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Radially symmetric solutions of semilinear elliptic equations, existence and Sobolev estimates

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1. Introduction

In this paper we consider radially symmetric solutions to the semilinear elliptic equation

$$\Delta u + f(|x|, u) = 0, \qquad x \in \Omega, \tag{1.1}$$

where $\Omega \equiv \{x \in \boldsymbol{R}^n: |x| < 1\}$, $n \geq 2$, and the function $f(t, u)$ is assumed to be continuous in $[0, 1] \times \boldsymbol{R}$. In order to discuss radially symmetric solutions $u = u(t)$, $t = |x|$, it is natural to convert equation (1.1) to the second order ordinary differential equation

$$u'' + \frac{n-1}{t} u' + f(t, u) = 0, \qquad 0 < t < 1, \tag{1.2}$$

$$u'(0) = 0. \tag{1.3}$$

In the present paper we establish the existence of infinitely many solutions of equation (1.2) under the boundary conditions

$$u'(0) = 0, \qquad au(1) + bu'(1) = 0, \tag{1.4}$$

for any coefficients a and b. Moreover we investigate the Sobolev norms of solutions of the problem (1.2)–(1.3) in conjunction with their zeros. We treat the nonlinear function $f(t, s)$ under superlinear growth conditions and sublinear growth conditions. In the present paper the function f is said to be *superlinear* (in a neighborhood of $s = \pm\infty$) if

$$\lim_{s \to \pm\infty} \frac{f(t, s)}{s} = \infty \qquad \text{uniformly in} \quad t \in [0, 1]. \tag{1.5}$$

On the other hand, f is said to be *sublinear* (in a neighborhood of $s = 0$) if

$$\lim_{s \to 0} \frac{f(t, s)}{s} = \infty \qquad \text{uniformly in} \quad t \in [0, 1]. \tag{1.6}$$

Equation (1.2) can be written as

$$(t^{n-1}u')' + t^{n-1}f(t, u) = 0, \qquad t \in (0, 1), \tag{1.2$'$}$$

and so we can treat the problem (1.2)$'$–(1.4) as a singular boundary value problem for a nonlinear Sturm-Liouville equation.

In the case where $f(t, s)$ is superlinear and odd in s and satisfies some growth conditions, Ambrosetti and Rabinowitz showed in [1] that equation (1.1) with Dirichlet boundary condition possesses infinitely many solutions in any bounded domain Ω. More precisely, they obtained an unbounded sequence of solutions in Sobolev space $H_0^1(\Omega)$. See [2, 6, 8, 9, 10] for the related results.

In the case where Ω is the unit ball and f is superlinear, the existence of infinitely many radially symmetric solutions has been investigated by Castro-Kurepa [4] and Struwe [11]. In fact, they treated equation (1.2) under Dirichlet boundary condition, namely,

$$u'(0) = 0, \qquad u(1) = 0. \tag{1.7}$$

Then Struwe [11] proved by means of a variational method that there is an integer k_0 such that for any $k \geq k_0$ the problem (1.2)–(1.7) admits a solution with exactly k zeros in the interval [0, 1]. On the other hand, Castro-Kurepa [4] obtained the same results under weaker assumptions on the nonlinear term $f(t, s)$ by applying so-called shooting method. In the case of $n = 1$ Berestycki [3] gave similar results with the aid of bifurcation theory. As mentioned above, many authors have dealt with equation (1.2) under Dirichlet boundary condition (1.7) in the superlinear case. However, it seems to the author that very little is known about the existence of infinitely many solutions of (1.2) under the other types of boundary conditions or in the sublinear case. In the present paper we establish the existence of infinitely many solutions under arbitrary boundary conditions in both superlinear and sublinear cases.

On the other hand, it is an interesting problem to study the relation between $H^1(\Omega)$ norms of radially symmetric solutions and the numbers of their zeros. Recently, the author [7] has obtained the following result for this problem. Consider the superlinear function $f(t, s) \equiv g(s)$ satisfying

$$g(0) = 0, \qquad g'(0) = \lim_{s \to 0} \frac{g(s)}{s} \quad \text{exists} \tag{1.8}$$

and

$$0 < a_1 \leq g(s)/|s|^{p-1}s \leq a_2 \tag{1.9}$$

for sufficiently large $|s|$ and some constants a_1, a_2 and $1 < p < (n + 2)/(n - 2)$.

Under appropriate additional conditions on g, the author obtained the estimate

$$c_1 k^{(p+1)/(p-1)} \leq \|u\|_{H_0^1(\Omega)} \leq c_2 k^{(p+1)/(p-1)} \tag{1.10}$$

for any solution u of the problem (1.2)–(1.7) having exactly k zeros. In this assertion condition (1.8) can not be removed so far as one employs the method as in [7]. However, as mentioned before, the existence result of infinitely many solutions has been obtained without assuming condition (1.8). See [4] and [11]. Therefore in the present paper we prove the estimate (1.10) without condition (1.8) by exploiting a new approach. It should be mentioned at this point that the estimate (1.10) can be derived under condition (1.3) which is weaker than (1.7). Actually, for the derivation of (1.10) we do not need any boundary condition at the end point $t = 1$.

Outline. In this paper we present four theorems. These are concerned with the existence and the Sobolev estimates for solutions of (1.2). The first and second theorems assert respectively that there exist infinitely many solutions of (1.2) under boundary conditions of the form (1.4) in the sublinear and superlinear cases. In fact, a positive integer k_0 can be found such that for any coefficients a and b in (1.4) there exist two sequences $\{u_k^+\}_{k \geq k_0}$ and $\{u_k^-\}_{k \geq k_0}$ of solutions to (1.2)–(1.4) such that both u_k^+ and u_k^- have exactly k zeros in the interval $[0, 1]$ and $u_k^+(0) > 0$, $u_k^-(0) < 0$. Moreover we have

$$\lim_{k\to\infty} \|u_k^{\pm}\|_{L^\infty(\Omega)} = 0, \qquad \lim_{k\to\infty} \|u_k^{\pm}\|_{H^1(\Omega)} = 0 \tag{1.11}$$

in the sublinear case;

$$\lim_{k\to\infty} \|u_k^{\pm}\|_{L^\infty(\Omega)} = \infty, \qquad \lim_{k\to\infty} \|u_k^{\pm}\|_{H^1(\Omega)} = \infty \tag{1.12}$$

in the superlinear case, respectively.

Our third and fourth theorems may be illustrated as follows: In these theorems the $H^1(\Omega)$ estmates for the solutions are given in terms of the numbers of zeros of the solutions. The third theorem is concerned with the sublinear case. More precisely, we suppose that

$$0 < a_1 \leq f(t, s)/|s|^{p-1} s \leq a_2 \tag{1.13}$$

for sufficiently small $|s|$ and some constants a_1, a_2 and $0 < p < 1$. Under some additional conditions on $f(t, s)$, we obtain the estimate

$$c_1 k^{-(1+p)/(1-p)} \leq \|u\|_{H^1(\Omega)} \leq c_2 k^{-(1+p)/(1-p)} \tag{1.14}$$

for any solution u of (1.2)–(1.3) with exactly k zeros. The fourth theorem is formulated for the superlinear case. In this theorem we assume (1.13) for sufficiently large $|s|$ and some constants a_1, a_2 and p with $1 < p < (n+2)/(n-2)$. Then we obtain the estimate of the form

$$c_1 k^{(p+1)/(p-1)} \leq \|u\|_{H^1(\Omega)} \leq c_2 k^{(p+1)/(p-1)} \tag{1.15}$$

for any solution u of (1.2)–(1.3) with exactly k zeros. This theorem improves and extends our earlier results given in [7]. In fact, we needed to assume condition (1.8) in [7] but do not necessitate to assume (1.8) in this paper. Also, these two theorems provide the estimates for the Sobolev norms of solutions in which boundary conditions at $t = 1$ do not affect. More precisely, all solutions of (1.2)–(1.4) are estimated in the form (1.14) or (1.15) in terms of the $H^1(\Omega)$ norm, although the constants c_1 and c_2 in (1.14) or (1.15) are independent of the coefficients a and b in the bundary condition (1.4).

We now sketch the proofs of the above-mentioned theorems. Our main tools for the proofs are combinations of shooting method, various a priori estimates for the solutions and new types of Prüfer transformations. We here employ the following Prüfer transformation:

$$|u'|^{(1-p)/(1+p)} u' = \rho \cos \varphi,$$

$$u = \rho \sin \varphi,$$

$$\varphi(0) = \frac{\pi}{2},$$

in the sublinear case, where $0 < p < 1$, and

$$u' = \rho \cos \varphi,$$

$$|u|^{(p-1)/2} u = \rho \sin \varphi,$$

$$\varphi(0) = \frac{\pi}{2},$$

in the superlinear case, where $1 < p < (n+2)/(n-2)$. By means of this transformation one can define new unknown functions $\rho(t)$ and $\varphi(t)$, and it is seen that these functions are well defined provided $u(t)$ has only simple zeros in the interval $[0, 1]$. We here say that a zero τ of u is a *simple zero* if $u(\tau) = 0$ and $u'(\tau) \neq 0$. We prepare several lemmas and a priori estimates for the solutions. We can then prove that $u(t)$ has only simple zeros, and so our Prüfer transformations make sense. Moreover we show that for any solution u,

$$k = \#\{t \in [0, 1] : u(t) = 0\} \tag{1.16}$$

if and only if

$$k\pi \leq \varphi(1) < (k+1)\pi. \tag{1.17}$$

That is, the value of $\varphi(1)$ determines the number of zeros of u. On the other hand, using the Prüfer transformation, we obtain the relation

$$\varphi'(t) = F(t, u(t), u'(t)), \tag{1.18}$$

where F is a function determined by equation (1.2) and can be written explicitly. Integrating (1.18) over $[0, 1]$, we have

$$\varphi(1) - \varphi(0) = \int_0^1 F(t, u(t), u'(t))\, dt. \tag{1.19}$$

The left-hand side represents the number of zeros of $u(t)$ by (1.17). We apply appropriate a priori inequalities, and so the right-hand side can be estimated in terms of the $H^1(\Omega)$ norm of $u(t)$. Thus we obtain the third and the fourth theorems. The first and second theorems are obtained by applying a shooting method together with the relation (1.19). To prove these four theorems, we prepare four propositions, Propositions 3.1, 4.1, 5.1 and 6.1, which are crucial for our discussions. In fact, Propositions 3.1 and 5.1 contain various a priori estimates for the solutions of (1.2)–(1.3) in the sublinear and superlinear cases, respectively. On the other hand, Propositions 4.1 and 6.1 give the relations between the total number of zeros of the solution u of (1.2)–(1.3) and its value $u(0)$ at $t = 0$.

This paper is organized as follows: In Section 2, we state our main results in Theorems 1, 2, 3 and 4 together with some examples. It turns out that the existence of infinitely many solutions to the problem (1.2)–(1.4) is proved and the precise estimates for the $H^1(\Omega)$ norm of the solutions are established. Our theorems are applied to Emden-Fowler type equations. We can discuss the existence of solutions and investigate the asymptotic distribution of the solutions in the Sobolev space $H^1(\Omega)$.

Section 3 deals with the sublinear case and several a priori inequalities are given for the solutions. These estimates together play an important role in proving our theorems.

Section 4 contains the proofs of Theorems 1 and 3. Here we introduce a new Prüfer transformation to treat the sublinear case. This transformation and the a priori estimates obtained in Section 3 are basic to our arguments in this section.

Section 5 concerns the superlinear case and several a priori estimates for the solutions are given.

Finally, in Section 6, we prove Theorems 2 and 4. To this end, we introduce another Prüfer transformation for the superlinear case. Using this transformation and the a priori estimates obtained in Section 5, we prove Theorems 2 and 4.

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2. Main results and examples

We begin this section by introducing some notations and definitions which are used in this paper. First $\Omega \equiv \{x \in \boldsymbol{R}^n : |x| < 1\}$ is the open unit ball in $\boldsymbol{R}^n$. We denote by $L^q(\Omega)$, $1 \le q \le \infty$, and by $H^1(\Omega)$ the usual Lebesgue and Sobolev spaces, respectively. The function spaces L^q and H denote respectively the subspaces of $L^q(\Omega)$ and $H^1(\Omega)$ which consist of radially symmetric functions. We define the norm $\|\cdot\|_q$ of L^q and the norm $\|\cdot\|_H$ of H by

$$\|u\|_q = \left(\int_0^1 |u(t)|^q t^{n-1}\, dt\right)^{1/q}, \qquad 1 \le q < \infty,$$

$$\|u\|_\infty = \text{ess. sup}_{0 \le t \le 1} |u(t)|,$$

and

$$\|u\|_H = \left(\int_0^1 (u(t)^2 + u'(t)^2)\, t^{n-1}\, dt\right)^{1/2},$$

respectively. Given $r > 0$ the symbols

$$B_\infty(r) = \{u \in L^\infty : \|u\|_\infty < r\}$$

and

$$B(r) = \{u \in H : \|u\|_H < r\}$$

stand for the open balls centered at 0 in L^∞ and H, respectively. Moreover we write S for the set of nontrivial solutions $u \in C^2(0, 1) \cap C^1[0, 1]$ of the problem (1.2)–(1.3). By a *nontrivial solution* we mean a solution u of (1.2)–(1.3) such that $u \not\equiv 0$. For $k \in \boldsymbol{N}$, S_k denotes the set of all solutions of (1.2)–(1.3) which have exactly k zeros in the unit interval $[0, 1]$. Lastly, given an integer n with $n \ge 2$, the symbol n^* means $n^* = \infty$ if $n = 2$ and $n^* = (n+2)/(n-2)$ if $n \ge 3$. In what follows, we impose one of the following two assumptions (A) and (B) on the function f.

ASSUMPTION (A) (Sublinearity of f). The nonlinear term f is assumed to be of the form $f(t, s) \equiv g(s) + h(t, s)$, and $g(s)$ and $h(t, s)$ are continuous functions satisfying the following conditions.

(A1) There exist constants $0 < p < 1$, $a_i > 0$ and $r_0 > 0$ such that for any $|s| \le r_0$,

$$a_1 |s|^{p+1} \le sg(s) \le a_2 |s|^{p+1}.$$

(A2) The function h satisfies the growth condition

$$\lim_{s\to 0} |h(t, s)|/|s|^{(p+1)/2} = 0,$$

and the convergence is uniform for $t \in [0, 1]$.

(A3) If $n \geq 3$, $g(s)$ also satisfies

$$\limsup_{s\to 0} \frac{sg(s)}{G(s)} < \frac{2n}{n-2}, \qquad \text{where} \quad G(s) \equiv \int_0^s g(\tau)\, d\tau.$$

The above assumption is put in the sublinear case, and the next assumption is imposed in the superlinear case.

Assumption (B) (Superlinearity of f). The nonlinear term f is assumed to be of the form $f(t, s) \equiv g(s) + h(t, s)$, and g and h are continuous functions satisfying the following conditions.

(B1) There exist constants $1 < p < n^*$, $a_i > 0$ and $R_0 > 0$ scuh that for any $|s| \geq R_0$,

$$a_1 |s|^{p+1} \leq sg(s) \leq a_2 |s|^{p+1}.$$

(B2) The function h satisfies the growth condition

$$\lim_{s\to \pm\infty} |h(t, s)|/|s|^{(p+1)/2} = 0.$$

and the convergence is uniform for $t \in [0, 1]$.

(B3) If $n \geq 3$, $g(s)$ also satisfies

$$\limsup_{s\to \pm\infty} \frac{sg(s)}{G(s)} < \frac{2n}{n-2}, \qquad \text{where} \quad G(s) \equiv \int_0^s g(\tau)\, d\tau.$$

The assumptions (A) and (B) restrict the growth order of the nonlinear term f in a neighborhood of $s = 0$ and $s = \pm\infty$, respectively. These assumptions imply that the function g is the main part of f and h is regarded as a small perturbation of g. Under the assumption (A) or (B), we wish to employ the so-called shooting method in order to prove the existence of infinitely many solutions to the boundary value problem (1.2)–(1.4). Our approach may be outlined as follows: We consider the solution $u(t, \gamma)$ of the initial value problem

$$u'' + \frac{n-1}{t} u' + f(t, u) = 0, \qquad 0 < t < 1, \tag{2.1}$$

$$u'(0) = 0, \qquad u(0) = \gamma. \tag*{(2.2)$_\gamma$}$$

We then vary the parameter $\gamma \in \boldsymbol{R}$ continuously and make an attempt to find infinitely many γ for which solutions $u(t, \gamma)$ of the boundary value problem (1.2)–(1.4) exist. To proceed this argument, we impose the next assumption.

ASSUMPTION $(C)_\gamma$. Suppose that there exists a unique and global solution $u(t)$ on $[0, 1]$ of the problem (2.1)–$(2.2)_\gamma$.

We now state our first main result which guarantees the existence of infinitely many solutions for (1.2)–(1.4) in the sublinear case.

THEOREM 1. *Suppose that assumptions* (A) *and* $(C)_\gamma$ *are valid for sufficiently small* $|\gamma|$. *Then there exists a positive integer* k_0 *such that for any* a *and* b *with* $a^2 + b^2 \neq 0$, *there exist two sequences* $\{u_k^+\}_{k \geq k_0}$ *and* $\{u_k^-\}_{k \geq k_0}$ *of solutions to* (1.2)–(1.4) *such that* u_k^+ *and* u_k^- *have exactly* k *zeros in the interval* $[0, 1]$, $u_k^+(0) > 0$, $u_k^-(0) < 0$,

$$\lim_{k \to \infty} \| u_k^\pm \|_\infty = 0 \qquad \textit{and} \quad \lim_{k \to \infty} \| u_k^\pm \|_H = 0.$$

In the next theorem we establish the existence of infinitely many solutions of (1.2)–(1.4) in the superlinear case

THEOREM 2. *Suppose that assumptions* (B) *and* $(C)_\gamma$ *are valid for sufficiently large* $|\gamma|$. *Then there exists a positive integer* k_0 *such that for any* a *and* b *with* $a^2 + b^2 \neq 0$, *there exist two sequences* $\{u_k^+\}_{k \geq k_0}$ *and* $\{u_k^-\}_{k \geq k_0}$ *of solutions to* (1.2)–(1.4) *such that* u_k^+ *and* u_k^- *have exactly* k *zeros in the interval* $[0, 1]$, $u_k^+(0) > 0$, $u_k^-(0) < 0$,

$$\lim_{k \to \infty} \| u_k^\pm \|_\infty = \infty \qquad \textit{and} \quad \lim_{k \to \infty} \| u_k^\pm \|_H = \infty.$$

The above two theorems assert the existence of infinitely many solutions of the boundary value problem (1.2)–(1.4). In the following two theorems we give the estimates with respect to the H norm of solutions of (1.2)–(1.3) in terms of the numbers of their zeros. We first deal with the sublinear case.

THEOREM 3. *Suppose that assumption* (A) *is valid. Then there exist constants* c_1, $c_2 > 0$ *and* $r > 0$ *such that*

$$c_1 k^{-(1+p)/(1-p)} \leq \| u \|_H \leq c_2 k^{-(1+p)/(1-p)} \tag{2.3}$$

for any $u \in S_k \cap B_\infty(r)$ *and any* $k \geq 1$.

In the next theorem we consider the superlinear case and give the estimates with respect to the H norm of solutions.

THEOREM 4. *Suppose that assumption* (B) *is valid. Then there exist constants* c_1, $c_2 > 0$ *and* $R > 0$ *such that*

$$c_1 k^{(p+1)/(p-1)} \leq \| u \|_H \leq c_2 k^{(p+1)/(p-1)} \tag{2.4}$$

for any $u \in S_k \cap B(R)^c$ *and any* $k \geq 1$, *where* $B(R)^c$ *means the complement in* H: $B(R)^c \equiv H \setminus B(R) = \{u \in H : \| u \|_H \geq R\}$.

Theorem 4 extends our earlier result given in [7]. In fact, if we put $h(t, s) \equiv 0$ in assumption (B), then the main result (Theorem 3) of [7] follows immediately from Theorem 4. Moreover, it should be mentioned that in [7, Theorem 3] the function $g(s)$ is assumed to satisfy $g(0) = 0$ and supposed to be differentiable at $s = 0$. In Theorem 4 these assumptions are removed.

REMARK 2.1. We here list some sufficient conditions for condition $(C)_\gamma$. In later sections, we will obtain certain a priori estimates, (3.10) or (5.9), for the solutions of (1.2)–(1.3) under the assumptions (A) or (B), respectively. These estimates imply that any solution of the problem (2.1)–$(2.2)_\gamma$ can be extended on all of the interval [0, 1]. Hence our requirements are only local existence and uniqueness of solutions. One of them is a local Lipschitz condition on $f(t, s)$ for the variable s. However this condition does not hold in the sublinear case (1.6). Actually, from (1.6) it follows that $f(t, 0) \equiv 0$, and so $f(t, s)$ is not locally Lipschitz continuous. For the sublinear case several sufficient conditions have been investigated by Coffman and Wong [5]. See Theorems A5, A6, A7 and Corollaries A4, A5 and A6 in the paper [5].

REMARK 2.2. Theorems 3 and 4 do not necessarily guarantee the existence of solutions on [0, 1] of the problem (1.2)–(1.3). These results give only the estimates with respect to the H norm of solutions u if they exist. Sufficient conditions for the existence of solutions are given in Theorems 1 and 2.

Next, we note that the restrictions in terms of $B_\infty(r)$ and $B(R)^c$ can not necessarilly be removed in Theorems 3 and 4, respectively. In fact, if we were able to remove the restriction in terms of $B_\infty(r)$ in Theorem 3, this theorem would imply that any solution with many zeros in [0, 1] must have a small H norm. But, as seen from Example 3 below, we will find a solution which has many zeros but has a large H norm. Using the same Example 3, we can also discuss the case of Theorem 4, too. Therefore we see that the restriction by means of $B_\infty(r)$ and $B(R)^c$ are essential in our discussions.

Finally, we illustrate the significance of our results by applying them to a few typical Emden-Fowler equations. It turns out that the existence and the asymptotic distribution of the solutions are discussed in the space H.

EXAMPLE 1 (Emden–Fowler equation). Consider the boundary value problem

$$u'' + \frac{n-1}{t}u' + |u|^p \operatorname{sgn} u = 0, \qquad 0 < t < 1, \tag{2.5}$$

$$u'(0) = 0, \qquad u(1) = 0, \tag{2.6}$$

where (2.5) is sublinear for $0 < p < 1$ and is superlinear for $1 < p < n^*$. In each case we can show that for each $k \geq 1$ the problem (2.5)–(2.6) possesses a unique solution which has exactly k zeros in $[0, 1]$ and satisfies $u(0) > 0$. If we denote the solution by $u_k(t)$, then

$$S_k = \{u_k, -u_k\} \quad \text{and} \quad S = \{\pm u_k : k \in N\}.$$

In the superlinear case, this assertion has already been established in [7]. The sublinear case can also be proved in the same way as in [7]. To show this, we consider (2.5) on $[0, \infty)$ together with the initial condition,

$$u'(0) = 0 \quad \text{and} \quad u(0) = 1. \tag{2.7}$$

It is easy to check that equation (2.5) subject to (2.7) possesses a unique global solution $w(t)$ on $[0, \infty)$. Furthermore the solution $w(t)$ is oscillatory, that is, it has an unbounded sequence of zeros in the interval $[0, \infty)$. To prove this assertion, we use the so-called Liouville transformations: For the case of $n = 2$ we employ

$$s = \log t, \qquad y(s) = u(t), \qquad t > 1,$$

and in the case of $n \geq 3$ we take

$$s = (n-2)t^{n-2}, \qquad y(s) = su(t), \qquad t > 0.$$

Then equation (2.5) is reduced to the equation of the form

$$y'' + e^{2s}|y|^p \operatorname{sgn} y = 0, \qquad s > 0, \qquad \text{if } n = 2$$

and

$$y'' + cs^\sigma |y|^p \operatorname{sgn} y = 0, \qquad s > 0, \qquad \text{if } n \geq 3,$$

where c is a positive constant and $\sigma = -p - 1 + 2/(n-2)$. Applying [12, p 345, Theorem 4.7] in the case of $n = 2$ and [12, p 343] in the case where $n \geq 3$, we see that all solutions of the above equations are oscillatory; hence so is $w(t)$. By the same discussion as in [7] we obtain the uniqueness and existence of $u_k(t)$. We now apply our theorems to find the asymptotic distribution in H of the solutions of (2.5)–(2.6). If $0 < p < 1$, we have

$$\lim_{k\to\infty} \|u_k\|_\infty = \lim_{k\to\infty} \|u_k\|_H = 0 \tag{2.8}$$

and

$$C_1 k^{-(1+p)/(1-p)} \leq \|u_k\|_H \leq C_2 k^{-(1+p)/(1-p)} \qquad \text{for } k \in N. \tag{2.9}$$

If $1 < p < n^* = (n+2)/(n-2)$, we have

(2.10) $$\lim_{k\to\infty} \|u_k\|_\infty = \lim_{k\to\infty} \|u_k\|_H = \infty$$

and

(2.11) $$C_3 k^{(p+1)/(p-1)} \le \|u_k\|_H \le C_4 k^{(p+1)/(p-1)} \qquad \text{for } k \in \boldsymbol{N}.$$

EXAMPLE 2 (Perturbed Emden-Fowler equations). We consider the Emden-Fowler equation with a perturbing term,

(2.12) $$u'' + \frac{n-1}{t}u' + |u|^p \operatorname{sgn} u + h(t) = 0, \qquad 0 < t < 1,$$

(2.13) $$u'(0) = 0, \qquad au(1) + bu'(1) = 0,$$

where $1 < p < n^*$ and h is continuous on $[0, 1]$. Using our theorems, we find an integer k_0 and two sequences $\{u_k^+\}_{k\ge k_0}$ and $\{u_k^-\}_{k\ge k_0}$ of solutions to (2.12)–(2.13) such that u_k^+ and u_k^- have exactly k zeros in $[0, 1]$, $u_k^+(0) > 0$, $u_k^-(0) < 0$ and they satisfy (2.10) and (2.11).

EXAMPLE 3 (Emden-Fowler equations involving both superlinearity and sublinearity). Finally, we treat the boundary value problem which involves both of the superlinearity and sublinearity:

(2.14) $$u'' + \frac{n-1}{t}u' + |u|^p \operatorname{sgn} u + |u|^q \operatorname{sgn} u = 0, \qquad 0 < t < 1,$$

(2.15) $$u'(0) = 0, \qquad au(1) + bu'(1) = 0,$$

where $0 < q < 1 < p < n^*$. Using the Liouville transformations introduced in Example 1 and applying Theorems A5, A6 and A7 in [5], one can easily check that the assumption $(\mathrm{C})_\gamma$ is valid for every $\gamma \in \boldsymbol{R}$. We now apply our theorems to this equation. Notice that the nonlinear terms in equation (2.14) are odd functions. Therefore there exist an integer k_0 and four sequences $\{u_k\}_{k\ge k_0}$, $\{-u_k\}_{k\ge k_0}$ $\{v_k\}_{k\ge k_0}$ and $\{-v_k\}_{k\ge k_0}$ of solutions to (2.14)–(2.15) such that u_k and v_k have exactly k zeros in $[0, 1]$, $u_k(0)$, $v_k(0) > 0$ and

$$\lim_{k\to\infty} \|u_k\|_\infty = \lim_{k\to\infty} \|u_k\|_H = 0,$$

$$\lim_{k\to\infty} \|v_k\|_\infty = \lim_{k\to\infty} \|v_k\|_H = \infty,$$

$$C_1 k^{-(1+q)/(1-q)} \le \|u_k\|_H \le C_2 k^{-(1+q)/(1-q)}$$

and

$$C_3 k^{(p+1)/(p-1)} \le \|v_k\|_H \le C_4 k^{(p+1)/(p-1)} \qquad \text{for } k \ge k_0.$$

3. A priori estimates in the sublinear case

In this section we are concerned with the sublinear case and give some basic a priori estimates of the solutions with respect to the H norm. These results are basic to the proofs of our results. The most important one is the following proposition.

PROPOSITION 3.1. *Suppose* (A1), (A2) *and* (A3). *Then for any* $m > n - 1$ *there exist constants* $C_i > 0$, $1 \le i \le 5$, *and* $r > 0$ *such that*

$$\begin{aligned}\max\nolimits_{0\le t\le 1} |u(t)|^{p+1} t^n &\le C_1 \max\nolimits_{0\le t\le 1} G(u(t)) t^n \\ &\le C_2 \max\nolimits_{0\le t\le 1} u'(t)^2 t^n \le C_3 \int_0^1 |u'|^2 t^m\, dt \\ &\le C_3 \int_0^1 |u'|^2 t^{n-1}\, dt \le C_4 \int_0^1 G(u) t^m\, dt \\ &\le C_4 \int_0^1 G(u) t^{n-1}\, dt \le C_5 \max\nolimits_{0\le t\le 1} |u(t)|^{p+1} t^n\end{aligned}$$

for any $u \in S \cap B_\infty(r)$.

In what follows, we always make assumptions (A1), (A2) and (A3) without further mention. By (A1) there exist constants $b_i > 0$, $1 \le i \le 4$, such that

$$|s|^{p+1} \le b_1 G(s) \le b_2 s g(s) \le b_3 G(s) \le b_4 |s|^{p+1} \tag{3.1}$$

for any $|s| \le r_0$. From (A2) it follows that given $\varepsilon > 0$ there exists a constant $r(\varepsilon) \in (0, 1)$ such that for $|s| \le r(\varepsilon)$ and $t \in [0, 1]$,

$$|h(t, s)| \le \varepsilon |s|^{(p+1)/2}. \tag{3.2}$$

Since $(p + 3)/2 > p + 1$, we infer from (3.2) that

$$|sh(t, s)| \le \varepsilon |s|^{p+1} \qquad \text{for} \quad |s| \le r(\varepsilon). \tag{3.3}$$

Further, (A3) implies that if $n \ge 3$ then there exist positive constants θ and r_1 such that

$$(2n - \theta) G(s) \ge (n - 2) s g(s) \qquad \text{for} \quad |s| \le r_1. \tag{3.4}$$

We may suppose without loss of generality that $r(\varepsilon) \le r_1 \le r_0 \le 1$ for all $\varepsilon > 0$.

In what follows, we denote various constants by $C_1, C_2, \cdots, C, C'$ and C'', which depend on neither a solution u nor the total number of zeros of u. To prove Proposition 3.1, we prepare several lemmas.

LEMMA 3.1. (i) *For any* $u \in S$, *we have*

$$\int_0^1 u'(t)^2 t^{n-1}\,dt = \int_0^1 uf(t, u)t^{n-1}\,dt + u'(1)u(1). \tag{3.5}$$

(ii) *There exist constants* C_1, $C_2 > 0$ *and* $r_2 \in (0, r_1)$ *such that*

$$\int_0^1 u'(t)^2 t^{n-1}\,dt \le C_1 \int_0^1 |u|^{p+1} t^{n-1}\,dt + u'(1)u(1) \tag{3.6}$$

and

$$\int_0^1 |u|^{p+1} t^{n-1}\,dt \le C_2 \int_0^1 u'(t)^2 t^{n-1}\,dt + C_2|u'(1)u(1)| \tag{3.7}$$

for any $u \in S \cap B_\infty(r_2)$.

PROOF. Multiplying (1.2) by $u(t)t^{n-1}$ and applying integration by parts, we obtain the desired relation (3.5). Next, by (3.1) and (3.3) there exists a constant $C_1 > 0$ such that

$$|sf(t, s)| \le C_1|s|^{p+1} \quad \text{for} \quad |s| \le r(1). \tag{3.8}$$

This inequality and (3.5) together imply the inequality (3.6). Lastly, substituting $\varepsilon = a_1/2$ into (3.3) and applying (A1) and (3.3) we obtain

$$sf(t, s) \ge sg(s) - |sh(t, s)| \ge \frac{a_1}{2}|s|^{p+1} \tag{3.9}$$

for $|s| \le r(a_1/2)$. The inequality (3.7) follows from (3.5) and (3.9). The proof is now complete.

LEMMA 3.2. *There exist constants* C_1, $C_2 > 0$ *and* $r_3 \in (0, r_2)$ *such that*

$$\tfrac{1}{2}u'(t)^2 + G(u(t)) \le C_1 G(u(0)) \tag{3.10}$$

and

$$|u(0)| \le \|u\|_\infty \le C_2|u(0)| \tag{3.11}$$

for $t \in [0, 1]$ *and* $u \in S \cap B_\infty(r_3)$.

PROOF. Multiplying (1.2) by $u'(t)$ and integrating the resultant identity over $[0, t]$, we have

$$\frac{1}{2}u'(t)^2 + G(u(t)) + (n-1)\int_0^t \frac{1}{s}\,u'(s)^2\,ds = G(u(0)) - \int_0^t u'h(s, u)\,ds. \tag{3.12}$$

We now set

$$E = \max_{0 \le t \le 1} \{\tfrac{1}{2} u'(t)^2 + G(u(t))\}.$$

Then it follows from (3.1) that

$$|u(t)| \le (b_1 E)^{1/(p+1)} \quad \text{and} \quad |u'(t)| \le (2E)^{1/2}$$

for $t \in [0, 1]$. Using the above inequalities and (3.2), we have

$$\int_0^t |u'(s)h(s, u)|\, ds \le \varepsilon \int_0^t |u'||u|^{(p+1)/2}\, ds \le \varepsilon CE \tag{3.13}$$

for any $u \in S \cap B_\infty(r(\varepsilon))$, where C is a positive constant independent of ε. It follows from (3.12) and (3.13) that

$$\tfrac{1}{2} u'(t)^2 + G(u(t)) \le \varepsilon CE + G(u(0))$$

for $t \in [0, 1]$. This implies

$$E \le \varepsilon CE + G(u(0)) \quad \text{for} \quad u \in S \cap B_\infty(r(\varepsilon)).$$

We then choose $\varepsilon = 1/2C$ to obtain $E \le C' G(u(0))$, which implies the inequality (3.10). Next, by (3.10), we have

$$G(u(t)) \le C_1 G(u(0)) \quad \text{for} \quad t \in [0, 1].$$

The inequality (3.11) follows from the above inequality and (3.1), and the proof is complete.

LEMMA 3.3. *There is a constant $r_4 \in (0, r_3)$ such that to each m with $m > n - 1$ there correspond constants C_1, $C_2 > 0$ for which the inequalities*

$$\int_0^1 G(u(t))t^m dt \le C_1 \int_0^1 u'(t)^2 t^m dt + C_1 |u'(1)u(1)| + C_1 u(1)^2 \tag{3.14}$$

and

$$\int_0^1 u'(t)^2 t^m dt \le C_2 \int_0^1 G(u(t))t^{m-2} dt + u'(1)u(1) \tag{3.15}$$

hold for any $u \in S \cap B_\infty(r_4)$.

PROOF. Let $m > 1$. Multiplying (1.2) by $u(t)t^m$ and integrating the resultant identity over $[0, 1]$, we have

$$\int_0^1 |u'|^2 t^m dt = \int_0^1 uf(t, u)t^m dt + \frac{(m - n + 1)(m - 1)}{2} \int_0^1 u^2 t^{m-2} dt \tag{3.16}$$

$$+ u'(1)u(1) - \frac{m-n+1}{2} u(1)^2.$$

On the other hand, by (3.1) and (3.9), we obtain

$$\int_0^1 uf(t, u)t^m\,dt \geq \frac{a_1}{2} \int_0^1 |u|^{p+1} t^m\,dt \geq C_1 \int_0^1 G(u)t^m\,dt \tag{3.17}$$

provided that $\|u\|_\infty \leq r(a_1/2)$. Assume now that $m > n-1$. Then the second term on the right-hand side of (3.16) is positive, and so the inequality (3.14) follows from (3.16) and (3.17). Next, by (3.8) and (3.1), there exist constants C, C' and C'' such that

$$\begin{aligned}\int_0^1 uf(t, u)t^m\,dt &+ \frac{(m-n+1)(m-1)}{2} \int_0^1 u^2 t^{m-2}\,dt \\ &\leq C \int_0^1 |u|^{p+1} t^m\,dt + C \int_0^1 u^2 t^{m-2}\,dt \\ &\leq C' \int_0^1 |u|^{p+1} t^{m-2}\,dt \leq C'' \int_0^1 G(u)t^{m-2}\,dt,\end{aligned}$$

provided that $\|u\|_\infty \leq r(1)$. This estimate, together with (3.16) implies (3.15). This completes the proof.

Using Lemmas 3.1 and 3.3, we obtain the next lemma which is useful for proving Proposition 3.1 below.

LEMMA 3.4. *For any m with $m > n-1$, there exist constants $C_i > 0$ and $r > 0$ such that*

$$\begin{aligned}\max_{0\leq t\leq 1} \left\{\frac{1}{2}u'(t)^2 t^n + G(u(t))t^n\right\} &\leq C_1 \int_0^1 G(u)t^{n-1}\,dt \\ &\leq C_2 \int_0^1 |u'|^2 t^m\,dt \leq C_3 \max_{0\leq t\leq 1} u'(t)^2 t^n\end{aligned}$$

for any $u \in S \cap B_\infty(r)$.

PROOF. Let $m > 0$. Multiplying equation (1.2) by $u'(t)t^m$ and integrating the resultant identity over $[0, T]$, we have

$$\begin{aligned}\frac{1}{2}u'(T)^2 T^m + G(u(T))T^m &= m \int_0^T G(u(t))t^{m-1}\,dt \\ &+ \left(\frac{m}{2} - n + 1\right) \int_0^T |u'|^2 t^{m-1}\,dt - \int_0^T u'h(t, u)t^m\,dt.\end{aligned} \tag{3.18}$$

First we substitute $m = n$ into (3.18) to obtain

$$\frac{1}{2}u'(T)^2 T^n + G(u(T))\,T^n \leq n\int_0^1 G(u)t^{n-1}\,dt + \int_0^1 |u'h(t, u)|\,t^n\,dt. \tag{3.19}$$

We here set

$$K(t) = \tfrac{1}{2}u'(t)^2 t^n + G(u(t))t^n$$

and

$$K = \max_{0\leq t\leq 1} K(t).$$

Then it follows from (3.1) that

$$|u(t)|^{(p+1)/2} t^{n/2} \leq (b_1 K)^{1/2} \tag{3.20}$$

and

$$|u'(t)|\,t^{n/2} \leq (2K)^{1/2} \tag{3.21}$$

for all $t \in [0, 1]$. Using (3.20), (3.21) and (3.2), we have

$$\int_0^1 |u'h(t, u)|\,t^n\,dt \leq \varepsilon\int_0^1 |u'|\,t^{n/2}|u|^{(p+1)/2} t^{n/2}\,dt \leq \varepsilon CK \tag{3.22}$$

for all $u \in S \cap B_\infty(r(\varepsilon))$, where C does not depend on u, K, and ε. It follows from (3.19) and (3.22) that

$$K(T) \leq n\int_0^1 G(u)t^{n-1}\,dt + \varepsilon CK$$

for any $T \in [0, 1]$, so that

$$K \leq n\int_0^1 G(u)t^{n-1}\,dt + \varepsilon CK.$$

Put $\varepsilon = 1/2C$, then we have

$$K \leq 2n\int_0^1 G(u)t^{n-1}\,dt. \tag{3.23}$$

This is the first inequality displayed in the statement of Lemma 3.4. We now prove the second inequality. Substituting $m = n$ and $T = 1$ into (3.18), we obtain

$$\frac{1}{2}u'(1)^2 + G(u(1)) = n\int_0^1 G(u)t^{n-1}\,dt \tag{3.24}$$

$$-\frac{n-2}{2}\int_0^1 |u'|^2 t^{n-1}\,dt - \int_0^1 u'h(t,u)t^n\,dt.$$

First we consider the case in which $n \geq 3$. From (3.4) and (3.5) it follows that

$$\begin{aligned}(3.25)\qquad n\int_0^1 G(u)t^{n-1}\,dt &- \frac{n-2}{2}\int_0^1 |u'|^2 t^{n-1}\,dt\\ &= \frac{1}{2}\int_0^1 \{2nG(u)-(n-2)ug(u)\}t^{n-1}\,dt\\ &\quad -\frac{n-2}{2}\int_0^1 uh(t,u)t^{n-1}\,dt - \frac{n-2}{2}u'(1)u(1)\\ &\geq \frac{\theta}{2}\int_0^1 G(u)t^{n-1}\,dt - \frac{n-2}{2}\int_0^1 uh(t,u)t^{n-1}\,dt - \frac{n-2}{2}u'(1)u(1)\end{aligned}$$

for $u \in S \cap B_\infty(r_1)$. Therefore by (3.24) and (3.25) we obtain

$$\begin{aligned}(3.26)\qquad \frac{1}{2}u'(1)^2 + G(u(1)) &\geq \frac{\theta}{2}\int_0^1 G(u)t^{n-1}\,dt\\ &-\frac{n-2}{2}\int_0^1 uh(t,u)t^{n-1}\,dt - \frac{n-2}{2}u'(1)u(1) - \int_0^1 u'h(t,u)t^n\,dt\end{aligned}$$

for any $u \in S \cap B_\infty(r_1)$. Next, let $n = 2$. In this case we directly obtain from (3.24) the inequality (3.26) with $\theta/2$ replaced by 2. On the other hand, the substitution of $T = 1$ into (3.18) yields

$$\begin{aligned}\frac{1}{2}u'(1)^2 + G(u(1)) &= m\int_0^1 G(u)t^{m-1}\,dt\\ &+\left(\frac{m}{2}-n+1\right)\int_0^1 |u'|^2 t^{m-1}\,dt - \int_0^1 u'h(t,u)t^m\,dt.\end{aligned}$$

From this and (3.14) we see that for any $m > n$ there is a constant $C > 0$ such that

$$\begin{aligned}(3.27)\qquad \frac{1}{2}u'(1)^2 + G(u(1)) &\leq C\int_0^1 |u'|^2 t^{m-1}\,dt - \int_0^1 u'h(t,u)t^m\,dt\\ &\quad + C|u'(1)u(1)| + Cu(1)^2\end{aligned}$$

for $u \in S \cap B_\infty(r_4)$. The estimates (3.26) and (3.27) together imply that for any $m > n$ one finds a constant $C > 0$ and

$$(3.28)\qquad \frac{\theta}{2}\int_0^1 G(u)t^{n-1}\,dt \le C\int_0^1 |u'|^2 t^{m-1}\,dt + \frac{n-2}{2}\int_0^1 |uh(t,u)|t^{n-1}\,dt + 2\int_0^1 |u'h(t,u)|t^n\,dt + C|u'(1)u(1)| + Cu(1)^2$$

for $u\in S\cap B_\infty(r_4)$. We then estimate each term on the right-hand side. Using the inequalities (3.1), (3.3), (3.22) and (3.23), we have

$$(3.29)\qquad \int_0^1 |uh(t,u)|t^{n-1}\,dt \le \varepsilon\int_0^1 |u|^{p+1}t^{n-1}\,dt \le \varepsilon b_1\int_0^1 G(u)t^{n-1}\,dt$$

and

$$(3.30)\qquad \int_0^1 |u'h(t,u)|t^n\,dt \le \varepsilon C\int_0^1 G(u)t^{n-1}\,dt$$

for $u\in S\cap B_\infty(r(\varepsilon))$. Combining (3.28), (3.29) and (3.30), we see that for any $m>n$ there is a constant $C>0$ such that

$$\int_0^1 G(u)t^{n-1}\,dt \le C\int_0^1 |u'|^2 t^{m-1}\,dt + \varepsilon C\int_0^1 G(u)t^{n-1}\,dt + C|u'(1)u(1)| + Cu(1)^2$$

for $u\in S\cap B_\infty(r(\varepsilon))$, where the constant C may depend on m but does not depend upon u, K and ε. Set $\varepsilon=1/2C$ and $r_5=r(1/2C)$, then we have

$$(3.31)\qquad \int_0^1 G(u)t^{n-1}\,dt \le C\int_0^1 |u'|^2 t^{m-1}\,dt + C|u'(1)u(1)| + Cu(1)^2$$

for $u\in S\cap B_\infty(r_5)$, where C and r_5 may depend only on m. To estimate the second and third terms on the right-hand side, we substitute $t=1$ into (3.20) and (3.21). Then it follows from (3.20), (3.21), (3.23) and (3.1) that

$$\begin{aligned}(3.32)\qquad |u'(1)u(1)| + u(1)^2 &\le CK^{(p+3)/2(p+1)} + CK^{2/(p+1)}\\ &\le C\left(\int_0^1 G(u)t^{n-1}\,dt\right)^{(p+3)/2(p+1)} + C\left(\int_0^1 G(u)t^{n-1}\,dt\right)^{2/(p+1)}\\ &\le C\|u\|_\infty^{(1-p)/2}\int_0^1 G(u)t^{n-1}\,dt + C\|u\|_\infty^{1-p}\int_0^1 G(u)t^{n-1}\,dt.\end{aligned}$$

Consequently, it follows from (3.31) and (3.32) that

$$\int_0^1 G(u)t^{n-1}\,dt \le C\int_0^1 |u'|^2 t^{m-1}\,dt + C(\|u\|_\infty^{(1-p)/2} + \|u\|_\infty^{1-p})\int_0^1 G(u)t^{n-1}\,dt$$

for $u \in S \cap B_\infty(r_5)$. The second inequality in Lemma 3.4 follows from the above inequality provided that $\|u\|_\infty$ is sufficiently small. Lastly, for $m > n-1$ we have

$$\int_0^1 |u'|^2 t^m\,dt \le (\max_{0\le t\le 1} u'(t)^2 t^n)\int_0^1 t^{m-n}\,dt,$$

which implies the last inequality in Lemma 3.4. Thus we obtain the desired estimates by choosing an appropriate radius $r > 0$, and the proof is now complete.

We are now in a position to give the proof of Proposition 3.1.

Proof of Proposition 3.1. It is easy to check the first, the fourth and the sixth inequalities stated in Proposition 3.1. The second and the third inequalities have already been proved in Lemma 3.4. Therefore it is sufficient to verify the fifth and the last inequalities. By Lemma 3.1 (ii) and (3.1) there is a constant $C_1 > 0$ such that

$$\int_0^1 |u'|^2 t^{n-1}\,dt \le C_1\int_0^1 G(u)t^{n-1}\,dt + u'(1)u(1) \tag{3.33}$$

for any $u \in S \cap B_\infty(r_2)$, where r_2 is a radius as mentioned in Lemma 3.1. To estimate the right-hand side of (3.33), we use Lemma 3.4 and (3.15), and so for any $m > n-1$ there exist positive constants C_2 and r such that

$$\int_0^1 G(u)t^{n-1}\,dt \le C_2\int_0^1 G(u)t^{m-2}\,dt + C_2|u'(1)u(1)| \tag{3.34}$$

for $u \in S \cap B_\infty(r)$. From (3.33) and (3.34) we see that for any $m > n-3$ there is a constant $C > 0$ such that

$$\int_0^1 |u'|^2 t^{n-1}\,dt \le C\int_0^1 G(u)t^m\,dt + C|u'(1)u(1)|. \tag{3.35}$$

On the other hand, letting $m = n$ in Lemma 3.4, we have

$$K \equiv \max_{0\le t\le 1}\left(\frac{1}{2}u'(t)^2 t^n + G(u(t))t^n\right) \le C'\int_0^1 |u'|^2 t^n\,dt \le C'\int_0^1 |u'|^2 t^{n-1}\,dt.$$

Using this inequality, we obtain

$$(3.36)\qquad \begin{aligned}|u'(1)u(1)| &\le |u'(1)|\,|u(1)|^{(1+p)/2}\,\|u\|_\infty^{(1-p)/2}\\ &\le \tfrac{1}{2}(u'(1)^2 + |u(1)|^{1+p})\,\|u\|_\infty^{(1-p)/2}\\ &\le C_1 K\|u\|_\infty^{(1-p)/2} \le C_2\|u\|_\infty^{(1-p)/2}\int_0^1 |u'|^2 t^{n-1}\,dt,\end{aligned}$$

where C_1 and C_2 are positive constants independent of u and K. Consequently, it follows from (3.35) and (3.36) that

$$\int_0^1 |u'|^2 t^{n-1}\,dt \le C_3\int_0^1 G(u)t^m\,dt + C_3\|u\|_\infty^{(1-p)/2}\int_0^1 |u'|^2 t^{n-1}\,dt$$

for any $u \in S \cap B_\infty(r)$, where C_3 and r depend only upon m. If $\|u\|_\infty$ is sufficiently small, the above inequality implies the fifth inequality stated in Proposition 3.1.

We next prove the last inequality. Using (3.32) and (3.34), for any $m > n - 1$ we have

$$\int_0^1 G(u)t^{n-1}\,dt \le C_1\int_0^1 G(u)t^{m-2}\,dt + C_2\|u\|_\infty^{(1-p)/2}\int_0^1 G(u)t^{n-1}\,dt,$$

and so

$$\int_0^1 G(u)t^{n-1}\,dt \le C\int_0^1 G(u)t^{m-2}\,dt$$

provided that $\|u\|_\infty$ is sufficiently small. In this inequality we put $m = n + 2$ to obtain

$$\int_0^1 G(u)t^{n-1}\,dt \le C\int_0^1 G(u)t^n\,dt \le C' \max_{0\le t\le 1} |u(t)|^{p+1}t^n.$$

This is nothing but the last inequality of Proposition 3.1. The proof is complete.

We next prepare two technical lemmas which will play an important role in the proof of Theorems 1 and 3.

LEMMA 3.5. *For any $\omega > 0$ there are constants C and $r > 0$ such that*

$$(3.37)\qquad \int_0^1 G(u)t^{n-1}\,dt \le C\left\{\frac{1}{2}u'(t)^2 + G(u(t))\right\}t^{-\omega}$$

for $t \in (0, 1]$ and $u \in S \cap B_\infty(r)$.

PROOF. Let $\omega > 0$ and define

$$E(t) \equiv \tfrac{1}{2}u'(t)^2 + G(u(t)).$$

Let $T \in (0, 1)$. Multiplying equation (1.2) by $u'(t)t^{-\omega}$ and integrating over $[T, 1]$, we have

$$E(T)T^{-\omega} = \omega \int_T^1 E(t)t^{-\omega-1}\,dt + (n-1)\int_T^1 |u'|^2 t^{-\omega-1}\,dt + \int_T^1 u'h(t, u)t^{-\omega}\,dt + E(1). \tag{3.38}$$

We then show the following two inequalities:

$$\int_T^1 |u'h(t, u)|t^{-\omega}\,dt \le \varepsilon C_1 \int_T^1 E(t)t^{-\omega-1}\,dt \tag{3.39}$$

and

$$E(1) \ge \left(\frac{\theta}{2} - \varepsilon C_2 - C_3\|u\|_\infty^{(1-p)/2}\right)\int_0^1 G(u)t^{n-1}\,dt \tag{3.40}$$

for any $\varepsilon > 0$ and $u \in S \cap B_\infty(r(\varepsilon))$. In these inequalities, C_i, $1 \le i \le 3$, are positive constants independent of ε, u and T, θ is a positive constant as defined in (3.4) in the case of $n \ge 3$ and $\theta/2$ should be replaced by 2 in the case of $n = 2$. First it follows from (3.2) that

$$\begin{aligned}\int_T^1 |u'h(t, u)|t^{-\omega}\,dt &\le \varepsilon \int_T^1 |u'|\,|u|^{(p+1)/2}t^{-\omega-1}\,dt \\ &\le \varepsilon \int_T^1 (|u'|^2 + |u|^{p+1})t^{-\omega-1}\,dt \le \varepsilon C \int_T^1 E(t)t^{-\omega-1}\,dt\end{aligned}$$

if u satisfies $\|u\|_\infty \le r(\varepsilon)$. This is the desired inequality (3.39). We next obtain (3.40) by using the inequalities (3.26), (3.29), (3.30) and (3.32) together. By (3.38), (3.39) and (3.40) we have

$$\begin{aligned}E(T)T^{-\omega} \ge &\left\{\frac{\theta}{2} - \varepsilon C_2 - C_3\|u\|_\infty^{(1-p)/2}\right\}\int_0^1 G(u)t^{n-1}\,dt \\ &+ (\omega - \varepsilon C_1)\int_T^1 E(t)t^{-\omega-1}\,dt + (n-1)\int_T^1 |u'|^2 t^{-\omega-1}\,dt\end{aligned}$$

for $u \in S \cap B_\infty(r(\varepsilon))$. We put

$$\varepsilon = \varepsilon_0 \equiv \min(\omega/2C_1,\ \theta/4C_2).$$

Then the second and third terms on the right-hand side are positive, and so we obtain

$$E(T)T^{-\omega} \geq \left(\frac{\theta}{4} - C_3 \|u\|_\infty^{(1-p)/2}\right) \int_0^1 G(u)t^{n-1}\, dt$$

for $u \in S \cap B_\infty(r(\varepsilon_0))$. Consequently, we obtain (3.37) from the above inequality as far as $\|u\|_\infty$ is sufficiently small. This completes the proof.

We say that a zero τ of u is a *simple zero* if $u(\tau) = 0$ and $u'(\tau) \neq 0$. Lemma 3.5, together with (3.11), implies that any nontrivial solution with small L^∞ norm has only simple zeros. This fact is important for defining our Prüfer transformations. In the following lemma, we give some technical estimates near $t = 0$ for the solutions belonging to S.

LEMMA 3.6. *There are constants* $\delta > 0$ *and* C_1, $C_2 > 0$ *such that*

$$\|u\|_\infty \leq C_1 |u(t)| \tag{3.41}$$

and

$$|u'(t)| \leq C_2 t |u(t)|^p \tag{3.42}$$

for all t *and* u *satisfying* $t \in [0, \delta \|u\|_{p+1}^{(1-p)/2}]$ *and* $u \in S \cap B_\infty(r_0)$, *where* r_0 *is the number appearing in condition* (A1).

PROOF. Let $u \in S \cap B_\infty(r_0)$. Multiplying equation (1.2) by t^{n-1} and integrating the resultant identity over $[0, t]$, we have

$$u'(t) = -t^{-(n-1)} \int_0^t f(s, u)s^{n-1}\, ds \tag{3.43}$$

and

$$u(t) - u(0) = -\int_0^t \left(s^{-(n-1)} \int_0^s f(\tau, u(\tau))\tau^{n-1}\, d\tau\right) ds. \tag{3.44}$$

We then estimate the right-hand side. For $t \in [0, \delta \|u\|_{p+1}^{(1-p)/2}]$, it holds that

$$\begin{aligned}\int_0^t \left(s^{-(n-1)} \int_0^s |f(\tau, u(\tau))|\tau^{n-1}\, d\tau\right) ds &\leq Ct^2 \|u\|_\infty^p \\ &\leq C\delta^2 \|u\|_{p+1}^{1-p} \|u\|_\infty^p \leq C\delta^2 \|u\|_\infty,\end{aligned}$$

where C is independent of δ and u. From this and (3.44) we have

$$|u(0)| - C\delta^2 \|u\|_\infty \leq |u(t)|,$$

and so

$$\left(\frac{1}{C_2} - C\delta^2\right) \|u\|_\infty \leq |u(t)|,$$

by (3.11). Therefore, choosing δ so small, we obtain (3.41) from the above inequality.

Next, (3.43) implies that

$$|u'(t)| \leq Ct \|u\|_\infty^p.$$

This, together with (3.41), implies (3.42). This completes the proof.

As seen from the proof of Lemma 3.6, we may assume that $\delta < 1$. In view of this, we may assume without loss of generality that

$$\delta \|u\|_{p+1}^{(1-p)/2} \leq \delta \|u\|_\infty^{(1-p)/2} < 1$$

holds for $u \in S \cap B_\infty(1)$.

Proposition 3.1 and Lemma 3.6 lead us to the following definition.

Definition 3.1. For $u \in S \cap B_\infty(1)$ we define the numbers

$$M(u) \equiv \max_{0 \leq t \leq 1} |u(t)|^{p+1} t^n$$

and

$$T(u) \equiv \delta \|u\|_{p+1}^{(1-p)/2}.$$

We conclude this section by proving the next lemma, which means that it is sufficient to compute the value of $M(u)$ instead of the H norm of $u \in S$ in our subsequent discussions.

Lemma 3.7. *There exist constants C_1, $C_2 > 0$ and $r > 0$ such that for any $u \in S \cap B_\infty(r)$ we have*

$$C_1 M(u) \leq \|u\|_H^2 \leq C_2 M(u).$$

Proof. It follows from Proposition 3.1 that

$$\|u\|_H^2 \geq \int_0^1 |u'|^2 t^{n-1}\, dt \geq CM(u).$$

This is nothing but the first inequality. Again, by Proposition 3.1, we have

$$\int_0^1 u^2 t^{n-1}\, dt \leq \|u\|_\infty^{1-p} \int_0^1 |u|^{p+1} t^{n-1}\, dt \leq C \int_0^1 G(u) t^{n-1}\, dt \leq C' M(u)$$

and

$$\int_0^1 |u'|^2 t^{n-1}\, dt \le C'' M(u) \qquad \text{for any} \quad u \in S \cap B_\infty(r).$$

Therefore these two inequalities together imply

$$\|u\|_H^2 = \int_0^1 (u^2 + |u'|^2) t^{n-1}\, dt \le (C' + C'') M(u).$$

This completes the proof of Lemma 3.7.

4. Proof of Theorems 1 and 3

In this section we prove Theorems 1 and 3. We suppose assumptions (A1), (A2) and (A3) throughout this section. We begin by introducing notations which will be used in the subsequent discussions.

DEFINITION 4.1. For a continuous function $u(t)$ on $[0, 1]$, $N[u]$ denotes the number of zeros of $u(t)$, that is,

$$N[u] = \#\{t \in [0, 1] : u(t) = 0\} \quad (\le +\infty).$$

Moreover, under assumption $(C)_\gamma$, we denote by $u(t, \gamma)$ the solution of the initial value problem (2.1)–$(2.2)_\gamma$.

The next proposition is crucial for proving our theorems.

PROPOSITION 4.1. *Under the assumptions of Theorem* 1, *it follows that* $N[u(\cdot, \gamma)] < +\infty$ *for sufficiently small* $|\gamma| > 0$, *and that*

$$\lim_{\gamma \to 0} N[u(\cdot, \gamma)] = +\infty.$$

In what follows, we choose a constant $r > 0$ so small that Lemmas 3.1, 3.2, 3.3, 3.6 and 3.7 are all valid for any $u \in S \cap B_\infty(r)$. We here observe that, in Lemmas 3.4, 3.5 and Proposition 3.1, the number r depends on an exponent m. When we use these lemmas and Proposition 3.1 in our discussion, we exchange a constant r for a smaller one in accordance with the choice of m if necessary. Therefore, in the following lemmas and definitions, the number r is supposed to be chosen in an appropriate way.

To prove Proposition 4.1, we introduce a new type of Prüfer transformation.

DEFINITION 4.2. For $u \in S \cap B_\infty(r)$ we define the functions $\rho(t)$ and $\varphi(t)$ by

$$|u'|^{\mu-1} u' = \rho \cos \varphi, \tag{4.1}$$

$$u = \rho \sin \varphi, \tag{4.2}$$

$$\varphi(0) = \frac{\pi}{2}, \tag{4.3}$$

where $\mu = 2/(1 + p)$. Notice that $\mu > 1$.

The relations (4.1), (4.2) and (4.3) determine the functions $\rho(t)$ and $\varphi(t)$ of class $C^1[0, 1]$ uniquely. In fact, by (3.37) and (3.11), any nontrivial solution u has only simple zeros, and so

$$\rho(t) = (|u'|^{2\mu} + |u|^2)^{1/2} > 0 \qquad \text{for} \quad t \in [0, 1].$$

Therefore $\rho(t)$ and $\varphi(t)$ are well-defined. Since the left-hand sides of (4.1) and (4.2) are continuously differentiable, $\rho(t)$ and $\varphi(t)$ are of class $C^1[0, 1]$ as well.

LEMMA 4.1. *For $u \in S \cap B_\infty(r)$ we have*

$$\begin{aligned}\varphi'(t) = \rho^{-2}\{&|u'|^{\mu+1} + \mu|u'|^{\mu-1}ug(u)\\ &+ \mu(n-1)\frac{1}{t}|u'|^{\mu-1}u'u + \mu|u'|^{\mu-1}uh(t, u)\}.\end{aligned} \tag{4.4}$$

PROOF. Differentiating (4.1) with respect to t, we have

$$\mu|u'|^{\mu-1}u'' = \rho' \cos\varphi - \rho\varphi' \sin\varphi.$$

Substituting this relation into (1.2) yields

$$\rho' \cos\varphi - \rho\varphi' \sin\varphi = \mu|u'|^{\mu-1}\left(-\frac{n-1}{t}u' - f\right). \tag{4.5}$$

Next, we differentiate (4.2) to obtain

$$u' = \rho' \sin\varphi + \rho\varphi' \cos\varphi. \tag{4.6}$$

Moreover, by (4.1), we get

$$u' = |\rho \cos\varphi|^{1/\mu} \operatorname{sgn}(\cos\varphi). \tag{4.7}$$

Combining this with (4.6) gives

$$\rho' \sin\varphi + \rho\varphi' \cos\varphi = |\rho\cos\varphi|^{1/\mu} \operatorname{sgn}(\cos\varphi). \tag{4.8}$$

Multiplying (4.5) by $-\sin\varphi$ and (4.8) by $\cos\varphi$, and then summing up these identities, we obtain

$$\rho\varphi' = \mu|u'|^{\mu-1}\left(\frac{n-1}{t}u' + f\right)\sin\varphi + \rho^{1/\mu}|\cos\varphi|^{1+1/\mu}.$$

Dividing both sides by ρ and using the resultant identity together with (4.1) and (4.2), we obtain the desired relation (4.4). This completes the proof.

From Lemma 4.1 we obtain the next key lemma which gives the relation between $N[u]$ and $\varphi(1)$.

LEMMA 4.2. (i) *Let* $u \in S_k \cap B_\infty(r)$ *and* $(t_i)_{i=1}^k \equiv (0 < t_1 < t_2 < \cdots < t_k \leq 1)$ *the associated sequence of its zeros. Then we have*

$$\varphi(t_j) = j\pi \qquad \text{for } 1 \leq j \leq k,$$

$$(j-1)\pi < \varphi(t) < j\pi \qquad \text{for } t_{j-1} < t < t_j,\ 1 \leq j \leq k+1,$$

and

$$k\pi \leq \varphi(1) < (k+1)\pi,$$

where we set $t_0 = 0$ *and* $t_{k+1} = 1$.

(ii) *Let* $u \in S \cap B_\infty(r)$. *Then* $k = N[u]$ *if and only if* $k\pi \leq \varphi(1) < (k+1)\pi$.

PROOF. First notice that $u(t) = 0$ if and only if $\varphi(t) = 0 \pmod{\pi}$. Let τ be a zero of $u(t)$. Then it is a simple zero, and so (4.4) implies

$$\varphi'(\tau) = \rho(\tau)^{-2} |u'(\tau)|^{\mu+1} > 0.$$

Consequently, $\varphi(t)$ is strictly increasing in the neighborhoods of zeros of u. Now, let $0 \leq s < \tau \leq 1$, $u(t) \neq 0$ for $t \in (s, \tau)$ and $u(\tau) = 0$. If $(j-1)\pi \leq \varphi(s) < j\pi$, then it follows that $(j-1)\pi < \varphi(t) < j\pi$ for $t \in (s, \tau)$ and $\varphi(\tau) = j\pi$. This proves the first assertion (i). The second assertion (ii) follows readily from (i), and the proof is complete.

We need the next lemma for the proof of Proposition 4.1.

LEMMA 4.3. *For each* $\alpha > 0$ *there is a constant* $C_\alpha > 0$ *such that*

$$\int_0^1 |u'|^\mu |u| \rho^{-2} t^{-1}\, dt \leq C_\alpha M(u)^{-\alpha} \qquad \text{for} \quad u \in S \cap B_\infty(r), \tag{4.9}$$

where $M(u)$ *is the number defined in Definition* 3.1.

PROOF. First, it follows from the definition of $\rho(t)$ that

$$\rho(t) \geq |u(t)| \qquad \text{and} \quad \rho(t) \geq |u'(t)|^\mu \tag{4.10}$$

for any $t \in [0, 1]$. By Lemma 3.6 and (4.10) we have

$$|u'|^\mu |u| \rho^{-2} t^{-1} \leq C |u'|^{\mu-1} |u|^{p+1} \rho^{-2} \leq C \rho^{-(1-p)/2}$$
$$\leq C |u(t)|^{-(1-p)/2} \leq C \| u \|_\infty^{-(1-p)/2}$$

for $t \in (0, T(u)]$, where $T(u)$ is defined in Definition 3.1. Hence we obtain

$$(4.11)\qquad \int_0^{T(u)} |u'|^{\mu}|u|\rho^{-2}t^{-1}\,dt \le CT(u)\,\|u\|_\infty^{-(1-p)/2}$$

$$\le C\|u\|_\infty^{(1-p)/2}\,\|u\|_\infty^{-(1-p)/2} = C,$$

where C is independent of u in $S\cap B_\infty(r)$.

Next, (4.10) asserts that for any $\alpha > 0$,

$$(4.12)\qquad \int_{T(u)}^1 |u'|^{\mu}|u|\rho^{-2}t^{-1}\,dt \le \int_{T(u)}^1 t^{-1}\,dt$$

$$\le T(u)^{-\alpha}\int_{T(u)}^1 t^{-1+\alpha}\,dt \le \frac{1}{\alpha}T(u)^{-\alpha} \le C_\alpha\,\|u\|_{p+1}^{-(1-p)\alpha/2}.$$

By Proposition 3.1

$$M(u) \le C\int_0^1 G(u)t^{n-1}\,dt \le C\,\|u\|_{p+1}^{p+1},$$

and so by (4.12) we have

$$(4.13)\qquad \int_{T(u)}^1 |u'|^{\mu}|u|\rho^{-2}t^{-1}\,dt \le C_\alpha M(u)^{-(1-p)\alpha/2(1+p)}$$

for $\alpha > 0$. Summing up (4.11) and (4.13), we obtain

$$\int_0^1 |u'|^{\mu}|u|\rho^{-2}t^{-1}\,dt \le C_\alpha M(u)^{-(1-p)\alpha/2(1+p)} + C$$

for $\alpha > 0$. Since r is taken to be sufficiently small, we may assume that $M(u) \le 1$. Consequently, it follows that

$$\int_0^1 |u'|^{\mu}|u|\rho^{-2}t^{-1}\,dt \le (C_\alpha + C)M(u)^{-(1-p)\alpha/2(1+p)}$$

for any $\alpha > 0$. Replacing $(1-p)\alpha/2(1+p)$ by α, we get the desired conclusion, and the proof is complete.

We are now ready to prove Proposition 4.1. The main tools of our proof are Proposition 3.1 and the new Prüfer transformation as introduced in Definition 4.2.

Proof of Proposition 4.1. Let $u\in S\cap B_\infty(r)$. Integrating the identity (4.4) displayed in Lemma 4.1 over $[0, 1]$, we get

$$(4.14)\qquad \varphi(1) - \frac{\pi}{2} \ge I_1(u) + I_2(u) - I_3(u) - I_4(u),$$

where $I_i(u)$, $1 \le i \le 4$, are respectively defined by

$$(4.15)\qquad I_1(u) \equiv \int_0^1 |u'|^{\mu+1}\rho^{-2}\,dt,$$

$$I_2(u) \equiv \mu \int_0^1 |u'|^{\mu-1} ug(u)\rho^{-2}\,dt, \tag{4.16}$$

$$I_3(u) \equiv \mu(n-1) \int_0^1 |u'|^{\mu}|u|\rho^{-2}t^{-1}\,dt, \tag{4.17}$$

$$I_4(u) \equiv \mu \int_0^1 |u'|^{\mu-1}|uh(t, u)|\rho^{-2}\,dt. \tag{4.18}$$

We then estimate the terms $I_i(u)$, $1 \le i \le 4$. First we give a lower bound of $I_1(u) + I_2(u)$. It follows from the definition of $\rho(t)$ that

$$\rho^2 = |u'|^{2\mu} + u^2, \tag{4.19}$$

and so that

$$\rho^{p+1} \le |u'|^2 + |u|^{p+1}. \tag{4.20}$$

This implies the estimate

$$\begin{aligned} I_1(u) + I_2(u) &= \int_0^1 |u'|^{\mu-1}\rho^{-2}(|u'|^2 + \mu ug(u))\,dt \\ &\ge C\int_0^1 |u'|^{\mu-1}\rho^{-2}(|u'|^2 + |u|^{p+1})\,dt \ge C\int_0^1 |u'|^{\mu-1}\rho^{-(1-p)}\,dt. \end{aligned} \tag{4.21}$$

To estimate the right-hand side, we fix any exponent m such that $m > \max\{n-1, (3+p)n/2(1+p)\}$. Then it follows from (4.20) and Proposition 3.1 that

$$\begin{aligned} \rho(t)^{(3+p)/2}t^m &\le (|u'|^2t^n + |u|^{p+1}t^n)^{(3+p)/2(1+p)} \\ &\le CM(u)^{(3+p)/2(1+p)} \qquad \text{for} \quad t \in [0, 1]. \end{aligned} \tag{4.22}$$

By (4.10) and (4.22) we have

$$|u'|^{\mu-1}\rho^{-(1-p)} \ge |u'|^2\rho^{-(3+p)/2} \ge CM(u)^{-(3+p)/2(1+p)}|u'|^2t^m. \tag{4.23}$$

From (4.21), (4.23) and Proposition 3.1, it follows that

$$\begin{aligned} I_1(u) + I_2(u) &\ge CM(u)^{-(3+p)/2(1+p)}\int_0^1 |u'|^2t^m\,dt \\ &\ge CM(u)^{-(1-p)/2(1+p)}, \end{aligned} \tag{4.24}$$

where C is a constant independent of u. This is the desired estimate. We have already estimated the term $I_3(u)$ in Lemma 4.3. Hence we estimate $I_4(u)$. It follows from (3.2) and (4.10) that

$$|u'|^{\mu-1}|uh(t, u)|\rho^{-2} \le |u'|^{\mu-1}|u|^{(p+3)/2}\rho^{-2} \le 1,$$

and therefore

$$I_4(u) \leq \mu. \tag{4.25}$$

We choose a constant α in Lemma 4.3 so that $0 < \alpha < (1-p)/2(1+p)$, and use (4.14), (4.24), (4.25) and Lemma 4.3. Then it follows that

$$\varphi(1) \geq C_1 M(u)^{-(1-p)/2(1+p)} - C_2 \tag{4.26}$$

for any $u \in S \cap B_\infty(r)$, where C_1 and C_2 are independent of u. On the other hand, we note that the inequalities (3.10) and (3.11) stated in Lemma 3.2 hold for any $u \in S$ with $|u(0)| \leq r_3$. These facts can easily be proved in the same way as in the proof of Lemma 3.2. It therefore follows from (3.11) that

$$M(u) \leq \|u\|_\infty^{p+1} \leq C|u(0)|^{p+1} \tag{4.27}$$

for any $u \in S$ with $|u(0)| \leq r_3$. The inequalities (4.26) and (4.27) together imply

$$\varphi(1) \geq C_3 |u(0)|^{-(1-p)/2} - C_4 \tag{4.28}$$

for $u \in S$ with $|u(0)|$ sufficiently small.

We now prove the first assertion that $N[u(\cdot, \gamma)] < \infty$ for $|\gamma|$ sufficiently small. By Lemma 3.5 and (3.11), any solution with small L^∞ norm has only simple zeros, and so it possesses at most a finite number of zeros. In fact, if it would have infinitely many zeros, then there would exist an accumulation point of zeros. However the accumulation point is not a simple zero, and this is a contradiction and therefore any solution with small L^∞ norm has at most a finite number of zeros. This fact, together with (4.27), implies that if $|u(0, \gamma)| = |\gamma| > 0$ is sufficiently small, then $N[u(\cdot, \gamma)] < \infty$. Recall that $u(t, \gamma)$ denotes the solution of (2.1)–$(2.2)_\gamma$. We here define $\varphi(t, \gamma)$ by the function $\varphi(t)$ introduced in Definition 4.2 with $u = u(t, \gamma)$. Then it follows from (4.28) that

$$\varphi(1, \gamma) \geq C_3 |\gamma|^{-(1-p)/2} - C_4 \qquad \text{for } |\gamma| \text{ sufficiently small.} \tag{4.29}$$

This inequality and Lemma 4.2 together imply that

$$\lim_{\gamma \to 0} N[u(\cdot, \gamma)] = +\infty.$$

The proof is thereby complete.

We choose a constant $\gamma_0 > 0$ such that $u(t, \gamma)$ is well-defined and $N[u(\cdot, \gamma)] < \infty$ for any $\gamma \in [-\gamma_0, \gamma_0] \backslash \{0\}$ and moreover the inequalities (3.10), (3.11), (4.27) and (4.28) hold for any $u(t, \gamma)$ with $0 < |\gamma| \leq \gamma_0$. We here present a lemma which guarantees a continuous dependence of solutions on the initial condition $(2.2)_\gamma$.

LEMMA 4.4. *The mapping* $\gamma \to u(\cdot, \gamma)$ *from* $[-\gamma_0, \gamma_0] \backslash \{0\}$ *into* $C^2[0, 1]$ *is* *continuous.*

PROOF. Assume that γ_k, $\gamma \in [-\gamma_0, \gamma_0] \setminus \{0\}$ and $\lim_{k \to \infty} \gamma_k = \gamma$. We write $u_k(t) = u(t, \gamma_k)$ and $u(t) = u(t, \gamma)$. We wish to prove that $\lim_{k \to \infty} u_k = u$ in $C^2[0, 1]$. Suppose that it be false. Then there would exist an $\varepsilon > 0$ and a subsequence (again denoted by (u_k) for simplicity) of (u_k) such that

$$\|u_k - u\|_{C^2[0,1]} \geq \varepsilon \qquad \text{for all} \quad k. \tag{4.30}$$

On the other hand, (4.27) asserts that $\sup_k \|u_k\|_\infty < \infty$. By (3.43) we obtain

$$|u_k'(t)| \leq Ct \|u_k\|_\infty^p \qquad \text{for all} \quad t \in [0, 1]. \tag{4.31}$$

This, together with equation (1.2), implies that $\sup_k \|u_k''\|_\infty < \infty$. Now (4.31) also implies that $\sup_k \|u_k'\|_\infty < \infty$ and therefore (u_k) is bounded in $C^2[0, 1]$. By Ascoli-Arzelà's theorem, we find a subsequence (u_{k_j}) and a function v such that $\{u_{k_j}\}$ converges to v in $C^1[0, 1]$. Applying this fact to integral equation (3.43), we see that (u_{k_j}) converges in $C^2[0, 1]$ and v is a solution of (2.1)–(2.2)$_\gamma$, from which it follows that $u = v$. This contradicts (4.30). The proof is now complete.

We now give the proofs of Theorems 1 and 3.

PROOF OF THEOREM 1. Let $a^2 + b^2 \neq 0$. We first consider the case where $b \neq 0$. According to the choice of γ_0, we have

$$\varphi(1, \gamma) < \infty \qquad \text{for} \quad 0 < |\gamma| \leq \gamma_0. \tag{4.32}$$

It suffices to find only a sequence (u_k^+), since another sequence (u_k^-) can also be found in the same way. To this end we choose a positive integer k_0 such that $\varphi(1, \gamma_0) < k_0 \pi$. Since $\lim_{\gamma \to \infty} \varphi(1, \gamma) = \infty$ by (4.29) and $\varphi(1, \gamma)$ is a continuous function of γ by Lemma 4.4, the intermediate value theorem implies that for $k \geq k_0$ there is a $\gamma \in (0, \gamma_0)$ such that $\varphi(1, \gamma) = k\pi$. Therefore we can define a sequence $(q_k)_{k \geq k_0}$ by

$$q_k = \sup \{\gamma \in (0, \gamma_0) : \varphi(1, \gamma) = (k + 1)\pi\}. \tag{4.33}$$

We also define a sequence $(p_k)_{k \geq k_0}$ by

$$p_k = \inf \{\gamma \in (q_k, \gamma_0) : \varphi(1, \gamma) = k\pi\}. \tag{4.34}$$

These sequences possess the following properties:

$$p_k > q_k \geq p_{k+1} > q_{k+1} > 0, \tag{4.35}$$

$$\lim_{k \to \infty} p_k = \lim_{k \to \infty} q_k = 0, \tag{4.36}$$

$$\varphi(1, p_k) = k\pi, \quad \varphi(1, q_k) = (k + 1)\pi \qquad \text{and} \tag{4.37}$$

(4.38) $\qquad k\pi < \varphi(1, \gamma) < (k+1)\pi \qquad \text{for} \quad \gamma \in (q_k, p_k) \quad \text{and} \quad k \geq k_0.$

The properties (4.35), (4.37) and (4.38) follow readily from the definitions of (p_k) and (q_k). We show the assertion (4.36). By (4.35) the sequence (p_k) converges. Suppose that $\lim_{k\to\infty} p_k = \gamma^* \neq 0$. Then $\gamma^* \in (0, \gamma_0)$ and (4.37) asserts that $\varphi(1, \gamma^*) = \lim_{k\to\infty} \varphi(1, p_k) = \infty$, which contradicts (4.32). We therefore obtain $\lim_{k\to\infty} p_k = 0$, and so (4.36) holds.

We here set $y(\gamma) = au(1, \gamma) + bu'(1, \gamma)$. The function $y(\cdot)$ is continuous on $(0, \gamma_0]$ by Lemma 4.4. From (4.37) and Definition 4.2 it follows that

$$y(p_k) = (-1)^k b\, \rho(1, p_k)^{1/\mu},$$
$$y(q_k) = (-1)^{k+1} b\, \rho(1, q_k)^{1/\mu}.$$

Since $b \neq 0$,

$$y(p_k)y(q_k) = -b^2 \{\rho(1, p_k)\, \rho(1, q_k)\}^{1/\mu} < 0$$

for $k \geq k_0$. By the intermediate value theorem there is a $\gamma_k \in (q_k, p_k)$ such that $y(\gamma_k) = 0$. We then define $u_k^+(t) = u(t, \gamma_k)$ for $k \geq k_0$. These are the desired solutions. In fact, they satisfy equation (1.2), boundary condition (1.4) and the sign condition $u_k^+(0) > 0$. By Lemma 3.7 and (4.27) we have

$$\| u_k^+ \|_H^2 \leq CM(u_k^+) \leq C \| u_k^+ \|_\infty^{p+1} \leq C\gamma_k^{p+1} \longrightarrow 0 \qquad \text{as} \quad k \longrightarrow \infty.$$

Since $k\pi < \varphi(1, \gamma_k) < (k+1)\pi$ by (4.38), $u_k^+(t)$ has exactly k zeros in the interval $[0, 1]$.

Next, we treat the case in which $b = 0$. Since we assumed that $a^2 + b^2 \neq 0$, we must have $a \neq 0$. Hence the boundary condition (1.4) can be written as

$$u'(0) = 0 \qquad \text{and} \quad u(1) = 0.$$

Since $\lim_{\gamma\to 0} \varphi(1, \gamma) = \infty$, we find a sequence $(\gamma_k)_{k \geq k_0}$ such that $\gamma_k > 0$, $\lim_{k\to\infty} \gamma_k = 0$ and $\varphi(1, \gamma_k) = k\pi$ for $k \geq k_0$. Therefore, in this case, we define $u_k^+(t) = u(t, \gamma_k)$ for $k \geq k_0$. These are the desired solutions of (1.2)–(1.4). The proof of Theorem 1 is thereby complete.

Proof of Theorem 3. We first show the lower estimates for the H norms of the solutions. Let $u \in S_k \cap B_\infty(r)$. Since $\varphi(1) \in [k\pi, (k+1)\pi)$ by Lemma 4.2, (4.26) yields

$$(k+1)\pi \geq C_1 M(u)^{-(1-p)/2(1+p)} - C_2,$$

Therefore, by Lemma 3.7, we have

$$Ck^{-(1+p)/(1-p)} \leq \| u \|_H,$$

where C is a constant independent of u. This is the desired lower estimate with respect to the H norm.

We next consider the upper estimates for the H norms of the solutions. Let $u \in S_k \cap B_\infty(r)$. Integrating both sides of (4.4) over $[0, 1]$ with respect to t, we obtain

$$\varphi(1) - \frac{\pi}{2} \leq I_1(u) + I_2(u) + I_3(u) + I_4(u), \tag{4.39}$$

where the terms $I_i(u)$, $1 \leq i \leq 4$, are defined by (4.15), (4.16), (4.17) and (4.18), respectively. We then estimate the terms $I_i(u)$, $1 \leq i \leq 4$. In order to estimate $I_1(u)$, we apply (4.10) and obtain

$$I_1(u) \leq \int_0^1 \rho(t)^{-(1-p)/2}\, dt. \tag{4.40}$$

Let $\omega > 0$. From (4.10), Lemma 3.5 and Proposition 3.1 it follows that

$$\rho^{p+1} \geq \frac{1}{2}(|u'(t)|^2 + |u(t)|^{p+1}) \geq Ct^\omega \int_0^1 G(u)t^{n-1}\, dt \geq Ct^\omega M(u),$$

so that

$$\rho(t)^{-(1-p)/2} \leq Ct^{-(1-p)\omega/2(1+p)} M(u)^{-(1-p)/2(1+p)} \tag{4.41}$$

for $t \in [0, 1]$ and $u \in S \cap B_\infty(r)$. Here the constants r and C depend only on ω. Choose a constant ω so small that $(1-p)\omega/2(1+p) < 1$. Then $t^{-(1-p)\omega/2(1+p)}$ is integrable over $[0, 1]$, and hence it follows from (4.40) and (4.41) that

$$I_1(u) \leq CM(u)^{-(1-p)/2(1+p)}. \tag{4.42}$$

We next estimate $I_2(u)$. By (4.10) we have

$$I_2(u) \leq C \int_0^1 \rho^{-(1-p)/2}\, dt,$$

and so (4.41) yields

$$I_2(u) \leq CM(u)^{-(1-p)/2(1+p)}. \tag{4.43}$$

We have already estimated $I_3(u)$ and $I_4(u)$ in Lemma 4.3 and (4.25), respectively. Combining (4.42), (4.43), Lemma 4.3 with $\alpha = (1-p)/2(1+p)$, (4.25) and (4.39), we obtain

$$\varphi(1) \leq C_1 M(u)^{-(1-p)/2(1+p)} + C_2.$$

Since $\varphi(1) \in [k\pi, (k+1)\pi)$ by Lemma 4.2, it follows that

$$k\pi \le C_1 M(u)^{-(1-p)/2(1+p)} + C_2.$$

At this point we may assume that $M(u) \le \|u\|_\infty \le 1$ for $u \in S \cap B_\infty(r)$. This implies that

$$k\pi \le (C_1 + C_2)\, M(u)^{-(1-p)/2(1+p)}.$$

Consequently, by Lemma 3.7, we obtain

$$\|u\|_H \le Ck^{-(1+p)/(1-p)}.$$

This completes the proof of Theorem 3.

5. A priori estimates in the superlinear case

The purpose of this section is to present several a priori inequalities of solutions in the superlinear case. Throughout this section we always suppose conditions (B1). (B2) and (B3). By (B1) we may assume that

$$a_1|s|^{p+1} \le sg(s) \le a_2|s|^{p+1} \qquad \text{for all} \quad s \in \boldsymbol{R}. \tag{5.1}$$

Indeed, we first define $\tilde{g}(s)$ by $\tilde{g}(s) = g(s)$ if $|s| \ge R_0$, $\tilde{g}(s) = g(R_0)R_0^{-p}s^p$ if $0 \le s < R_0$ and $\tilde{g}(s) = g(-R_0)R_0^{-p}|s|^p$ if $-R_0 < s < 0$. Then $\tilde{g}(s)$ is continuous on $\boldsymbol{R}$ and satisfies conditions (5.1) and (B3). We next define $\tilde{h}(t, s)$ by $\tilde{h}(t, s) = g(s) - \tilde{g}(s) + h(t, s)$. Then $f(t, s)$ can be represented as $f(t, s) \equiv \tilde{g}(s) + \tilde{h}(t, s)$. The function $\tilde{h}(t, s)$ is continuous on $[0, 1] \times \boldsymbol{R}$ and satisfies (B2). Thus we may suppose (5.1) for the function $g(s)$ without loss of generality. From this condition one finds constants $b_i > 0$, $1 \le i \le 4$, such that

$$|s|^{p+1} \le b_1 G(s) \le b_2 sg(s) \le b_3 G(s) \le b_4|s|^{p+1} \tag{5.2}$$

for all $s \in \boldsymbol{R}$. On the other hand, if $n \ge 3$ then (B3) states that there exist constants θ, $C > 0$ such that

$$(2n - \theta)G(s) + C \ge (n-2)sg(s) \qquad \text{for all} \quad s \in \boldsymbol{R}. \tag{5.3}$$

Lastly, (B2) assures that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|h(t, s)| \le \varepsilon|s|^{(p+1)/2} + C_\varepsilon \qquad \text{for} \quad s \in \boldsymbol{R}. \tag{5.4}$$

Moreover, since $(p+3)/2 < p+1$, we have

$$|sh(t, s)| \le \varepsilon|s|^{p+1} + C_\varepsilon \qquad \text{for} \quad s \in \boldsymbol{R}. \tag{5.5}$$

The next proposition is the main result of this section and plays an important role in the proofs of Theorems 2 and 4.

PROPOSITION 5.1. *Suppose that conditions* (B1), (B2) *and* (B3) *hold. Then* *for any $m > n - 1$ there exist constants $C_i > 0$, $1 \le i \le 5$, such that the estimates*

$$\begin{aligned}
&\max_{0\le t\le 1} |u(t)|^{p+1} t^n \le C_1 \max_{0\le t\le 1} G(u(t))t^n \\
&\le C_2 \max_{0\le t\le 1} u'(t)^2 t^n + C_2 \le C_3 \int_0^1 |u'|^2 t^m \, dt + C_3 \\
&\le C_3 \int_0^1 |u'|^2 t^{n-1} \, dt + C_3 \le C_4 \int_0^1 G(u) t^m \, dt + C_4 \\
&\le C_4 \int_0^1 G(u) t^{n-1} \, dt + C_4 \le C_5 \max_{0\le t\le 1} |u(t)|^{p+1} t^n + C_5
\end{aligned}$$

are valid for any $u \in S$.

To prove Proposition 5.1, we prepare several lemmas. Noting that $p > 1$ and using conditions (5.2), (5.3), (5.4) and (5.5) in the proofs of Lemmas 3.1, 3.2 and 3.3 instead of (3.1), (3.2), (3.3) and (3.4), we obtain the following three lemmas.

LEMMA 5.1. (i) *For $u \in S$ we have*

$$\int_0^1 |u'|^2 t^{n-1} \, dt = \int_0^1 uf(t, u) t^{n-1} \, dt + u'(1)u(1). \tag{5.6}$$

(ii) *There exist constants C_1, $C_2 > 0$ such that*

$$\int_0^1 |u'|^2 t^{n-1} \, dt \le C_1 \int_0^1 |u|^{p+1} t^{n-1} \, dt + u'(1)u(1) + C_1 \tag{5.7}$$

and

$$\int_0^1 |u|^{p+1} t^{n-1} \, dt \le C_2 \int_0^1 |u'|^2 t^{n-1} \, dt + C_2 |u'(1)u(1)| + C_2 \tag{5.8}$$

are valid for $u \in S$.

LEMMA 5.2. *There exist constants C_1, $C_2 > 0$ such that*

$$\tfrac{1}{2} u'(t)^2 + G(u(t)) \le C_1 G(u(0)) + C_1 \tag{5.9}$$

and

$$|u(0)| \le \|u\|_\infty \le C_2 |u(0)| + C_2 \tag{5.10}$$

for $t \in [0, 1]$ and $u \in S$.

LEMMA 5.3. *For any $m > n - 1$ there exist constants C_1, $C_2 > 0$ such that*

$$\int_0^1 G(u)t^m dt \leq C_1 \int_0^1 |u'|^2 t^m dt + C_1 |u'(1)u(1)| + C_1 u(1)^2 + C_1 \tag{5.11}$$

and

$$\int_0^1 |u'|^2 t^m dt \leq C_2 \int_0^1 G(u)t^{m-2} dt + u'(1)u(1) + C_2 \tag{5.12}$$

hold for any $u \in S$.

From Lemmas 5.1 and 5.3 we obtain the next lemma which is basic to the verification of Proposition 5.1.

LEMMA 5.4. *For any* $m > n - 1$ *there are constants* $C_i > 0$, $1 \leq i \leq 3$, *such that*

$$\max_{0 \leq t \leq 1} \left\{ \frac{1}{2} u'(t)^2 t^n + G(u(t))t^n \right\} \leq C_1 \int_0^1 G(u)t^{n-1} dt + C_1$$

$$\leq C_2 \int_0^1 |u'|^2 t^m dt + C_2 \leq C_3 \max_{0 \leq t \leq 1} u'(t)^2 t^n + C_3$$

for $u \in S$.

PROOF. Using Lemmas 5.1, 5.2 and 5.3, we follow the lines of the proof of Lemma 3.4, where we need some modifications to complete the proof of Lemma 5.4. First we rewrite the inequality (3.22) as

$$\begin{aligned} \int_0^1 |u'(t)h(t, u)| t^n dt &\leq \varepsilon \int_0^1 |u'| t^{n/2} |u|^{(p+1)/2} t^{n/2} dt + C_\varepsilon \\ &\leq \varepsilon CK + C_\varepsilon \quad \text{for any} \quad \varepsilon > 0, \end{aligned} \tag{5.13}$$

where C_ε can be chosen as a constant depending only on ε. Therefore the inequality (3.23) can be modified in the form

$$K \leq 2n \int_0^1 G(u)t^{n-1} dt + C, \tag{5.14}$$

where C is independent of u and K. In what follows, C denotes various constants independent of u, K and ε, while C_ε means a constant depending only on ε. As in the inequalities (5.13) and (5.14), we must add appropriate constants "$\pm C_\varepsilon$" or "$\pm C$" to the corresponding inequalities. Using (5.14), we may rewrite (3.32) as

$$|u'(1)u(1)| + u(1)^2 \leq CK^{(p+3)/2(p+1)} + CK^{2/(p+1)} \tag{5.15}$$

$$\le C\left(\int_0^1 G(u)t^{n-1}\,dt\right)^{(p+3)/2(p+1)} + C\left(\int_0^1 G(u)t^{n-1}\,dt\right)^{2/(p+1)} + C$$

$$\le \varepsilon\int_0^1 G(u)t^{n-1}\,dt + C_\varepsilon \qquad \text{for any} \quad \varepsilon > 0.$$

Here $(p+3)/2(p+1)$, $2/(p+1) < 1$ and we used Young's inequality. From the modified form of (3.31) and (5.15) it follows that

$$\int_0^1 G(u)t^{n-1}\,dt \le C\int_0^1 |u'|^2 t^{m-1}\,dt + \varepsilon C\int_0^1 G(u)t^{n-1}\,dt + C_\varepsilon$$

for any $\varepsilon > 0$. Choosing ε sufficiently small, we have

$$\int_0^1 G(u)t^{n-1}\,dt \le C\int_0^1 |u'|^2 t^{m-1}\,dt + C,$$

which is the second inequality of Lemma 5.4. It is easy to check the last inequality of Lemma 5.4, and this completes the proof.

We are now ready to prove Proposition 5.1.

Proof of Proposition 5.1. We follow the lines of the proof of Proposition 3.1. As mentioned in the proof of Lemma 5.4, we need to add constant terms "$\pm C$" or "$\pm C_\varepsilon$" to the corresponding inequalities treated in the proof of Proposition 3.1. Moreover it is necessary to modify all the inequalities which include the terms "$\|u\|_\infty$". First appropriate constant terms "$+C$" must be added to the right-hand sides of (3.33), (3.34) and (3.35); it is seen that for any $m > n-3$ there is a constant $C > 0$ such that

$$\int_0^1 |u'|^2 t^{n-1}\,dt \le C\int_0^1 G(u)t^{n-1}\,dt + u'(1)u(1) + C, \tag{5.16}$$

$$\int_0^1 G(u)t^{n-1}\,dt \le C\int_0^1 G(u)t^m dt + C|u'(1)u(1)| + C, \tag{5.17}$$

$$\int_0^1 |u'|^2 t^{n-1}\,dt \le C\int_0^1 G(u)t^m\,dt + C|u'(1)u(1)| + C, \tag{5.18}$$

for $u \in S$. Next, we choose $m = n$ in Lemma 5.4 and get

$$\max_{0\le t\le 1}\left\{\frac{1}{2}u'(t)^2 + G(u(t))\right\}t^n \le C\int_0^1 |u'|^2 t^n\,dt + C \le C\int_0^1 |u'|^2 t^{n-1}\,dt + C.$$

From this we obtain

$$|u'(1)| \leq \left(C \int_0^1 |u'|^2 t^{n-1}\, dt + C \right)^{1/2}$$

and

$$|u(1)| \leq \left(C \int_0^1 |u'|^2 t^{n-1}\, dt + C \right)^{1/(p+1)}.$$

Since $1/2 + 1/(p+1) < 1$, we have

$$(5.19) \quad |u'(1)u(1)| \leq \left(C \int_0^1 |u'|^2 t^{n-1}\, dt + C \right)^{1/2 + 1/(p+1)} \leq \varepsilon \int_0^1 |u'|^2 t^{n-1}\, dt + C_\varepsilon$$

for $\varepsilon > 0$ and some constant C_ε depending upon ε. By (5.18) and (5.19), for any $m > n - 3$ there is a constant C such that

$$\int_0^1 |u'|^2 t^{n-1}\, dt \leq C \int_0^1 G(u) t^m\, dt + \varepsilon C \int_0^1 |u'|^2 t^{n-1}\, dt + C_\varepsilon$$

for $\varepsilon > 0$ and some constant C_ε. Here C is a constant independent of ε. Let $\varepsilon > 0$ be sufficiently small. Then the above inequality implies the fifth inequality of Proposition 5.1. To prove the last inequality of Proposition 5.1, we use (5.15) and (5.17). Then we see that for any $m > n - 3$ and $\varepsilon > 0$,

$$\int_0^1 G(u) t^{n-1}\, dt \leq C \int_0^1 G(u) t^m dt + \varepsilon C \int_0^1 G(u) t^{n-1}\, dt + C_\varepsilon,$$

where C depends only on m, but C_ε depends on both m and ε. Choosing $\varepsilon = 1/2C$, we obtain

$$\int_0^1 G(u) t^{n-1} dt \leq C \int_0^1 G(u) t^m dt + C.$$

Letting $m = n$, we get

$$\int_0^1 G(u) t^{n-1} dt \leq C \int_0^1 G(u) t^n dt + C \leq C' \max_{0 \leq t \leq 1} |u(t)|^{p+1} t^n + C.$$

This is nothing but the last inequality of Proposition 5.1, and the proof of Proposition 5.1 is complete.

In the next lemma which corresponds to Lemma 3.6, we present some technical estimates for solutions in a neighborhood of $t = 0$.

LEMMA 5.5. *There are positive constants C_1, C_2, R_1 and δ such that if $u \in S \cap B_\infty(R_1)^c$, then*

(5.20)
$$\|u\|_{\infty} \le C_1 |u(t)|$$

and

(5.21)
$$|u'(t)| \le C_2 t |u(t)|^p$$

for $t \in [0, \delta \|u\|_{\infty}^{-(p-1)/2}]$, *where*

$$B_{\infty}(R)^c \equiv L^{\infty} \backslash B_{\infty}(R) \equiv \{u \in L^{\infty}: \|u\|_{\infty} \ge R\}.$$

PROOF. We follow the lines of the proof of Lemma 3.6 with some slight modifications. Let $u \in S$. We wish to estimate the right-hand sides of the identities (3.43) and (3.44). For $t \in [0, \delta \|u\|_{\infty}^{-(p-1)/2}]$ it follows that

$$\int_0^t \left(s^{-(n-1)} \int_0^s |f(\tau, u(\tau))| \tau^{n-1} \, d\tau \right) ds \le Ct^2 \|u\|_{\infty}^p + C \le C\delta^2 \|u\|_{\infty} + C,$$

where C is independent of δ and u. This inequality and (3.44) together imply

$$|u(0)| - C\delta^2 \|u\|_{\infty} \le |u(t)| + C,$$

and therefore (5.10) yields

$$\left(\frac{1}{C_2} - C\delta^2 \right) \|u\|_{\infty} \le |u(t)| + C + 1.$$

Choose δ so small. Then we find $C' > 0$ such that

$$\|u\|_{\infty} \le C' |u(t)| + C'.$$

Set $R_1 = 2C'$. Then the above inequality yields

$$\|u\|_{\infty} \le 2C' |u(t)| \qquad \text{for} \quad u \in B_{\infty}(R_1)^c.$$

This is the desired inequality (5.20). Next, it follows from the relation (3.43) that

$$|u'(t)| \le Ct \|u\|_{\infty}^p + Ct.$$

If $\|u\|_{\infty} \ge 1$, then we have $|u'(t)| \le 2Ct \|u\|_{\infty}^p$. This, together with (5.20), implies (5.21). The proof is now complete.

In view of Lemma 5.5, we introduce a notation corresponding to Definition 3.1.

DEFINITION 5.1. We define

$$L(u) \equiv \delta \|u\|_{\infty}^{-(p-1)/2} \qquad \text{for} \quad u \in S \cap B_{\infty}(1)^c,$$

where δ is a constant determined in Lemma 5.5.

In Definition 5.1 we may assume that δ, $L(u) < 1$ without loss of generality. In the next lemma we see that any solution u with large L^∞ norm has only simple zeros. This fact is essential for defining our new Prüfer transformations later.

LEMMA 5.6. (i) *There exist constants* R_2 $(\geq R_1)$ *and* $C > 0$ *such that if* $u \in S \cap B_\infty(R_2)^c$, *then*

$$\tag{5.22} \tfrac{1}{2}u'(t)^2 + G(u(t)) \geq C\,\|u\|_\infty^\sigma \qquad \text{for all} \quad t \in [0, 1],$$

where $\sigma \equiv p + 1 - n(p-1)/2$ (> 0).

(ii) *Any solution* $u \in S \cap B_\infty(R_2)^c$ *has only simple zeros and possesses at most a finite number of zeros in the interval* $[0, 1]$.

PROOF. (i): Substituting $m = n$ into (3.18), we obtain

$$\tag{5.23} \begin{aligned} \left(\frac{1}{2}u'(t)^2 + G(u(t))\right)t^n &= n\int_0^t G(u)s^{n-1}\,ds \\ &\quad + \left(-\frac{n}{2} + 1\right)\int_0^t |u'|^2 s^{n-1}\,ds - \int_0^t u'h(s, u)s^n\,ds \end{aligned}$$

for $0 \leq t \leq 1$. We first use (5.4) to estimate the last term on the right-hand side as

$$\tag{5.24} \begin{aligned} \int_0^t |u'h(s, u)|s^n\,ds &\leq \int_0^t |u'|\,(\varepsilon|u|^{(p+1)/2} + C_\varepsilon)s^n\,ds \\ &\leq \varepsilon\int_0^t |u'|^2 s^{n-1}\,ds + \varepsilon\int_0^t |u|^{p+1}s^{n-1}\,ds + C_\varepsilon. \end{aligned}$$

Here and from now on, we denote by C_ε various constants dependent only on ε. We define

$$E(t) \equiv \tfrac{1}{2}u'(t)^2 + G(u(t)).$$

From (5.23) and (5.24) it follows that

$$\tag{5.25} \begin{aligned} E(t) \geq E(t)t^n \geq n\int_0^t G(u)s^{n-1}\,ds &+ \left(-\frac{n}{2} + 1 - \varepsilon\right)\int_0^t |u'|^2 s^{n-1}\,ds \\ &- \varepsilon\int_0^t |u|^{p+1}s^{n-1}\,ds - C_\varepsilon. \end{aligned}$$

On the other hand, multiplying equation (1.2) by $u(t)t^{n-1}$ and integrating the resultant identity over $[0, t]$, we have

$$(5.26)\qquad \int_0^t |u'|^2 s^{n-1}\,ds = \int_0^t ug(u)s^{n-1}\,ds + \int_0^t uh(s,u)s^{n-1}\,ds + u'(t)u(t)t^{n-1}.$$

Combining (5.25) and (5.26), we have

$$(5.27)\qquad \begin{aligned} E(t) \geq n\int_0^t G(u)s^{n-1}\,ds &+ \left(-\frac{n}{2}+1-\varepsilon\right)\int_0^t ug(u)s^{n-1}\,ds \\ &+ \left(-\frac{n}{2}+1-\varepsilon\right)\int_0^t uh(s,u)s^{n-1}\,ds - \varepsilon\int_0^t |u|^{p+1}s^{n-1}\,ds \\ &+ \left(-\frac{n}{2}+1-\varepsilon\right)u'(t)u(t)t^{n-1} - C_\varepsilon. \end{aligned}$$

We now estimate the third and the fifth terms on the right-hand side. It follows from (5.2) and (5.5) that for any $0 < \varepsilon < 1$,

$$(5.28)\qquad \begin{aligned} &\left|-\frac{n}{2}+1-\varepsilon\right|\int_0^t |uh(s,u)|s^{n-1}\,ds \\ &\qquad \leq \frac{n\varepsilon}{2}\int_0^t |u|^{p+1}s^{n-1}\,ds + C_\varepsilon \leq \varepsilon C_0\int_0^t ug(u)s^{n-1}\,ds + C_\varepsilon, \end{aligned}$$

where C_0 is independent of ε. From the definition of $E(t)$ and (5.2), we have

$$|u'(t)| \leq (2E(t))^{1/2} \quad \text{and} \quad |u(t)| \leq (b_1E(t))^{1/(p+1)}$$

and so

$$(5.29)\qquad \left|\left(-\frac{n}{2}+1-\varepsilon\right)u'(t)u(t)t^{n-1}\right| \leq CE(t)^{(p+3)/2(p+1)} \leq E(t) + C.$$

We here used Young's inequality together with the fact that $(p+3)/2(p+1) < 1$. From the inequalities (5.2), (5.27), (5.28) and (5.29), it follows that

$$(5.30)\qquad 2E(t) \geq n\int_0^t G(u)s^{n-1}\,ds - \frac{n-2+\varepsilon C}{2}\int_0^t ug(u)s^{n-1}\,ds - C_\varepsilon,$$

where C is independent of ε. On the other hand, we find constants a, $C_1 > 0$ such that

$$(5.31)\qquad 2nG(s) - (n-2+a)sg(s) + C_1 \geq a|s|^{p+1}$$

for all $s \in \boldsymbol{R}$. Indeed, if $n \geq 3$ then this follows from (B3) and (5.2); if $n = 2$ then this inequality is obtained from (5.2) only. We choose ε so small that $\varepsilon C \leq a$. It then follows from (5.30) and (5.31) that

$$(5.32) \qquad E(t) \geq C_2 \int_0^t |u|^{p+1} s^{n-1}\, ds - C_3.$$

If $u \in S \cap B_\infty(R_1)^c$, it follows from Lemma 5.5 that for any $t \in [L(u), 1]$,

$$\int_0^t |u|^{p+1} s^{n-1}\, ds \geq \int_0^{L(u)} |u|^{p+1} s^{n-1}\, ds \geq C \| u \|_\infty^{p+1} L(u)^n = C' \| u \|_\infty^\sigma,$$

where $\sigma \equiv p + 1 - n(p-1)/2$ is positive since $1 < p < n^*$. Combining this inequality with (5.32), then we have

$$(5.33) \qquad E(t) \geq C_4 \| u \|_\infty^\sigma - C_3 \qquad \text{for} \quad t \in [L(u), 1].$$

We choose R_2 $(\geq R_1)$ so large that $C_4 R_2^\sigma \geq 2C_3$. Thus (5.33) implies that if $u \in S \cap B_\infty(R_2)^c$, then

$$(5.34) \qquad E(t) \geq \tfrac{1}{2} C_4 \| u \|_\infty^\sigma \qquad \text{for} \quad t \in [L(u), 1].$$

Next, let us consider the case where $t \in [0, L(u)]$. In this case the application of Lemma 5.5 implies

$$(5.35) \qquad E(t) \geq G(u(t)) \geq C |u(t)|^{p+1} \geq C' \| u \|_\infty^{p+1}$$

for $t \in [0, L(u)]$ and $u \in S \cap B_\infty(R_1)^c$. Therefore the first assertion (i) follows from (5.34) and (5.35).

(ii): From (5.22) we see straightforwardly that $u \in S \cap B_\infty(R_2)^c$ has only simple zeros. If it has infinitely many zeros, then there exists an accumulation point of zeros in $[0, 1]$. However the accumulation point is not a simple zero. This is a contradiction, and so we obtain the second assertion (ii). The proof is thereby complete.

In the next lemma corresponding to Lemma 3.7, we find a relation between the quantities $M(u)$ and $\| u \|_H$ for any solution u in S.

LEMMA 5.7. *There are two positive constants C_1 and C_2 such that*

$$(5.36) \qquad M(u) \leq C_1 \| u \|_H^2 + C_1 \leq C_2 M(u) + C_2$$

for any $u \in S$.

PROOF. Using Proposition 5.1, we find constants C_1, $C_2 > 0$ such that

$$\| u \|_H^2 \geq \int_0^1 |u'|^2 t^{n-1}\, dt \geq C_1 M(u) - C_2,$$

which is the first inequality of (5.36). Again, applying Proposition 5.1 and noting that $(p+1)/2 > 1$, we have

$$\begin{aligned}\|u\|_H^2 &= \int_0^1 (u^2 + |u'|^2)t^{n-1}\,dt \\ &\le \left(\int_0^1 |u|^{p+1}t^{n-1}\,dt\right)^{2/(p+1)} + \int_0^1 |u'|^2 t^{n-1}\,dt \\ &\le (CM(u) + C)^{2/(p+1)} + CM(u) + C \le C'M(u) + C',\end{aligned}$$

which is the second inequality of (5.36). This completes the proof.

Using Lemma 5.6, we obtain the next lemma, which gives a relation between the quantities $M(u)$ and $\|u\|_\infty$ for a solution belonging to the class S.

LEMMA 5.8. *There exist constants $C > 0$ and $R_3 (\ge R_2)$ such that*

$$CM(u) \ge \|u\|_\infty^\sigma \qquad \text{for} \quad u \in S \cap B_\infty(R_3)^c, \tag{5.37}$$

where $\sigma \equiv p + 1 - n(p-1)/2$.

PROOF. Multiplying (5.22) by t^{n-1} and integrating over $[0, 1]$, we see with the aid of Proposition 5.1 that

$$C_1 M(u) + C_2 \ge \|u\|_\infty^\sigma \qquad \text{for} \quad u \in S \cap B_\infty(R_2)^c.$$

Choose R_3 $(\ge R_2)$ so large that $2C_2 \le R_3^\sigma$. Then the above inequality implies (5.37) and the proof is complete.

REMARK 5.1. We can replace the conditions $u \in S \cap B_\infty(R_i)^c$, $1 \le i \le 3$, by a single condition $u \in S \cap B(R)^c$ for some $R > 0$ in Lemmas 5.5, 5.6 and 5.8, respectively. In other words, for $u \in S \cap B(R)^c$ these lemmas remain valid. Here R is chosen as a positive constant dependent on R_1, R_2 and R_3. In fact, it follows from Lemma 5.7 that

$$\|u\|_H^2 \le CM(u) + C \le C\|u\|_\infty^{p+1} + C \qquad \text{for any} \quad u \in S.$$

We here choose R so large that the above inequality implies that $\|u\|_\infty \ge R_3 \ge R_2 \ge R_1$ for all $u \in S$ with $\|u\|_H \ge R$. Therefore we conclude that the conditions $u \in S \cap B_\infty(R_i)^c$, $1 \le i \le 3$, can be replaced by $u \in S \cap B(R)^c$ in Lemmas 5.5, 5.6 and 5.8, respectively.

6. Proof of Theorems 2 and 4

In this section we apply the results given in Section 5 and prove Theorems 2 and 4. We first present the following proposition which plays an important role in the proof of Theorem 2.

PROPOSITION 6.1. *Under the assumptions of Theorem 2, it follows that*

$N[u(\cdot, \gamma)] < +\infty$ *for* $|\gamma|$ *sufficiently large and*

$$\lim_{\gamma \to \pm\infty} N[u(\cdot, \gamma)] = +\infty.$$

In what follows, we assume without further mention that conditions (B1), (B2) and (B3) are valid. To prove Proposition 6.1 mentioned above, Theorems 2 and 4, we introduce new Prüfer transformations below.

DEFINITION 6.1. For $u \in S \cap B(R)^c$ we define the functions $\rho(t)$ and $\varphi(t)$ by

$$u' = \rho \cos \varphi, \tag{6.1}$$

$$|u|^{\nu-1} u = \rho \sin \varphi, \tag{6.2}$$

$$\varphi(0) = \frac{\pi}{2}, \tag{6.3}$$

where $\nu = (p+1)/2$. Notice that $\nu > 1$.

The relations (6.1), (6.2) and (6.3) determine the functions $\rho(t)$ and $\varphi(t)$ of class $C^1[0, 1]$ uniquely. In fact, as mentioned in Lemma 5.6, any solution $u \in S \cap B(R)^c$ has only simple zeros, and so it follows that

$$\rho(t) = (|u'|^2 + |u|^{p+1})^{1/2} > 0 \qquad \text{for all} \quad t \in [0, 1].$$

and the continuous functions $\rho(t)$ and $\varphi(t)$ are uniquely defined. Since the left-hand sides of (6.1) and (6.2) are continuously differentiable, the functions $\rho(t)$ and $\varphi(t)$ are also of the class $C^1[0, 1]$.

LEMMA 6.1. *For* $u \in S \cap B(R)^c$ *we have*

$$\begin{aligned} \varphi'(t) = \rho^{-2}\{&|u|^{\nu-1} ug(u) + \nu |u|^{\nu-1} |u'|^2 \\ &+ \frac{n-1}{t} u' |u|^{\nu-1} u + |u|^{\nu-1} uh(t, u)\}. \end{aligned} \tag{6.4}$$

PROOF. Differentiating (6.1) with respect to t, we obtain

$$u'' = \rho' \cos \varphi - \rho\varphi' \sin \varphi,$$

which, together with (1.2), implies

$$\rho' \cos \varphi - \rho\varphi' \sin \varphi = -\frac{n-1}{t} u' - f. \tag{6.5}$$

Next, we differentiate (6.2) to obtain

$$\rho' \sin \varphi + \rho\varphi' \cos \varphi = \nu |u|^{\nu-1} u'. \tag{6.6}$$

Multiplying (6.5) by $-\sin\varphi$, (6.6) by $\cos\varphi$ and then summing up these identities, we obtain

$$\rho\varphi' = \sin\varphi\left\{\frac{n-1}{t}u' + f\right\} + \nu|u|^{\nu-1}u'\cos\varphi.$$

Dividing both sides of this relation by ρ and using the relations (6.1) and (6.2), we get (6.4). This completes the proof.

LEMMA 6.2. *Let $u \in S \cap B(R)^c$. Then for any zero τ of u there is an $\varepsilon > 0$ such that*

$$\varphi'(t) > 0 \qquad \text{if} \quad 0 < |t - \tau| < \varepsilon.$$

Therefore $\varphi(t)$ is strictly increasing in some neighborhoods of zeros of $u(t)$.

PROOF. Since $u \in S \cap B(R)^c$ and $u(\tau) = 0$, Lemma 5.6 asserts that τ is a simple zero of $u(t)$. Therefore $u'(\tau) \neq 0$ and $\tau \neq 0$. We set $a = u'(\tau)$ and choose $\varepsilon_1 > 0$ so that

$$\frac{|a|}{2} \leq |u'(t)| \leq 2|a|$$

and

$$\frac{|a|}{2}|t-\tau| \leq |u(t)| \leq 2|a|\,|t-\tau|$$

for all $t \in [\tau - \varepsilon_1, \tau + \varepsilon_1]$. Here we may assume that $\tau - \varepsilon_1 > 0$ since $\tau > 0$. Using the above inequalities, one finds positive constants d_0, d_1 and d_2 such that

$$\nu|u(t)|^{\nu-1}|u'(t)|^2 \geq d_0|t-\tau|^{\nu-1}, \tag{6.7}$$

$$|u(t)|^{\nu}|h(t, u)| \leq d_1|t-\tau|^{\nu}, \tag{6.8}$$

$$\frac{n-1}{t}|u'(t)|\,|u(t)|^{\nu} \leq d_2|t-\tau|^{\nu}, \tag{6.9}$$

for $t \in [\tau - \varepsilon_1, \tau + \varepsilon_1]$. Combining (6.4), (6.7), (6.8) and (6.9), we obtain the estimate

$$\begin{aligned}\rho(t)^2\varphi'(t) &\geq \nu|u|^{\nu-1}|u'|^2 - \frac{n-1}{t}|u'|\,|u|^{\nu} - |u|^{\nu}|h(t, u)| \\ &\geq d_0|t-\tau|^{\nu-1} - (d_1 + d_2)|t-\tau|^{\nu}\end{aligned}$$

for $t \in [\tau - \varepsilon_1, \tau + \varepsilon_1]$. Therefore, if $|t - \tau| > 0$ is sufficiently small then we obtain $\rho(t)^2 \varphi'(t) > 0$. This completes the proof.

By means of Lemma 6.2 we obtain the next key lemma which gives us the relation between the number of zeros of a solution u in $S \cap B(R)^c$ and the value of $\varphi(1)$.

LEMMA 6.3. (i) *Let* $u \in S_k \cap B(R)^c$ *and* $(t_i)_{i=1}^k \equiv (0 < t_1 < t_2 < \cdots < t_k \leq 1)$ *the sequence of its zeros. Then we have the same assertion as in* (i) *of Lemma* 4.2.

(ii) *Let* $u \in S \cap B(R)^c$. *Then the same assertion as in* (ii) *of Lemma* 4.2 *is valid.*

PROOF. In view of Lemma 6.2, we obtain the conclusion in the same way as in the proof of Lemma 4.2.

For the proof of Proposition 6.1 we need the following lemma which contains a technical estimate for the solutions in $S \cap B(R)^c$.

LEMMA 6.4. *For any* $\alpha > 0$ *there is a* $C_\alpha > 0$ *such that*

$$\int_0^1 |u'||u|^\nu \rho^{-2} t^{-1}\, dt \leq C_\alpha M(u)^\alpha + C_\alpha \tag{6.10}$$

for $u \in S \cap B(R)^c$.

PROOF. Let $u \in S \cap B(R)^c$. From the definition of $\rho(t)$ it follows that

$$\rho(t) \geq |u(t)|^{(p+1)/2} \qquad \text{and} \quad \rho(t) \geq |u'(t)| \tag{6.11}$$

for all $t \in [0, 1]$. By Lemma 5.5 and (6.11) we have

$$|u'||u|^\nu \rho^{-2} t^{-1} \leq C|u(t)|^{\nu-1} \leq C\|u\|_\infty^{(p-1)/2}$$

for all $0 < t \leq L(u)$. This implies that

$$\int_0^{L(u)} |u'||u|^\nu \rho^{-2} t^{-1}\, dt \leq C\|u\|_\infty^{(p-1)/2} L(u) \leq C', \tag{6.12}$$

where C' is independent of u in $S \cap B(R)^c$.

On the other hand, using (6.11), we obtain

$$\int_{L(u)}^1 |u'||u|^\nu \rho^{-2} t^{-1}\, dt \leq \int_{L(u)}^1 t^{-1} dt \leq L(u)^{-\alpha} \int_{L(u)}^1 t^{-1+\alpha} dt$$
$$\leq \alpha^{-1} L(u)^{-\alpha} \qquad \text{for any} \quad \alpha > 0.$$

Moreover Lemma 5.8 implies

$$L(u)^{-\alpha} = C_\alpha \|u\|_\infty^{(p-1)\alpha/2} \leq C_\alpha M(u)^{(p-1)\alpha/2\sigma}$$

and hence

$$(6.13)\qquad \int_{L(u)}^{1} |u'||u|^{\nu}\rho^{-2}t^{-1}\,dt \le C_{\alpha}M(u)^{(p-1)\alpha/2\sigma}.$$

Consequently, combining (6.12) and (6.13) gives

$$\int_0^1 |u'||u|^{\nu}\rho^{-2}t^{-1}\,dt \le C_{\alpha}M(u)^{(p-1)\alpha/2\sigma} + C'$$

for any $\alpha > 0$. Rewriting $(p-1)\alpha/2\sigma$ as α, we get (6.10). The proof is complete

We now use the Prüfer transformations introduced in Definition 6.1 and a priori estimates given in Section 5 to complete the proof of Proposition 6.1.

PROOF OF PROPOSITION 6.1. We first observe the trivial inequality $\|u\|_{\infty} \ge |u(0)|$. If $|u(0, \gamma)| = |\gamma|$ is sufficiently large, the assertion (ii) of Lemma 5.6 states that $u(\cdot, \gamma)$ has at most a finite number of zeros in $[0, 1]$. That is, $N[u(\cdot, \gamma)] < +\infty$. We then demonstrate that $\lim_{\gamma\to\pm\infty} N[u(\cdot, \gamma)] = +\infty$. Let $u \in S \cap B(R)^c$. Integrating both sides of (6.4) over $[0, 1]$, we obtain

$$(6.14)\qquad \varphi(1) - \frac{\pi}{2} \ge I_1(u) + I_2(u) - I_3(u) - I_4(u),$$

where $I_i(u)$ $(1 \le i \le 4)$ are defined by

$$(6.15)\qquad I_1(u) \equiv \int_0^1 |u|^{\nu-1}ug(u)\rho^{-2}\,dt,$$

$$(6.16)\qquad I_2(u) \equiv \nu\int_0^1 |u|^{\nu-1}|u'|^2\rho^{-2}\,dt,$$

$$(6.17)\qquad I_3(u) \equiv (n-1)\int_0^1 |u'||u|^{\nu}\rho^{-2}t^{-1}\,dt,$$

$$(6.18)\qquad I_4(u) \equiv \int_0^1 |u|^{\nu}|h(t, u)|\rho^{-2}\,dt.$$

We then estimate the terms $I_i(u)$, $1 \le i \le 4$. We first compute a lower bound of $I_1(u) + I_2(u)$. From the definition of $\rho(t)$ we infer that

$$(6.19)\qquad I_1(u) + I_2(u) = \int_0^1 \rho^{-2}|u|^{\nu-1}(\nu|u'|^2 + ug(u))\,dt$$

$$\ge C\int_0^1 \rho^{-2}|u|^{\nu-1}(|u'|^2 + |u|^{p+1})\,dt \ge C\int_0^1 |u|^{(p-1)/2}\,dt.$$

To estimate the right-hand side, we choose a constant m such that

$$m > \max(n-1,\ n(p+3)/2(p+1)).$$

Then we have

$$|u(t)|^{(p+3)/2}\, t^m \leq (|u|^{p+1}\, t^n)^{(p+3)/2(p+1)} \leq M(u)^{(p+3)/2(p+1)},$$

and so

$$|u(t)|^{(p-1)/2} \geq M(u)^{-(p+3)/2(p+1)}\, |u(t)|^{p+1}\, t^m \tag{6.20}$$

for all $t \in [0, 1]$. By virtue of (6.19), (6.20) and Proposition 5.1 and the choice of m, we have

$$\begin{aligned} I_1(u) + I_2(u) &\geq CM(u)^{-(p+3)/2(p+1)} \int_0^1 |u|^{p+1}\, t^m\, dt \\ &\geq C_1 M(u)^{(p-1)/2(p+1)} - C_2 M(u)^{-(p+3)/2(p+1)}. \end{aligned}$$

Since $\|u\|_H^2 \leq CM(u) + C$ from Lemma 5.7, we may assume that $M(u) \geq 1$ for $u \in S \cap B(R)^c$. Accordingly, we obtain

$$I_1(u) + I_2(u) \geq C_1 M(u)^{(p-1)/2(p+1)} - C_2. \tag{6.21}$$

We have already estimated the value of $I_3(u)$ in Lemma 6.4. It remains only to compute the upper bound of $I_4(u)$. We infer from (5.4) and (6.11) that

$$I_4(u) \leq C \int_0^1 |u|^{\nu}(|u|^{(p+1)/2} + 1)\rho^{-2}\, dt \leq C + C \int_0^1 \rho^{-1}\, dt. \tag{6.22}$$

We wish to estimate the right-hand side. Since $u \in S \cap B(R)^c$, we get $\|u\|_\infty \geq R_3$ as mentioned in Remark 5.1. Hence it follows from Lemma 5.6 that

$$\begin{aligned} \rho(t)^2 = |u'|^2 + |u|^{p+1} &\geq C\left(\frac{1}{2}u'(t)^2 + G(u(t))\right) \\ &\geq C'\, \|u\|_\infty^\sigma \geq C' R_3^\sigma \qquad \text{for} \quad t \in [0, 1]. \end{aligned} \tag{6.23}$$

Combining (6.22) and (6.23), one finds a positive constant C independent of u such that

$$I_4(u) \leq C \qquad \text{for all} \quad u \in S \cap B(R)^c. \tag{6.24}$$

Using (6.14), (6.21), Lemma 6.4 and (6.24), we obtain

$$\varphi(1) - \frac{\pi}{2} \geq C_1 M(u)^{(p-1)/2(p+1)} - C_\alpha M(u)^\alpha - C_\alpha.$$

We choose α so small that $0 < \alpha < (p-1)/2(p+1)$. Then we get

$$\varphi(1) \geq CM(u)^{(p-1)/2(p+1)} - C'. \tag{6.25}$$

This, together with Lemma 5.8, yields

$$\varphi(1) \geq C_1 \|u\|_\infty^{(p-1)\sigma/2(p+1)} - C' \geq C_1 |u(0)|^{(p-1)\sigma/2(p+1)} - C' \tag{6.26}$$

for all $u \in S \cap B(R)^c$.

On the other hand, Lemmas 5.7 and 5.8 together imply

$$\|u\|_\infty^\sigma \leq C\|u\|_H^2 + C \qquad \text{for} \quad u \in S \cap B_\infty(R_3)^c. \tag{6.27}$$

By (6.27) and the trivial inequality $\|u\|_\infty \geq |u(0)|$, there is a constant $R' > 0$ such that if $u \in S$ and $|u(0)| \geq R'$ then $u \in S \cap B(R)^c$. From this fact and (6.26) we infer that

$$\varphi(1) \geq C_1 |u(0)|^{(p-1)\sigma/2(p+1)} - C' \tag{6.28}$$

holds provided that $u \in S$ and $|u(0)| \geq R'$.

Recall that $u(t, \gamma)$ denotes the solution of (2.1)–$(2.2)_\gamma$. We here define $\varphi(t, \gamma)$ by the function $\varphi(t)$ of Definition 6.1 with $u = u(t, \gamma)$. It follows from (6.28) that

$$\varphi(1, \gamma) \geq C_1 |\gamma|^{(p-1)\sigma/2(p+1)} - C' \qquad \text{for} \quad |\gamma| \geq R'. \tag{6.29}$$

This inequality and Lemma 6.3 together imply

$$\lim_{\gamma \to \pm\infty} N[u(\cdot, \gamma)] = +\infty,$$

and this completes the proof of Proposition 6.1.

We can choose a constant $\gamma_0 (\geq R')$ such that $u(t, \gamma)$ is well defined and $N[u(\cdot, \gamma)] < \infty$ for any γ with $|\gamma| \geq \gamma_0$. In the same way as in the proof of Lemma 4.4, we obtain the next lemma.

LEMMA 6.5. *The mapping $\gamma \to u(\cdot, \gamma)$ from $(-\infty, -\gamma_0] \cup [\gamma_0, +\infty)$ into $C^2[0, 1]$ is continuous.*

We now prove Theorems 2 and 4.

PROOF OF THEOREM 2. We prove Theorem 2 in the same way as in the proof of Theorem 1. Let $a^2 + b^2 \neq 0$. We first deal with the case where $b \neq 0$. The choice of γ_0 implies that $\varphi(1, \gamma) < \infty$ for γ with $|\gamma| \geq \gamma_0$. Choose a positive integer $k_0 \in N$ such that $\varphi(1, \gamma_0) < k_0 \pi$. As in the proof of Theorem 1, we apply the intermediate value theorem and define two sequences $(q_k)_{k \geq k_0}$ and $(p_k)_{k \geq k_0}$ by

$$q_k = \inf\{\gamma \in (\gamma_0, \infty) : \varphi(1, \gamma) = (k+1)\pi\} \tag{6.30}$$

and

$$(6.31) \qquad p_k = \sup\{\gamma \in (\gamma_0, q_k) : \varphi(1, \gamma) = k\pi\},$$

respectively. These sequences possess the following properties:

$$(6.32) \qquad \gamma_0 < p_k < q_k \le p_{k+1} < q_{k+1}, \qquad k \ge k_0,$$

$$(6.33) \qquad \lim_{k\to\infty} p_k = \lim_{k\to\infty} q_k = +\infty,$$

$$(6.34) \qquad \varphi(1, p_k) = k\pi, \qquad \varphi(1, q_k) = (k+1)\pi \qquad \text{and}$$

$$(6.35) \qquad k\pi < \varphi(1, \gamma) < (k+1)\pi \qquad \text{for} \quad \gamma \in (p_k, q_k).$$

We next define a continuous function $y(\gamma)$ on $[\gamma_0, \infty)$ by

$$y(\gamma) = au(1, \gamma) + bu'(1, \gamma).$$

It follows from Definition 6.1 and (6.34) that

$$y(p_k) = (-1)^k b\rho(1, p_k) \qquad \text{and} \quad y(q_k) = (-1)^{k+1} b\rho(1, q_k).$$

Since $b \ne 0$, we see that for any $k \ge k_0$,

$$y(p_k)\, y(q_k) = -b^2 \rho(1, p_k) \rho(1, q_k) < 0.$$

By the intermediate value theorem there is a $\gamma_k \in (p_k, q_k)$ such that $y(\gamma_k) = 0$. We then define $u_k^+(t) = u(t, \gamma_k)$ for $k \ge k_0$. These are the desired solutions. In fact, it follows from (6.27) that

$$C\,\|u(\cdot, \gamma_k)\|_H^2 + C \ge \|u(\cdot, \gamma_k)\|_\infty^\sigma \ge |\gamma_k|^\sigma \longrightarrow \infty \qquad \text{as} \quad k \longrightarrow \infty,$$

namely,

$$\lim_{k\to\infty} \|u_k^+\|_H = \lim_{k\to\infty} \|u_k^+\|_\infty = +\infty.$$

It is clear that $u_k^+(t)$ is a solution of (1.2)–(1.4) with exactly k zeros in $[0, 1]$ and satisfies $u_k^+(0) > 0$.

We next consider the case in which $b = 0$. In the same way as in the proof of Theorem 1 we find a sequence $(\gamma_k)_{k \ge k_0}$ such that $\lim_{k\to\infty} \gamma_k = +\infty$ and $\varphi(1, \gamma_k) = k\pi$ for $k \ge k_0$. In this case we define $u_k^+(t) = u(t, \gamma_k)$ for $k \ge k_0$. These are the desired solutions. We can also find (u_k^-) in the same way as above, and this completes the proof of Theorem 2.

Proof of Theorem 4. We first show the upper estimates for the H norms of solutions. Let $u \in S_k \cap B(R)^c$. Since $\varphi(1) \in [k\pi, (k+1)\pi)$ by Lemma 6.3, it follows from (6.25) that

$$(k+1)\pi \ge CM(u)^{(p-1)/2(p+1)} - C'.$$

This inequality, together with Lemma 5.7, implies

$$\|u\|_H \leq Ck^{(p+1)/(p-1)}.$$

We next treat the lower estimates for the H norms of solutions. Let $u \in S_k \cap B(R)^c$. Integrating the identity (6.4) over $[0, 1]$, we obtain

$$\varphi(1) - \frac{\pi}{2} \leq I_1(u) + I_2(u) + I_3(u) + I_4(u), \tag{6.36}$$

where the integrals $I_i(u)$, $1 \leq i \leq 4$, are defined by (6.15), (6.16), (6.17) and (6.18), respectively. We then estimate the terms $I_i(u)$, $1 \leq i \leq 4$, one by one. We first deal with $I_1(u)$. To this end, we set $\xi = n(p-1)/2(p+1)$. Note that $\xi < 1$ because of the assumption $1 < p < n^*$. Using the inequality (6.11), we have

$$\begin{aligned} I_1(u) &\leq C \int_0^1 |u|^{(p-1)/2}\, dt \\ &= C \int_0^1 (|u|^{p+1} t^n)^{(p-1)/2(p+1)} t^{-\xi}\, dt \leq CM(u)^{(p-1)/2(p+1)}. \end{aligned} \tag{6.37}$$

We next compute the term $I_2(u)$. By (6.11) and (6.37), we obtain

$$I_2(u) \leq \nu \int_0^1 |u|^{(p-1)/2}\, dt \leq CM(u)^{(p-1)/2(p+1)}. \tag{6.38}$$

We have already obtained the upper estimates for $I_3(u)$ and $I_4(u)$ in Lemma 6.4 and (6.24), respectively.

Combining (6.36), (6.37), (6.38), Lemma 6.4 with $\alpha = (p-1)/2(p+1)$ and (6.24), we obtain the estimate

$$\varphi(1) - \frac{\pi}{2} \leq CM(u)^{(p-1)/2(p+1)} + C.$$

Since $\varphi(1) \in [k\pi, (k+1)\pi)$ by Lemma 6.3, it follows from Lemma 5.7 that

$$k \leq C_1 \|u\|_H^{(p-1)/(p+1)} + C_2.$$

Since $u \in B(R)^c$, we may assume without loss of generality that $\|u\|_H \geq 1$, and so we get

$$k \leq (C_1 + C_2)\|u\|_H^{(p-1)/(p+1)}.$$

This completes the proof of Theorem 4.

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