HIROSHIMA MATH. J.
24 (1994), 15–24

Maximal conditions for locally finite Lie algebras and double chain conditions

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(Received April 8, 1992)

Introduction

A distinction between the minimal conditions for locally finite Lie algebras over a field of characteristic 0 has been exactly drawn by the author [5] and Stewart [7, 9]. However, very little is known concerning maximal conditions for Lie algebras. The first purpose of this paper is to distinguish between the maximal conditions for locally finite Lie algebras over a field of characteristic 0. The second one is to distinguish between various double chain conditions (i.e. maximal and minimal conditions) for Lie algebras.

In Section 2 we shall first prove that $\mathrm{L(wser)}\mathfrak{F} \cap \mathrm{Max}\text{-}\lhd = \mathfrak{F}$ over any field (Theorem 2.1), which is a generalization of [8, Theorem 6.5] and which suggests that under stronger conditions than local finiteness all the maximal conditions may be equivalent to each other. Secondly we shall prove that $\mathrm{L}\mathfrak{F} \cap \mathrm{Max}\text{-}\lhd^2 = \mathrm{L}\mathfrak{F} \cap \mathrm{Max}\text{-ser}$ over a field of characteristic 0 (Theorem 2.2) and shall exactly distinguish between the maximal conditions for locally finite Lie algebras over a field of characteristic 0 (Corollary 2.3).

In Section 3 we shall prove that $\mathrm{Max}\text{-}\lhd \cap \mathrm{Min}\text{-si} \leq \mathrm{Max}\text{-si}$ over any field (Theorem 3.2) and shall consequently draw a distinction between various double chain conditions for Lie algebras (Corollary 3.3). In addition, we shall prove that $\mathrm{Max}\text{-}\lhd^2 \leq \mathrm{Max}\text{-s}\mathfrak{F}$ over a field of characteristic 0 (Proposition 3.5), where $\mathrm{Max}\text{-s}\mathfrak{F}$ is the class of Lie algebras satisfying the maximal condition for finite-dimensional subideals.

In Section 4 we shall present a method of constructing Lie algebras satisfying neither the maximal condition nor the minimal condition for 2-step subideals and shall consequently prove that there exists a Lie algebra L over any field such that $L \in \mathrm{L}\mathfrak{F} \cap \mathrm{Max}\text{-}\lhd \cap \mathrm{Min}\text{-}\lhd$ and $L \notin \mathrm{Max}\text{-}\lhd^2 \cup \mathrm{Min}\text{-}\lhd^2$ (Theorem 4.6(1)).

1

Throughout this paper we are always concerned with Lie algebras which are not necessarily finite-dimensional over an arbitrary field $\mathfrak{k}$ unless otherwise specified. Notation and terminology is mainly based on [2]. In this section we explain some symbols and terms which we use here.

Let L be a Lie algebra over a field $\mathfrak{k}$, n a non-negative integer and α an ordinal. The symbol $H \leq L$ (resp. $H \lhd L$, $H \lhd^n L$, $H \operatorname{si} L$, $H \lhd^\alpha L$, $H \operatorname{asc} L$) denotes that H is a subalgebra (resp. an ideal, an n-step subideal, a subideal, an α-step ascendant subalgebra, an ascendant subalgebra) of L. Angular brackets $\langle\ \rangle$ denote the subalgebra generated by their contents. A subalgebra H of L is said to be a serial subalgebra (resp. a weakly serial subalgebra) of L, which is denoted by $H \operatorname{ser} L$ (resp. $H \operatorname{wser} L$), if there exist a totally ordered set Σ and a family $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ of subalgebras (resp. subspaces) of L such that

(a) $H \subseteq V_\sigma \subseteq \Lambda_\sigma$ for all $\sigma \in \Sigma$,

(b) $\Lambda_\sigma \subseteq V_\tau$ if $\sigma < \tau$,

(c) $L \backslash H = \bigcup_{\sigma \in \Sigma} (\Lambda_\sigma \backslash V_\sigma)$,

(d) $V_\sigma \lhd \Lambda_\sigma$ (resp. $[\Lambda_\sigma, H] \subseteq V_\sigma$) for all $\sigma \in \Sigma$.

The transfinite derived series (resp. the transfinite lower central series, the transfinite upper central series) of L is denoted by $\{L^{(\alpha)} : \alpha \geqq 0\}$ (resp. $\{L^\alpha : \alpha \geqq 1\}$, $\{\zeta_\alpha(L) : \alpha \geqq 0\}$). The intersection of all the terms of the transfinite derived series (resp. the transfinite lower central series) of L is denoted by $L^{(*)}$ (resp. L^*).

A class $\mathfrak{X}$ is a collection of Lie algebras together with their isomorphic copies and 0-dimensional Lie algebras. Let $\mathfrak{X}$ be a class of Lie algebras. A subalgebra H of L is called an $\mathfrak{X}$-subalgebra if $H \in \mathfrak{X}$. The symbol $\mathfrak{A}$ (resp. $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{A}^n$, E$\mathfrak{A}$, $\mathfrak{N}_n$, $\mathfrak{N}$, RE$\mathfrak{A}$, R$\mathfrak{N}$) denotes the class of Lie algebras which are abelian (resp. finite-dimensional, finitely generated, soluble of derived length $\leqq n$, soluble, nilpotent of class $\leqq n$, nilpotent, residually soluble, residually nilpotent).

The symbol Max-$\lhd$ (resp. Max-$\lhd^n$, Max-si, Max-$\lhd^\alpha$, Max-asc, Max-ser, Max) denotes the class of Lie algebras satisfying the maximal condition for ideals (resp. n-step subideals, subideals, α-step ascendant subalgebras, ascendant subalgebras, serial subalgebras, subalgebras). The classes Min-$\lhd$, Min-$\lhd^n$, Min-si, Min-$\lhd^\alpha$, Min-asc, Min-ser and Min of Lie algebras satisfying the minimal conditions are defined in the same way.

Moreover, we here use the following classes of Lie algebras.

$L \in \mathfrak{A}\mathfrak{F}$ if L has an abelian ideal of finite codimension.

$L \in \mathfrak{D}(\text{asc}, \text{si})$ (resp. $\mathfrak{D}(\text{ser}, \text{si})$) if every ascendant (resp. serial) subalgebra of L is a subideal of L.

$L \in \mathfrak{L}^\infty$ (resp. $\mathfrak{L}_\infty$) if the join (resp. the intersection) of any family of subideals of L is always a subideal of L. In particular, if $L \in \mathfrak{L}^\infty \cap \mathfrak{L}_\infty$ then L has the complete lattice of subideals.

$L \in \mathfrak{S}$ (resp. $\mathfrak{S}^*$) if $H \lhd L$ (resp. $H \operatorname{ser} L$) always implies that either $H = 0$

or $H = L$. In this case L is said to be simple (resp. absolutely simple).

$L \in \mathfrak{Z}_\alpha$ if $L = \zeta_\alpha(L)$.

$L \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$ (resp. $\acute{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$) if $L^{(*)} = 0$ (resp. $L^* = 0$).

$L \in \mathrm{L}\mathfrak{X}$ if every finite subset of L is contained in some $\mathfrak{X}$-subalgebra of L. In particular, Lie algebras belonging to the class $\mathrm{L}\mathfrak{F}$ (resp. $\mathrm{LE}\mathfrak{A}$, $\mathrm{L}\mathfrak{N}$) are said to be locally finite (resp. locally soluble, locally nilpotent) Lie algebras.

$L \in \mathrm{L(wser)}\mathfrak{F}$ if every finite subset of L is contained in some finite-dimensional weakly serial subalgebra of L.

$L \in \acute{\mathrm{N}}\mathfrak{F}$ if L is generated by finite-dimensional ascendant subalgebras.

2

In this section we confine our attention to locally finite Lie algebras in order to investigate the relation between various maximal conditions.

The classes Max-$\lhd^\alpha$ (α is an ordinal), Max-si, Max-asc, Max-ser, Max and $\mathfrak{F}$ are related by the series of inclusions

$$\text{Max-}\lhd \geq \text{Max-}\lhd^2 \geq \cdots \geq \text{Max-si} \geq \text{Max-}\lhd^\omega \geq \text{Max-}\lhd^{\omega+1}$$

$$\geq \cdots \geq \text{Max-asc} \geq \text{Max-ser} \geq \text{Max} \geq \mathfrak{F}.$$

Stewart [8, Theorem 6.5] has proved that over a field $\mathfrak{k}$ of characteristic 0

$$\acute{\mathrm{N}}\mathfrak{F} \cap \text{Max-}\lhd = \mathfrak{F},$$

which implies that for any $\acute{\mathrm{N}}\mathfrak{F}$-algebra over a field $\mathfrak{k}$ of characteristic 0 all the maximal conditions are equivalent to each other. It is well known ([2, p. 258]) that the class $\acute{\mathrm{N}}\mathfrak{F}$ is a proper subclass of the class of neoclassical Lie algebras over a field $\mathfrak{k}$ of characteristic 0. It follows from [11, Theorem 2a)] that over a field $\mathfrak{k}$ of characteristic 0,

$$\acute{\mathrm{N}}\mathfrak{F} < \mathrm{L(wser)}\mathfrak{F}.$$

Therefore the following theorem is a generalization of [8, Theorem 6.5].

THEOREM 2.1. *Over any field* $\mathfrak{k}$, $\mathrm{L(wser)}\mathfrak{F} \cap \text{Max-}\lhd = \mathfrak{F}$.

PROOF. Let $L \in \mathrm{L(wser)}\mathfrak{F} \cap \text{Max-}\lhd$ and let H be a finite-dimensional weakly serial subalgebra of L. We denote by $\lambda_{\mathfrak{N}}(H)$ the intersection of the ideals I of H for which H/I is nilpotent. Using [3, Proposition 2.11] we get $\lambda_{\mathfrak{N}}(H) \lhd L$. Put $K = \sum \lambda_{\mathfrak{N}}(H)$, the summation being over all finite-dimensional weakly serial subalgebras H of L. If $K \notin \mathfrak{F}$, then we can inductively construct a strictly ascending chain $\lambda_{\mathfrak{N}}(H_1) < \lambda_{\mathfrak{N}}(H_1) + \lambda_{\mathfrak{N}}(H_2) < \cdots$ such that $H_k \in \mathfrak{F}$ and $H_k \,\mathrm{wser}\, L$ for all $k \geqq 1$. This is a contradiction. Therefore we have $K \in \mathfrak{F}$. Since $L \in \mathrm{L}(\mathrm{wser})\mathfrak{F}$, we get $L/K \in \mathrm{L}\mathfrak{N}$. It follows from [2, Theorem 8.6.5] that $L/K \in \mathrm{L}\mathfrak{N} \cap \mathrm{Max}\text{-}\lhd \leq \mathfrak{F}$. Thus we obtain $L \in \mathfrak{F}$.

By virtue of the above theorem combined with the use of Theorem 4.6 (1), which will be proved in Section 4, we can see that $\mathrm{L}(\mathrm{wser})\mathfrak{F} < \mathrm{L}\mathfrak{F}$ over any field $\mathfrak{k}$.

We now restrict our attention to locally finite Lie algebras. Then we have the following result, which is the main theorem of this section.

THEOREM 2.2. *Over a field* $\mathfrak{k}$ *of characteristic* 0,

$$\mathrm{L}\mathfrak{F} \cap \mathrm{Max}\text{-}\lhd^2 = \mathrm{L}\mathfrak{F} \cap \mathrm{Max}\text{-ser}\,.$$

PROOF. Let $L \in \mathrm{L}\mathfrak{F} \cap \mathrm{Max}\text{-}\lhd^2$ and let $H \operatorname{ser} L$. We denote by $\lambda(H)$ the intersection of ideals I of H for which H/I is locally nilpotent. Then we know by [10, Theorem 5 and Corollary 6] that $\lambda(H) \lhd L$ and $H/\lambda(H) \leq \rho(L/\lambda(H))$, where $\rho(L/\lambda(H))$ is the Hirsch-Plotkin radical of $L/\lambda(H)$. Owing to [2, Theorem 8.6.5], we have $\rho(L/\lambda(H)) \in \mathrm{L}\mathfrak{N} \cap \mathrm{Max}\text{-}\lhd \leq \mathfrak{F}$. Now we want to claim that $L \in \mathrm{Max}\text{-ser}$. Assume, to the contrary, that there exists a strictly ascending chain of serial subalgebras of L, say, $H_1 < H_2 < \cdots$. Since $L \in \mathrm{L}\mathfrak{F}$, we can easily see that $H_k/\lambda(H_k) \in \mathrm{L}\mathfrak{N}$ for all $k \geqq 1$. It follows that $H_{k-1}/H_{k-1} \cap \lambda(H_k) \in \mathrm{L}\mathfrak{N}$ for all $k \geqq 2$. Hence $\lambda(H_1) \leq \lambda(H_2) \leq \cdots$ and therefore $\lambda(H_n) = \lambda(H_{n+1}) = \cdots$ for some $n \geqq 1$. Put $K = \lambda(H_n)$. Then $H_n/K < H_{n+1}/K < \cdots$ and $H_k/K \leq \rho(L/K) \in \mathfrak{F}$ for all $k \geqq n$. This is a contradiction.

It will be proved in Theorem 4.6(1) below that over any field $\mathfrak{k}$,

$$\mathrm{L}\mathfrak{F} \cap \mathrm{Max}\text{-}\lhd > \mathrm{L}\mathfrak{F} \cap \mathrm{Max}\text{-}\lhd^2\,.$$

Moreover, considering the Lie algebra constructed in the proof of [5, Theorem 1.7(2)], we can see that over any field $\mathfrak{k}$,

$$\mathrm{L}\mathfrak{F} \cap \mathrm{Max}\text{-ser} > \mathrm{L}\mathfrak{F} \cap \mathrm{Max} = \mathfrak{F} \qquad \text{(in particular, Max-ser} > \text{Max)}\,.$$

Combining Theorem 2.2 and the above results, we consequently obtain the following corollary, which exactly draws a distinction between the maximal conditions for locally finite Lie algebras over a field $\mathfrak{k}$ of characteristic 0.

COROLLARY 2.3. *Over a field* $\mathfrak{k}$ *of characteristic* 0,

$$\begin{aligned}\mathrm{L}\mathfrak{F} \cap \mathrm{Max}\text{-}\lhd &> \mathrm{L}\mathfrak{F} \cap \mathrm{Max}\text{-}\lhd^2 = \mathrm{L}\mathfrak{F} \cap \mathrm{Max}\text{-}\lhd^3 = \cdots = \mathrm{L}\mathfrak{F} \cap \mathrm{Max}\text{-si}\\ &= \mathrm{L}\mathfrak{F} \cap \mathrm{Max}\text{-}\lhd^{\omega} = \mathrm{L}\mathfrak{F} \cap \mathrm{Max}\text{-}\lhd^{\omega+1} = \cdots = \mathrm{L}\mathfrak{F} \cap \mathrm{Max}\text{-asc}\\ &= \mathrm{L}\mathfrak{F} \cap \mathrm{Max}\text{-ser} > \mathrm{L}\mathfrak{F} \cap \mathrm{Max} = \mathfrak{F}\,.\end{aligned}$$

REMARK. In the case where the ground field $\mathfrak{k}$ is of characteristic $p > 0$, it is well known that

$$\mathrm{L}\mathfrak{F} \cap \text{Max-}\lhd^2 > \mathrm{L}\mathfrak{F} \cap \text{Max-}\lhd^3 \qquad (\text{in particular, } \text{Max-}\lhd^2 > \text{Max-}\lhd^3).$$

For example, in [1, §5] Amayo and Stewart have constructed a locally finite Lie algebra over $\mathfrak{k}$ whose 2-step subideals are only finite in number and which has an infinite-dimensional, abelian 2-step subideal.

A Lie algebra belonging to the class $\mathfrak{L}^{\infty} \cap \mathfrak{L}_{\infty}$ has the complete lattice of subideals. Some subclasses of $\mathfrak{L}^{\infty} \cap \mathfrak{L}_{\infty}$ have been given in [4, §3]. As another corollary to Theorem 2.2, we can furthermore present a subclass of $\mathfrak{L}^{\infty} \cap \mathfrak{L}_{\infty}$ in the following

COROLLARY 2.4. *Over a field* $\mathfrak{k}$ *of characteristic* 0,

$$\mathrm{L}\mathfrak{F} \cap \text{Max-}\lhd^2 \leq \mathfrak{L}^{\infty} \cap \mathfrak{L}_{\infty}.$$

PROOF. It is not hard to verify that Max-ser $\leq \mathfrak{D}$(ser, si). Hence, owing to [6, Theorem 2.3], we get Max-ser $\leq \mathfrak{L}_{\infty}$. Furthermore, we know by [4, Theorem 8] that Max-si $\leq \mathfrak{L}^{\infty}$. Therefore the assertion immediately follows from Theorem 2.2.

Finally we present the structure of Lie algebras belonging to the class $\mathrm{L}\mathfrak{F} \cap \text{Max-}\lhd^2$ in the following

PROPOSITION 2.5. *Let L be a Lie algebra over a field $\mathfrak{k}$ of characteristic* 0. *If $L \in \mathrm{L}\mathfrak{F} \cap \text{Max-}\lhd^2$, then L has a descending series of subideals whose factors are absolutely simple.*

PROOF. Let $L \in \mathrm{L}\mathfrak{F} \cap \text{Max-}\lhd^2$. Using Theorem 2.2 we get $L \in \mathfrak{D}(\text{ser}, \text{si})$. We recursively define the terms of a descending series $\{L_\alpha : \alpha \geqq 0\}$ of subideals of L whose factors are absolutely simple. Define L_0 by L and suppose that the terms L_β $(\beta < \alpha)$ have been defined for some ordinal $\alpha > 0$. If α is a limit ordinal, then we may define L_α by $\bigcap_{\beta<\alpha} L_\beta$. Assume that α is not a limit ordinal and that $L_{\alpha-1} \neq 0$. Clearly $L_{\alpha-1}$ has a proper maximal ideal, say, M. Then by [10, Theorem 8] we get $L_{\alpha-1}/M \in \mathrm{L}\mathfrak{F} \cap \mathfrak{S} \leq \mathfrak{S}^*$. Hence we may define L_α by M. Now by set-theoretical considerations we can find an ordinal σ such that $L_\sigma = 0$. This completes the proof.

REMARK. Let $\grave{\mathrm{E}}(\text{si})\mathfrak{S}^*$ denote the class of Lie algebras which have descending series of subideals with absolutely simple factors. Since infinite-dimensional, abelian Lie algebras lie in $\grave{\mathrm{E}}(\text{si})\mathfrak{S}^*$, we conclude from Proposition 2.5 that $\mathrm{L}\mathfrak{F} \cap \text{Max-}\lhd^2 < \mathrm{L}\mathfrak{F} \cap \grave{\mathrm{E}}(\text{si})\mathfrak{S}^*$ over a field $\mathfrak{k}$ of characteristic 0.

3

Let Δ_1 and Δ_2 be any two of the relations $\leq$, $\lhd^\alpha$ (α is an ordinal), si, asc, ser. Then we say that a Lie algebra L satisfies the double chain condition

Max-Δ_1 and Min-Δ_2 if L belongs to the class Max-$\Delta_1 \cap$ Min-Δ_2. In this section we shall investigate the relation between various double chain conditions.

Let us recall the class Max-s$\mathfrak{F}$ of Lie algebras satisfying the maximal condition for finite-dimensional subideals. We here need the following lemma, which is directly deduced from [2, Corollary 9.3.3(b)].

LEMMA 3.1. *Over any field* $\mathfrak{k}$, Min-si $\leq$ Max-s$\mathfrak{F}$.

The main theorem of this section is the following

THEOREM 3.2. *Over any field* $\mathfrak{k}$, Max-$\lhd \cap$ Min-si $\leq$ Max-si.

PROOF. Let $L \in$ Max-$\lhd \cap$ Min-si and let H si L. Since $H^{(*)}$ is a perfect subideal of L, it is well known (see [2, Proposition 1.3.5]) that $H^{(*)} \lhd L$. Since $L \in$ Min-si, we can easily verify that $H/H^{(*)} \in \mathrm{E}\mathfrak{A} \cap$ Min-si $\leq \mathfrak{F}$. Assume that L has a strictly ascending chain of subideals, say, $H_1 < H_2 < \cdots$. Then there exists an integer $n \geqq 1$ such that $H_n^{(*)} = H_{n+1}^{(*)} = \cdots$. Put $K = H_n^{(*)}$. We get a strictly ascending chain $H_n/K < H_{n+1}/K < \cdots$ of finite-dimensional subideals of L/K. However, Lemma 3.1 implies that $L/K \in$ Min-si $\leq$ Max-s$\mathfrak{F}$. This is a contradiction. Thus we obtain $L \in$ Max-si.

In the case where the ground field $\mathfrak{k}$ is of characteristic 0, it is well known ([2, Theorem 8.1.4] and [9, Theorem]) that

$$\text{Min-}\lhd^2 = \text{Min-si} = \text{Min-asc} \leq \mathfrak{D}(\text{asc, si}).$$

In the case where the ground field $\mathfrak{k}$ is of characteristic $p > 0$, it is well known ([2, Proposition 8.1.5] and [9, Theorem]) that

$$\text{Min-}\lhd^3 = \text{Min-si} = \text{Min-asc} \leq \mathfrak{D}(\text{asc, si}).$$

Furthermore, the Lie algebra constructed in [1, § 5] satisfies neither the maximal condition nor the minimal condition for 3-step subideals but satisfies the double chain condition Max-$\lhd^2$ and Min-$\lhd^2$.

In the case where the ground field $\mathfrak{k}$ is of arbitrary characteristic, the Lie algebra constructed in the proof of Theorem 4.6(1) below satisfies neither the maximal condition nor the minimal condition for 2-step subideals but satisfies the double chain condition Max-$\lhd$ and Min-$\lhd$. Furthermore, the Lie algebra constructed in the proof of [5, Theorem 1.7(2)] satisfies neither the maximal condition nor the minimal condition for subalgebras but satisfies the double chain condition Max-ser and Min-ser.

As a direct consequence of Theorem 3.2 and the above results we obtain the following corollary, which draws a distinction between various double chain conditions for Lie algebras.

COROLLARY 3.3. (1) *Over a field* $\mathfrak{k}$ *of characteristic* 0,

$$\text{Max-}\lhd \cap \text{Min-}\lhd > \text{Max-}\lhd \cap \text{Min-}\lhd^2 = \text{Max-}\lhd^2 \cap \text{Min-}\lhd^2 = \cdots$$

$$= \text{Max-si} \cap \text{Min-}\lhd^2 = \text{Max-}\lhd^{\omega} \cap \text{Min-}\lhd^2 = \text{Max-}\lhd^{\omega+1} \cap \text{Min-}\lhd^2$$

$$= \cdots = \text{Max-asc} \cap \text{Min-}\lhd^2 > \text{Max} \cap \text{Min-}\lhd^2 .$$

(2) *Over a field* $\mathfrak{k}$ *of characteristic* $p > 0$,

$$\text{Max-}\lhd \cap \text{Min-}\lhd > \text{Max-}\lhd \cap \text{Min-}\lhd^2 > \text{Max-}\lhd \cap \text{Min-}\lhd^3$$

$$= \text{Max-}\lhd^2 \cap \text{Min-}\lhd^3 = \cdots = \text{Max-si} \cap \text{Min-}\lhd^3 = \text{Max-}\lhd^{\omega} \cap \text{Min-}\lhd^3$$

$$= \text{Max-}\lhd^{\omega+1} \cap \text{Min-}\lhd^3 = \cdots = \text{Max-asc} \cap \text{Min-}\lhd^3 > \text{Max} \cap \text{Min-}\lhd^3 .$$

It seems to be a hard problem whether we can precisely add the class Max-ser $\cap$ Min-$\lhd^2$ (resp. Max-ser $\cap$ Min-$\lhd^3$) to the series of inclusions given in Corollary 3.3(1) (resp. (2)).

Finally we prove the following result, which claims that if the ground field $\mathfrak{k}$ is of characteristic 0, then in Lemma 3.1 we can replace the class Min-si ($=$Min-$\lhd^2$) with the class Max-$\lhd^2$.

PROPOSITION 3.4. *Over a field* $\mathfrak{k}$ *of characteristic* 0, Max-$\lhd^2 \leq$ Max-s$\mathfrak{F}$.

PROOF. Let $L \in$ Max-$\lhd^2$ and let H be a finite-dimensional subideal of L. Then $H^{\omega} \lhd L$ and $H/H^{\omega} \leq \beta(L/H^{\omega})$, where $\beta(L/H^{\omega})$ is the Baer radical of L/H^{ω}. By using [2, Theorems 6.2.1, 6.3.3 and 8.6.5], we get $\beta(L/H^{\omega}) \in \text{L}\mathfrak{N} \cap$ Max-$\lhd \leq \mathfrak{F}$. Then, as in the proof of Theorem 2.2, we can prove that L has no strictly ascending chains of finite-dimensional subideals.

REMARK. In the case where the ground field $\mathfrak{k}$ is of characteristic 0, it is not known whether the class Max-$\lhd^2$ coincides with the class Max-si. We should note that the class Max-si is a proper subclass of Max-s$\mathfrak{F}$. In fact, let L be a direct sum of infinitely many infinite-dimensional, simple Lie algebras. Since L has no non-trivial finite-dimensional subideals, we have $L \in$ Max-s$\mathfrak{F}$. However, it is clear that $L \notin$ Max-si.

4

In this section we shall introduce a method of constructing locally finite Lie algebras which do not satisfy the maximal condition for 2-step subideals but satisfy the maximal condition for ideals. No such Lie algebras may be found in well-known examples, because of the following

PROPOSITION 4.1 ([2, Lemma 8.6.1, Theorem 8.6.5 and Corollary 11.1.8]). *Let* $L \in \mathrm{L}\mathfrak{F} \cap \mathrm{Max}\text{-}\triangleleft$. *If* $L \in \mathrm{E}\mathfrak{A} \cup \mathrm{L}\mathfrak{N} \cup \mathfrak{A}\mathfrak{F}$, *then* $L \in \mathfrak{F}$.

Let L be a Lie algebra over a field $\mathfrak{k}$ and let ε be the identity map of L. For each $x \in L$ we denote by x^ε the image under the identity map ε. Think of ε as a $\mathfrak{k}$-linear transformation of L. Then we can define an L-action on the image L^ε as follows:

$$x^\varepsilon y = [x, y]^\varepsilon \qquad (x, y \in L).$$

Regard the L-module L^ε as an abelian Lie algebra and denote by $L^\sim$ the split extension of L^ε by L. Then we obtain the following lemma, the proof of which is a nice exercise in the use of the definition of $L^\sim$.

LEMMA 4.2. (1) *If* $H \triangleleft L^\sim$, *then* $H \cap L^\varepsilon = I^\varepsilon$ *for some* $I \triangleleft L$.
(2) $(L^\sim)^\alpha = (L^\alpha)^\varepsilon + L^\alpha$ *for all ordinals* $\alpha \geqq 1$.
(3) $(L^\sim)^{(\alpha)} = (L^{(\alpha)})^\varepsilon + L^{(\alpha)}$ *for all ordinals* α.
(4) $\zeta_\alpha(L^\sim) = \zeta_\alpha(L)^\varepsilon + \zeta_\alpha(L)$ *for all ordinals* α.

We can show that several structures of Lie algebras are hereditary under the $\sim$-formation.

PROPOSITION 4.3. *Let* $\mathfrak{X}$ *be one of the following classes of Lie algebras*:

$$\mathfrak{F},\ \mathfrak{G},\ \mathrm{Max}\text{-}\triangleleft,\ \mathrm{Min}\text{-}\triangleleft,\ \mathfrak{N}_c,\ \mathfrak{A}^d,\ \mathfrak{Z}_\alpha,\ \mathrm{R}\mathfrak{N},\ \mathrm{RE}\mathfrak{A},\ \dot{\mathrm{E}}(\triangleleft)\hat{\mathfrak{A}},\ \dot{\mathrm{E}}(\triangleleft)\mathfrak{A},$$

where c *and* d *are non-negative integers and* α *is an ordinal. If* $L \in \mathfrak{X}$, *then* $L^\sim \in \mathfrak{X}$.

PROOF. In the case $\mathfrak{X} = \mathfrak{F}$ the assertion clearly holds. If $L = \langle x_1, \dots, x_n \rangle$, then $L^\sim = \langle x_1^\varepsilon, \dots, x_n^\varepsilon, x_1, \dots, x_n \rangle$. Hence in the case $\mathfrak{X} = \mathfrak{G}$ the assertion holds. We now consider the case $\mathfrak{X} = \mathrm{Max}\text{-}\triangleleft$ (resp. $\mathrm{Min}\text{-}\triangleleft$). Let $L \in \mathrm{Max}\text{-}\triangleleft$ (resp. $\mathrm{Min}\text{-}\triangleleft$). Since $L^\sim/L^\varepsilon \cong L$, $\{(H + L^\varepsilon)/L^\varepsilon : H \triangleleft L^\sim\}$ satisfies the maximal (resp. minimal) condition. We know by Lemma 4.2(1) that $\{H \cap L^\varepsilon : H \triangleleft L^\sim\}$ satisfies the maximal (resp. minimal) condition. Therefore by [2, Theorem 1.7.3] we get $L^\sim \in \mathrm{Max}\text{-}\triangleleft$ (resp. $\mathrm{Min}\text{-}\triangleleft$). In the other cases the assertion is directly deduced from Lemma 4.2.

LEMMA 4.4. *Let* $\mathfrak{X}$ *be a class of Lie algebras such that* $L \in \mathfrak{X}$ *always implies* $L^\sim \in \mathfrak{X}$. *If* $L \in \mathrm{L}\mathfrak{X}$, *then* $L^\sim \in \mathrm{L}\mathfrak{X}$.

PROOF. Let $L \in \mathrm{L}\mathfrak{X}$ and let X be a finite subset of $L^\sim$. Take elements x_i, y_i $(1 \leqq i \leqq n)$ such that $X = \{x_i^\varepsilon + y_i : 1 \leqq i \leqq n\}$. Then there exists an

$\mathfrak{X}$-subalgebra H of L such that $\{x_i, y_i : 1 \leqq i \leqq n\} \subseteq H$. Since $X \subseteq H^{\varepsilon} + H \cong H^{\sim} \in \mathfrak{X}$, we get $L^{\sim} \in \mathrm{L}\mathfrak{X}$.

By combining Proposition 4.3 and Lemma 4.4, we can immediately obtain the following result.

PROPOSITION 4.5. *Let $\mathfrak{X}$ be one of the following classes of Lie algebras*:

$$\mathrm{L}\mathfrak{F},\ \mathrm{L}\mathfrak{N},\ \mathrm{LE}\mathfrak{A},\ \mathrm{LR}\mathfrak{N},\ \mathrm{LRE}\mathfrak{A},\ \mathrm{L\acute{E}}(\lhd)\hat{\mathfrak{A}},\ \mathrm{L\grave{E}}(\lhd)\mathfrak{A}\,.$$

If $L \in \mathfrak{X}$, then $L^{\sim} \in \mathfrak{X}$.

Let L be an infinite-dimensional Lie algebra. Then L^{ε} is an infinite-dimensional, abelian ideal of $L^{\sim}$. It follows that $L^{\sim}$ satisfies neither the maximal condition nor the minimal condition for 2-step subideals. Consequently we arrive at the main theorem of this section.

THEOREM 4.6. (1) *Over any field $\mathfrak{k}$, there exists a Lie algebra L such that*

$$L \in \mathrm{L}\mathfrak{F} \cap \text{Max-}\lhd \cap \text{Min-}\lhd \quad \textit{and} \quad L \notin \text{Max-}\lhd^2 \cup \text{Min-}\lhd^2\,.$$

(2) *Over a field $\mathfrak{k}$ of characteristic* 0, *there exists a Lie algebra L such that*

$$L \in \mathfrak{G} \cap \text{Max-}\lhd \cap \text{Min-}\lhd \quad \textit{and} \quad L \notin \text{Max-}\lhd^2 \cup \text{Min-}\lhd^2\,.$$

(3) *Over a field $\mathfrak{k}$ of characteristic* 0, *there exists a Lie algebra L such that*

$$L \in \mathfrak{G} \cap \mathrm{R}\mathfrak{N} \cap \text{Max-}\lhd \quad \textit{and} \quad L \notin \text{Max-}\lhd^2 \cup \text{Min-}\lhd\,.$$

PROOF. (1) Let L be an infinite-dimensional, locally finite, simple Lie algebra over a field $\mathfrak{k}$. For example, the Lie algebra of all trace-zero transformations of finite rank of an infinite-dimensional vector space over $\mathfrak{k}$ (see [7, §4]) or the Lie algebra over $\mathfrak{k}$ constructed in the proof of [5, Theorem 1.7 (2)] is such a Lie algebra. Then we know by Proposition 4.3 and 4.5 that $L^{\sim} \in \mathrm{L}\mathfrak{F} \cap \text{Max-}\lhd \cap \text{Min-}\lhd$. Since $L \notin \mathfrak{F}$, we get $L^{\sim} \notin \text{Max-}\lhd^2 \cup \text{Min-}\lhd^2$.

(2) Let W be a generalized Witt algebra $\mathscr{W}_{\boldsymbol{Z}}$ (see [2, § 10.3]), that is, let W be a Lie algebra over $\mathfrak{k}$ with basis $\{w_i : i \in \boldsymbol{Z}\}$ and multiplication

$$[w_i, w_j] = (i - j)w_{i+j} \qquad (i, j \in \boldsymbol{Z})\,.$$

Then we know by [2, Theorem 10.3.1] that $W \in \mathfrak{S} \leq \text{Max-}\lhd \cap \text{Min-}\lhd$. Clearly we have $W = \langle w_{-2}, w_1, w_2 \rangle \in \mathfrak{G}$. It follows from Proposition 4.3 that $W^{\sim} \in \mathfrak{G} \cap \text{Max-}\lhd \cap \text{Min-}\lhd$.

(3) Let W be a generalized Witt algebra $\mathscr{W}_{\boldsymbol{Z}}$ as in (2) and L the subalgebra of W generated by $\{w_1, w_2\}$. Then we know by [2, Theorem 8.7.1] that $L > L^2 > \cdots$ and $L \in \mathrm{R}\mathfrak{N} \cap \text{Max-si}$. It follows from Lemma 4.2(2) and Proposition 4.3 that $L^{\sim} > (L^{\sim})^2 > \cdots$ and $L^{\sim} \in \mathfrak{G} \cap \mathrm{R}\mathfrak{N} \cap \text{Max-}\lhd$.

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