \acute{\mathrm{E}}(\lhd)\hat{\mathfrak{A}} = \mathfrak{M} \cap \mathfrak{B} = \mathfrak{N}$.

PROOF. Using [8, Theorem 2.9(2)] we have $\mathfrak{N} \leq \mathrm{R}\mathfrak{N} \leq \acute{\mathrm{E}}(\lhd)\hat{\mathfrak{A}} \leq \acute{\mathfrak{G}}\mathrm{r}$. It is clear that $\mathfrak{N} \leq \mathfrak{B} \leq \acute{\mathfrak{G}}\mathrm{r}$. Therefore it is enough to prove that $\mathfrak{M} \cap \acute{\mathfrak{G}}\mathrm{r} \leq \mathfrak{N}$. Let $L \in \mathfrak{M} \cap \acute{\mathfrak{G}}\mathrm{r}$. Then $L \in \mathfrak{M}_n$ for some integer $n > 0$. Since $L \in \mathfrak{M} \leq \mathfrak{L}_\infty$, by Corollary 2.4 we have $L \in \mathfrak{B}$. Let $x_i \in L\,(1 \leqq i \leqq n)$ and set $H = \langle x_i \colon 1 \leqq i \leqq n \rangle$. By [2, Theorem 7.1.5(c)] we have $H \,\mathrm{si}\, L$, so that $H \lhd^n L$. Hence $L \in \mathfrak{D}_{n,n}$. Owing to [2, Theorem 7.2.5], we have $L \in \mathfrak{N}$.

PROPOSITION 4.5. $\mathfrak{L}_\infty \cap \mathfrak{G}\mathfrak{N}\mathfrak{G} \cap \acute{\mathfrak{G}}\mathrm{r} = \mathfrak{N}$.

PROOF. By making use of Corollary 3.5(1), Lemma 4.4 and [2, Theorem 7.1.5(b)], we can easily see that

$$\mathfrak{B} \cap \mathfrak{G}\mathfrak{N} = \mathfrak{B} \cap (\mathfrak{B} \cap \mathfrak{G})\mathfrak{N} = \mathfrak{B} \cap \mathfrak{F}\mathfrak{N} = \mathfrak{B} \cap \mathfrak{M} = \mathfrak{N}.$$

Therefore by Corollary 2.4 we have $\mathfrak{L}_\infty \cap \mathfrak{G}\mathfrak{N}\mathfrak{G} \cap \acute{\mathfrak{G}}\mathrm{r} = \mathfrak{L}_\infty \cap \mathfrak{N}\mathfrak{G} \cap \mathfrak{B}$. Thus the result is immediately deduced from Lemma 4.3.

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*Niigata College of Pharmacy*