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Universal R-matrices and the center of the quantum generalized Kac-Moody algebras

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ABSTRACT. We extend the result in [13] to those for the quantization of generalized Kac-Moody algebras introduced in [10]. The existence of the universal R-matrix is proved, and a structure theorem for the center is given.

0. Introduction

The quantum groups—more precisely, the quantization of the universal enveloping algebras of Kac-Moody algebras—were independently introduced by Drinfel'd ([6]) and Jimbo ([7]) through their investigation of R-matrices which are the solutions to the Yang-Baxter equation. Its importance partly comes from the fact that there exists a solution to the Yang-Baxter equation inside the quantum group, called the *universal R-matrix*, so that one can obtain various R-matrices as its specialization on the representations of the quantum group.

On the other hand, the notion of Kac-Moody algebras was generalized to the so-called *generalized Kac-Moody algebras* ([1]), and it was used crucially in Borcherds' proof of the moonshine conjecture ([2]). In [10], the first-named author extended the quantum groups to those for the generalized Kac-Moody algebras, and proved some fundamental results on their structures and their representations.

In this paper, we continue the investigation by extending the results in [13] to the quantum groups of generalized Kac-Moody algebras. In the first half of this paper, we construct an analogue of the Killing form and prove the existence of the universal R-matrix. The proofs are very similar to those in [13] and the analogue of the Killing form plays a crucial role. In the second half, we investigate the structure of the center of the quantum groups for generalized Kac-Moody algebras. The case of quantized universal en-

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veloping algebras of ordinary Kac-Moody algebras was already treated in [4], [8], [13]. Hence we restrict ourselves to the non-ordinary case. We show that the center consists only of certain obvious elements in almost all cases. The proof is based on the reduction to the small rank cases.

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1. The Quantum Algebra $U_q(\mathfrak{g})$

Let $\mathbf{F}$ be a field of characteristic 0 and let $q \in \mathbf{F}$ be transcendental over the prime subfield $\mathbf{Q}$. We assume that $\mathbf{F}$ contains an n-th root of q for any positive integer n.

Let I be a countable (possibly infinite) index set and let $A = (a_{ij})_{i,j \in I}$ be a Borcherds-Cartan matrix with $a_{ij} \in \mathbf{Q}$ for all $i, j \in I$. That is, $A = (a_{ij})_{i,j \in I}$ is a rational square matrix satisfying (i) $a_{ii} = 2$ or $a_{ii} \leq 0$ for all $i \in I$, (ii) $a_{ij} \leq 0$ for $i \neq j$ and $a_{ij} \in \mathbf{Z}$ if $a_{ii} = 2$, (iii) $a_{ij} = 0$ implies $a_{ji} = 0$. Let $I^{re} = \{i \in I | a_{ii} = 2\}$, $I^{im} = \{i \in I | a_{ii} \leq 0\}$, and let $\underline{m} = (m_i | i \in I)$ be a collection of positive integers such that $m_i = 1$ for all $i \in I^{re}$. We call $\underline{m}$ the *charge* of the Borcherds-Cartan matrix A. We denote by $\mathfrak{g} = \mathfrak{g}(A, \underline{m})$ the generalized Kac-Moody algebra associated with the Borcherds-Cartan matrix A and the charge $\underline{m}$ ([1], [9], [10]).

A rational Borcherds-Cartan matrix $A = (a_{ij})_{i,j \in I}$ is called *symmetrizable* if there is a diagonal matrix $D = diag(s_i | i \in I)$ with $s_i \in \mathbf{Z}_{>0}$ such that DA is symmetric. From now on, we assume that A is a symmetrizable Borcherds-Cartan matrix.

Let $\mathfrak{h} = (\bigoplus_{i \in I} \mathbf{Q}h_i) \oplus (\bigoplus_{i \in I} \mathbf{Q}d_i)$ be the vector space with a basis $\{h_i, d_i | i \in I\}$, and let

$$P^\vee = \left(\bigoplus_{i \in I} \mathbf{Z}h_i\right) \oplus \left(\bigoplus_{i \in I} \mathbf{Z}d_i\right) \tag{1.1}$$

be the $\mathbf{Z}$-lattice of $\mathfrak{h}$. For each $j \in I$, we define the linear functionals $\alpha_j \in \mathfrak{h}^*$ by

$$\alpha_j(h_i) = a_{ij}, \qquad \alpha_j(d_i) = \delta_{ij} \ (i, j \in I). \tag{1.2}$$

Set $Q = \bigoplus_{i \in I} \mathbf{Z}\alpha_i$, $Q_+ = \sum_{i \in I} \mathbf{Z}_{\geq 0}\alpha_i$, and $Q_- = -Q_+$. Let $\rho \in \mathfrak{h}^*$ be a linear functional satisfying $\rho(h_i) = \frac{1}{2}a_{ii}$ for all $i \in I$. For each $i \in I^{re}$, we define the *simple reflection* $r_i \in GL(\mathfrak{h})$ by $r_i(h) = h - \alpha_i(h)h_i$. The subgroup W of $GL(\mathfrak{h})$

generated by the r_i's is called the *Weyl group* of the above Borcherds-Cartan data. It is a Coxeter group with canonical generator system $\{r_i | i \in I^{re}\}$. We denote its length function by $l\colon W \to \mathbf{Z}_{\geq 0}$. The contragredient action of W on $\mathfrak{h}^*$ is given by $r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$. Since A is symmetrizable, there exists a nondegenerate symmetric bilinear form $(\ |\ )$ on $\mathfrak{h}$ satisfying $(s_i h_i | h) = \alpha_i(h)$ $(i \in I, h \in \mathfrak{h})$.

For each $i \in I$, let $\xi_i = q^{s_i} - q^{-s_i}$, $q_i = q^{(s_i a_{ii})/2}$, and define the *q-integer* by

$$[n]_i = \begin{cases} \dfrac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}} & \text{if } a_{ii} \neq 0, \\ n & \text{if } a_{ii} = 0. \end{cases}$$

We also define $[n]_i! = \prod_{k=1}^n [k]_i$.

Definition 1.1. ([10]) The *quantum algebra* $U_q(\mathfrak{g})$ associated with a symmetrizable Borcherds-Cartan matrix $A = (a_{ij})_{i,j \in I}$ and a charge $\underline{m} = (m_i | i \in I)$ is an associative algebra with 1 over $\mathbf{F}$ generated by the elements q^h $(h \in P^\vee)$, e_{ik}, f_{ik} $(i \in I, k = 1, 2, \cdots, m_i)$ with the defining relations

(R1) $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$ $(h, h' \in P^\vee)$,

(R2) $q^h e_{ik} q^{-h} = q^{\alpha_i(h)} e_{ik}$ $(h \in P^\vee,\ i \in I,\ k = 1, 2, \ldots, m_i)$,

(R3) $q^h f_{ik} q^{-h} = q^{-\alpha_i(h)} f_{ik}$ $(h \in P^\vee,\ i \in I,\ k = 1, 2, \ldots, m_i)$,

(R4) $[e_{ik}, f_{jl}] = \delta_{ij}\delta_{kl} \dfrac{K_i - K_i^{-1}}{\xi_i}$, where $K_i = q^{s_i h_i}$ $(i, j \in I,\ k = 1, 2, \ldots, m_i,$ $l = 1, 2, \ldots, m_j)$,

(R5) $\sum_{s+t=1-a_{ij}} (-1)^s e_{ik}^{(s)} e_{jl} e_{ik}^{(t)} = 0$ if $a_{ii} = 2$ and $i \neq j$ $(k = 1, l = 1, 2, \ldots, m_j)$, where $e_{ik}^{(n)} = e_{ik}^n / [n]_i!$,

(R6) $\sum_{s+t=1-a_{ij}} (-1)^s f_{ik}^{(s)} f_{jl} f_{ik}^{(t)} = 0$ if $a_{ii} = 2$ and $i \neq j$ $(k = 1,\ l = 1, 2, \ldots, m_j)$, where $f_{ik}^{(n)} = f_{ik}^n / [n]_i!$,

(R7) $[e_{ik}, e_{jl}] = 0$ if $a_{ij} = 0$.

(R8) $[f_{ik}, f_{jl}] = 0$ if $a_{ij} = 0$.

The algebra $U_q(\mathfrak{g})$ has a Hopf algebra structure with comultiplication Δ, counit ε, and antipode S defined by

$$\begin{aligned} &\Delta(q^h) = q^h \otimes q^h, \\ &\Delta(e_{ik}) = e_{ik} \otimes 1 + K_i \otimes e_{ik}, \\ &\Delta(f_{ik}) = f_{ik} \otimes K_i^{-1} + 1 \otimes f_{ik}, \\ &\varepsilon(q^h) = 1, \qquad \varepsilon(e_{ik}) = \varepsilon(f_{ik}) = 0, \\ &S(q^h) = q^{-h}, \\ &S(e_{ik}) = -K_i^{-1} e_{ik}, \qquad S(f_{ik}) = -f_{ik} K_i \end{aligned} \tag{1.3}$$

for $h \in P^\vee$, $i \in I$, $k = 1, \cdots, m_i$. We denote by U^0 the subalgebra of $U = U_q(\mathfrak{g})$ with 1 generated by q^h $(h \in P^\vee)$ and U^+ (resp. U^-) the subalgebra of U generated by the elements e_{ik} (resp. f_{ik}) for $i \in I$, $k = 1, \ldots, m_i$. We also denote by $U^{\geq 0}$ (resp. $U^{\leq 0}$) the subalgebra of U generated by the elements q^h and e_{ik} (resp. f_{ik}) for $h \in P^\vee$, $i \in I$, $k = 1, \ldots, m_i$. For each $\beta \in Q_+$, let

$$U^{\pm}_{\pm\beta} = \{x \in U^{\pm} | q^h x q^{-h} = q^{\pm\beta(h)} x \text{ for all } h \in P^\vee\}.$$

Then we have:

PROPOSITION 1.2. ([10])
(a) $U \cong U^- \otimes U^0 \otimes U^+$.
(b) $U^0 = \bigoplus_{h \in P^\vee} \mathbf{F} q^h$.
(c) $U^{\pm} = \bigoplus_{\beta \in Q_+} U^{\pm}_{\pm\beta}$.
(d) (R5) *and* (R7) (*resp.* (R6) *and* (R8)) *are the fundamental relations for* U^+ (*resp.* U^-).

Define a structure of directed set on Q_+ by $\beta_1 \geq \beta_2$ if and only if $\beta_1 - \beta_2 \in Q_+$, and set $U^{+,\beta} = \bigoplus_{\gamma \in Q_+, \gamma \not\leq \beta} U^+_\gamma$ for $\beta \in Q_+$. We define a completion $\hat{U}$ of U by

$$\hat{U} = \varprojlim_{\beta} U/UU^{+,\beta}.$$

Then $\hat{U}$ is an algebra containing U. The comultiplication Δ and the counit ε are naturally extended to those of $\hat{U}$ ([13]).

A $U_q(\mathfrak{g})$-module V is called a *highest weight module* with highest weight $\lambda \in \mathfrak{h}^*$ if there is a nonzero vector $v_\lambda \in V$ such that (i) $e_{ik} v_\lambda = 0$ $(i \in I, k = 1, \cdots, m_i)$, (ii) $q^h v_\lambda = q^{\lambda(h)} v_\lambda$ $(h \in P^\vee)$, (iii) $V = U_q(\mathfrak{g}) v_\lambda$. Let $\lambda \in \mathfrak{h}^*$ and consider the left ideal $I(\lambda)$ of $U_q(\mathfrak{g})$ generated by e_{ik} $(i \in I, k = 1, \ldots, m_i)$ and $q^h - q^{\lambda(h)} 1$ $(h \in P^\vee)$. Let $M(\lambda) = U_q(\mathfrak{g})/I(\lambda)$ and define a $U_q(\mathfrak{g})$-module structure on $M(\lambda)$ by the left multiplication. Then $M(\lambda)$ becomes a highest weight module with highest weight λ and highest weight vector $v_\lambda = 1 + I(\lambda)$. The $U_q(\mathfrak{g})$-module $M(\lambda)$ is called the *Verma module* and it has a unique maximal submodule $J(\lambda)$. Hence the quotient $V(\lambda) = M(\lambda)/J(\lambda)$ is irreducible.

Let T denote the set of all imaginary roots α_i $(i \in I^{im})$ counted with multiplicity m_i.

PROPOSITION 1.3. ([1], [10]) *Suppose* $\lambda(h_i) \geq 0$ *for all* $i \in I$ *and* $\lambda(h_i) \in \mathbf{Z}$ *for all* $i \in I^{re}$. *Then we have*

(a) $$\operatorname{ch} M(\lambda) = \frac{e^\lambda}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}} = e^\lambda \sum_{\beta \in Q_+} (\dim U^-_{-\beta}) e^{-\beta},$$

(b) $$\operatorname{ch} V(\lambda) = \frac{\sum_{\substack{w \in W \\ F \subset T}} (-1)^{l(w)+|F|} e^{w(\lambda+\rho-s(F))-\rho}}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}},$$

where Δ_+ denotes the set of all positive roots of $\mathfrak{g}$, $\mathfrak{g}_\alpha$ *denotes the root space, and F runs over all the finite subsets of T such that $\lambda(h_i) = 0$ for $\alpha_i \in F$ and that $\alpha_i(h_j) = 0$ for α_i, $\alpha_j \in F$* with *$i \neq j$. We denote by $|F|$ the number of elements in F and $s(F)$ the sum of elements in F.*

COROLLARY 1.4. *Let $\gamma = \sum_{i \in I} n_i \alpha_i \in Q_+$. Suppose $\lambda(h_i) > 0$ for all $i \in I$, $\lambda(h_i) \in \mathbf{Z}$ for all $i \in I^{re}$, and $\lambda(h_i) \geq n_i$ for all $i \in I^{re}$. Then we have a linear isomorphism $U_{-\gamma}^- \xrightarrow{\sim} V(\lambda)_{\lambda-\gamma}$ given by $u \mapsto uv_\lambda$.*

PROOF. The surjectivity of the map $U_{-\gamma}^- \to V(\lambda)_{\lambda-\gamma}$ is obvious. Hence it suffices to show $\dim U_{-\gamma}^- = \dim V(\lambda)_{\lambda-\gamma}$. By our assumption, we have

$$\begin{aligned}\operatorname{ch} V(\lambda) &= \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}} \\ &= \left(\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)-\rho}\right)\left(\sum_{\beta \in Q_+} (\dim U_{-\beta}^-) e^{-\beta}\right).\end{aligned}$$

Therefore, it suffices to show that if $w(\lambda + \rho) - \rho - \beta = \lambda - \gamma$ for $w \in W$, $\beta \in Q_+$, then $w = 1$. Equivalently, if $w \neq 1$, then $\gamma + w(\lambda + \rho) - (\lambda + \rho) \notin Q_+$. Let us prove this by induction on the length $l(w)$ of w. If $w = r_i$ $(i \in I^{re})$, then

$$\gamma + r_i(\lambda + \rho) - (\lambda + \rho) = \gamma - (\lambda(h_i) + 1)\alpha_i \notin Q_+.$$

If $w = w'r_i$ and $l(w) = l(w') + 1$, then

$$\begin{aligned}\gamma + w(\lambda + \rho) - (\lambda + \rho) &= \gamma + w'r_i(\lambda + \rho) - (\lambda + \rho) \\ &= \gamma + w'(\lambda + \rho) - (\lambda + \rho) - (\lambda(h_i) + 1)w'(\alpha_i) \notin Q_+,\end{aligned}$$

which completes the proof. □

2. The Killing Form on $U_q(\mathfrak{g})$

The Hopf algebra structure of $U_q(\mathfrak{g})$ defines an algebra structure on $(U^{\geq 0})^*$ with the multiplication given by $(\phi_1\phi_2)(x) = (\phi_1 \otimes \phi_2)(\Delta(x))$ for $\phi_1, \phi_2 \in (U^{\geq 0})^*$, $x \in U^{\geq 0}$. For $h \in P^\vee$ and $i \in I$, $k = 1, 2, \ldots, m_i$, we define the linear functionals ϕ_h, $\psi_{ik} \in (U^{\geq 0})^*$ by

$$\begin{aligned}&\phi_h(xq^{h'}) = \varepsilon(x)q^{-(h|h')} \qquad (x \in U^+, h' \in P^\vee), \\ (2.1)\qquad &\psi_{ik}(xq^h) = 0 \qquad (x \in U_\beta^+, \beta \in Q_+ \backslash \{\alpha_i\}), \\ &\psi_{ik}(e_{il}q^h) = \delta_{kl}.\end{aligned}$$

Then it is easy to verify that there is an algebra homomorphism $\zeta\colon U^{\leq 0} \to (U^{\geq 0})^*$ given by $\zeta(q^h) = \phi_h$, $\zeta(f_{ik}) = -\frac{1}{\xi_i}\psi_{ik}$ $(h \in P^\vee, i \in I, k = 1, \ldots, m_i)$. Define a bilinear form $(\ |\ ): U^{\geq 0} \times U^{\leq 0} \to \mathbf{F}$ by

$$(x|y) = \langle \zeta(y), x \rangle \qquad (x \in U^{\geq 0}, y \in U^{\leq 0}). \tag{2.2}$$

Then we have:

PROPOSITION 2.1. *The bilinear form* $(\ |\ )$ *on* $U^{\geq 0} \times U^{\leq 0}$ *defined by* (2.2) *satisfies*

$$\begin{aligned} &(x|y_1 y_2) = (\Delta(x)|y_1 \otimes y_2) \qquad (x \in U^{\geq 0}, y_1, y_2 \in U^{\leq 0}), \\ &(x_1 x_2|y) = (x_2 \otimes x_1|\Delta(y)) \qquad (x_1, x_2 \in U^{\geq 0}, y \in U^{\leq 0}), \\ &(q^h|q^{h'}) = q^{-(h|h')} \qquad (h, h' \in P^\vee), \\ &(q^h|f_{ik}) = 0, \qquad (e_{ik}|q^h) = 0, \\ &(e_{ik}|f_{jl}) = -\frac{1}{\xi_i}\delta_{ij}\delta_{kl} \end{aligned} \tag{2.3}$$

for $i, j \in I$, $k = 1, 2, \cdots, m_i$, $l = 1, 2, \cdots, m_j$.

Moreover, the bilinear form on $U^{\geq 0} \times U^{\leq 0}$ *satisfying* (2.3) *is uniquely determined.*

The proof is similar to that of [13, Proposition 2.1.1].

The following lemmas can be proved inductively using (2.3).

LEMMA 2.2.
(a) $(S(x)|S(y)) = (x|y)$ for $x \in U^{\geq 0}$, $y \in U^{\leq 0}$.
(b) $(xq^h|yq^{h'}) = q^{-(h|h')}(x|y)$ $(h, h' \in P^\vee, x \in U^+, y \in U^-)$.
(c) $(U^+_\gamma|U^-_{-\beta}) = 0$ if $\gamma \neq \beta$.

For $n \in \mathbf{Z}_{>0}$, we denote by $\Delta_n: U_q(\mathfrak{g}) \to U_q(\mathfrak{g})^{\otimes(n+1)}$ the algebra homomorphism defined by $\Delta_1 = \Delta$, $\Delta_n = (\Delta \otimes 1) \circ \Delta_{n-1}$, and we write

$$\Delta_n(x) = \sum_{(x)_n} x_{(0)} \otimes x_{(1)} \otimes \cdots \otimes x_{(n)}.$$

LEMMA 2.3. *For* $x \in U^{\geq 0}$, $y \in U^{\leq 0}$, *we have*

$$\begin{aligned} yx &= \sum_{(x)_2, (y)_2} (x_{(0)}|S(y_{(0)}))(x_{(2)}|y_{(2)})x_{(1)}y_{(1)}, \\ xy &= \sum_{(x)_2, (y)_2} (x_{(0)}|y_{(0)})(x_{(2)}|S(y_{(2)}))y_{(1)}x_{(1)}. \end{aligned} \tag{2.4}$$

The following lemma is an immediate consequence of Corollary 1.4.

LEMMA 2.4. *Let* $\beta \in Q_+ \backslash \{0\}$ *and* $y \in U^-_{-\beta}$. *If* $e_{ik}y = ye_{ik}$ *for all* $i \in I$, $k = 1, 2, \cdots, m_i$, *then* $y = 0$.

Now we can state the main theorem of this section.

THEOREM 2.5. *For $\beta \in Q_+$, the bilinear form $(\ |\ )$: $U_\beta^{\geq 0} \times U_{-\beta}^{\leq 0} \to \mathbf{F}$ defined by* (2.2) *is nondegenerate.*

The proof is the same as that of [13, Proposition 2.1.4].

3. Universal R-matrix

In this section, we would like to give an explicit formula for the universal R-matrix of the quantum algebra $U_q(\mathfrak{g})$. We first recall the definition of quasi-triangular Hopf algebras and the pre-triangular Hopf algebras ([6], [13]). A Hopf algebra $\mathscr{H}$ together with an element $\mathscr{R} \in \mathscr{H} \otimes \mathscr{H}$ is called a *quasi-triangular Hopf algebra* if it satisfies:

(T1) $\mathscr{R}$ is invertible,

(T2) $\mathscr{R} \circ \Delta(a) = \Delta'(a) \circ \mathscr{R}$ for all $a \in \mathscr{H}$,

(T3) $(\Delta \otimes 1)(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{23}$,

(T4) $(1 \otimes \Delta)(\mathscr{R}) = \mathscr{R}_{13}\mathscr{R}_{12}$,

where $\Delta' = \tau \circ \Delta$ with $\tau(a \otimes b) = b \otimes a$ $(a, b \in \mathscr{H})$ and $\mathscr{R}_{ij}$ is an element of $\mathscr{H} \otimes \mathscr{H} \otimes \mathscr{H}$ such that the (i, j) component is given by $\mathscr{R}$ and the remaining component is 1. The element $\mathscr{R}$ is called the *universal R-matrix* of $\mathscr{H}$ since it satisfies the Yang-Baxter equation

$$\mathscr{R}_{12}\mathscr{R}_{13}\mathscr{R}_{23} = \mathscr{R}_{23}\mathscr{R}_{13}\mathscr{R}_{12}.$$

A Hopf algebra together with an element $\mathscr{C} \in \mathscr{H} \otimes \mathscr{H}$ and an algebra automorphism Φ: $\mathscr{H} \otimes \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is called a *pre-triangular Hopf algebra* if it satisfies:

(P1) $\mathscr{C}$ is invertible,

(P2) $\mathscr{C} \circ \Delta(a) = \Phi(\Delta'(a)) \circ \mathscr{C}$ for all $a \in \mathscr{H}$,

(P3) $\Phi_{23} \circ \Phi_{13}(\mathscr{C}_{12}) = \mathscr{C}_{12}$,

(P4) $\Phi_{12} \circ \Phi_{13}(\mathscr{C}_{23}) = \mathscr{C}_{23}$,

(P5) $\Phi_{23}(\mathscr{C}_{13}) \circ \mathscr{C}_{23} = (\Delta \otimes 1)(\mathscr{C})$,

(P6) $\Phi_{12}(\mathscr{C}_{13}) \circ \mathscr{C}_{12} = (1 \otimes \Delta)(\mathscr{C})$.

A pre-triangular Hopf algebra $\mathscr{H}$ becomes a quasi-triangular Hopf algebra if there is an invertible element $\mathscr{Z} \in \mathscr{H} \otimes \mathscr{H}$ satisfying

$$\Phi(a \otimes b) = \mathscr{Z}(a \otimes b)\mathscr{Z}^{-1},$$

$$(\Delta \otimes 1)(\mathscr{Z}) = \mathscr{Z}_{23}\mathscr{Z}_{13}, \tag{3.1}$$

$$(1 \otimes \Delta)(\mathscr{Z}) = \mathscr{Z}_{12}\mathscr{Z}_{13}.$$

In this case, the universal R-matrix is given by $\mathscr{R} = \mathscr{Z}^{-1}\mathscr{C}$.

We define an algebra automorphism Φ: $U \otimes U \to U \otimes U$ by

$$\Phi(q^h \otimes q^{h'}) = q^h \otimes q^{h'},$$
$$\Phi(e_{ik} \otimes 1) = e_{ik} \otimes K_i, \qquad \Phi(1 \otimes e_{ik}) = K_i \otimes e_{ik}, \tag{3.2}$$
$$\Phi(f_{ik} \otimes 1) = f_{ik} \otimes K_i^{-1}, \qquad \Phi(1 \otimes f_{ik}) = K_i^{-1} \otimes f_{ik}.$$

It can be shown that Φ can be naturally extended to an automorphism of $\hat{U} \hat{\otimes} \hat{U} = (U \otimes U)\hat{}$.

For $\beta = \sum_{i \in I} n_i \alpha_i \in Q_+$, we denote by $C_\beta \in U_\beta^+ \otimes U_\beta^+$ the canonical element of the bilinear form $(\ |\ ): U_\beta^+ \times U_{-\beta}^- \to \mathbf{F}$, and let $h_\beta = \sum_{i \in I} n_i s_i h_i$, $K_\beta = q^{h_\beta}$ so that $(h_\beta | h) = \beta(h)$ $(h \in P^\vee)$. We define

$$\mathscr{C} = \sum_{\beta \in Q_+} q^{(h_\beta | h_\beta)} (K_\beta^{-1} \otimes K_\beta) C_\beta \in \hat{U} \hat{\otimes} \hat{U}. \tag{3.3}$$

We would like to show that $(\hat{U}, \mathscr{C}, \Phi)$ satisfies the conditions (P1)–(P6).

By direct calculations, we can prove the following lemmas.

LEMMA 3.1.
(a) $\mathscr{C} \Delta(q^h) = \Phi(\Delta'(q^h))\mathscr{C}$ $(h \in P^\vee)$.
(b) $(\Phi_{23} \circ \Phi_{13})(\mathscr{C}_{12}) = \mathscr{C}_{12}$,
(c) $(\Phi_{12} \circ \Phi_{13})(\mathscr{C}_{23}) = \mathscr{C}_{23}$.

LEMMA 3.2. *Let*

$$\mathscr{C}' = \sum_{\beta \in Q_+} q^{(h_\beta | h_\beta)} (1 \otimes K_\beta)(S \otimes 1) C_\beta \in \hat{U} \hat{\otimes} \hat{U}.$$

Then $\mathscr{C}\mathscr{C}' = \mathscr{C}'\mathscr{C} = 1$ *if and only if for any* $\beta \in Q_+$ *we have*

$$\sum_{\substack{\gamma, \delta \in Q_+ \\ \gamma + \delta = \beta}} C_\gamma (K_\delta \otimes 1)(S \otimes 1)(C_\delta) = \delta_{\beta, 0},$$
$$\sum_{\substack{\gamma, \delta \in Q_+ \\ \gamma + \delta = \beta}} (K_\gamma \otimes 1)(S \otimes 1)(C_\gamma) C_\delta = \delta_{\beta, 0}. \tag{3.4}$$

LEMMA 3.3. *We have*

$$\mathscr{C} \Delta(e_{ik}) = \Phi(\Delta'(e_{ik}))\mathscr{C}, \qquad \mathscr{C} \Delta(f_{ik}) = \Phi(\Delta'(f_{ik}))\mathscr{C}'$$

if and only if

$$[1 \otimes e_{ik}, C_{\beta + \alpha_i}] = C_\beta (e_{ik} \otimes K_i^{-1}) - (e_{ik} \otimes K_i) C_\beta,$$
$$[f_{ik} \otimes 1, C_{\beta + \alpha_i}] = C_\beta (K_i \otimes f_{ik}) - (K_i^{-1} \otimes f_{ik}) C_\beta. \tag{3.5}$$

LEMMA 3.4. *We have*

$$\Phi_{23}(\mathscr{C}_{13})\mathscr{C}_{23} = (\Delta \otimes 1)\mathscr{C}, \qquad \Phi_{12}(\mathscr{C}_{13})\mathscr{C}_{12} = (1 \otimes \Delta)\mathscr{C}$$

if and only if

$$(3.6)\qquad \begin{aligned}(\Delta\otimes 1)(C_\beta) &= \sum_{\substack{\gamma,\delta\in Q_+\\ \gamma+\delta=\beta}} q^{-(h_\gamma|h_\delta)}(K_\delta\otimes 1\otimes 1)(C_\gamma)_{13}(C_\delta)_{23},\\ (1\otimes\Delta)(C_\beta) &= \sum_{\substack{\gamma,\delta\in Q_+\\ \gamma+\delta=\beta}} q^{-(h_\gamma|h_\delta)}(1\otimes 1\otimes K_{-\delta})(C_\gamma)_{13}(C_\delta)_{12}.\end{aligned}$$

Hence, in order to show that $(\hat{U}, \mathscr{C}, \Phi)$ satisfies the conditions (P1)–(P6), it remains to show that (3.4), (3.5), and (3.6) hold. But they can be proved in an almost the same manner as in [13, Proposition 4.3.3]. Therefore, we have:

THEOREM 3.5. *Let Φ: $\hat{U}\hat{\otimes}\hat{U}$ be the algebra automorphism defined by* (3.2), *and let $\mathscr{C}$ be the element of $\hat{U}\hat{\otimes}\hat{U}$ defined by* (3.3). *Then the triple $(\hat{U}.\mathscr{C}, \Phi)$ satisfies the conditions* (P1)–(P6).

REMARK. Let $\{h_i, d_i|i\in I\}$ and $\{h^i, d^i|i\in I\}$ be the dual bases of $\mathfrak{h}$ with respect to the bilinear form $(\ |\ )$ and set $\mathscr{X} = q^{\sum h_i\otimes h^i+\sum d_i\otimes d^i}$. Then $\mathscr{R} = \mathscr{X}^{-1}\mathscr{C}$ gives rise to an R-matrix for any $\mathfrak{h}$-diagonalizable integrable representation V of the quantum algebra $U_q(\mathfrak{g})$. Therefore, the formula (3.3) can be viewed as an explicit formula for the universal R-matrix of $U_q(\mathfrak{g})$.

4. The center of $U_q(\mathfrak{g})$

In this section, we will describe the center of the quantum algebra $U_q(\mathfrak{g})$. Let us denote by $\mathfrak{z}(U)$ the center of $U = U_q(\mathfrak{g})$. For each $i\in I$ with $a_{ii}\neq 0$, define the *simple reflection* $r_i\in GL(\mathfrak{h})$ by

$$(4.1)\qquad r_i(h) = h - \frac{2}{a_{ii}}\alpha_i(h)h_i,$$

and let $\tilde{W} = \langle r_i|i\in I, a_{ii}\neq 0\rangle$ be the subgroup of $GL(\mathfrak{h})$ generated by the r_i's ($i\in I, a_{ii}\neq 0$). Let $(U^0)^{\tilde{W}}$ be the subspace of U^0 consisting of the elements $\sum_{h\in P^\vee} c_h q^h$ ($c_h\in \mathbf{F}$) such that $c_h\neq 0$ implies $w(h)\in P^\vee$ and $c_{w(h)} = c_h$ for any $w\in\tilde{W}$. We define an algebra automorphism ϕ: $U^0\to U^0$ by $\phi(q^h) = q^{-\rho(h)}q^h$ ($h\in P^\vee$), and let η be the linear map given by

$$(4.2)\qquad \eta\colon U \xrightarrow{\sim} U^-\otimes U^0\otimes U^+ \xrightarrow{\varepsilon\otimes 1\otimes\varepsilon} U^0.$$

The linear map ξ: $\phi\circ(\eta|_{\mathfrak{z}})$: $\mathfrak{z}\to U^0$ is called the *Harish-Chandra homomorphism.*

PROPOSITION 4.1.

(a) *ξ is an algebra homomorphism.*

(b) *ξ is injective.*

(c) *$Im(\xi)\subset (U^0)^{\tilde{W}}$.*

PROOF. (a) can be proved in a standard way (for example, see [Di]), and (b) can be proved as in [13, Theorem 3.1.2].

For (c), let $M(\lambda)$ be the Verma module over $U_q(\mathfrak{g})$ with highest weight λ. Then it is easy to see that $z|_{M(\lambda)} = \chi_{\lambda+\rho}(\xi(z))I$ for all $z \in \mathfrak{z}$, where $\chi_\lambda\colon U^0 \to \mathbf{F}$ $(\lambda \in \mathfrak{h}^*)$ is the algebra homomorphism defined by $\chi_\lambda(q^h) = q^{\lambda(h)}$ $(h \in P^\vee)$.

Moreover, if $a_{ii} \neq 0$ and $(\lambda + \rho)(h_i) \in \frac{a_{ii}}{2}\mathbf{Z}_{\geq 0}$, then $\mathrm{Hom}_U(M(r_i(\lambda + \rho) - \rho), M(\lambda)) \neq 0$. Indeed, if v_λ is a highest weight vector of $M(\lambda)$ with highest weight λ, then $f_{ik}^{(2/a_{ii})(\lambda+\rho)(h_i)}v_\lambda$ is a highest weight vector with highest weight $r_i(\lambda + \rho) - \rho$.

Let $i \in I$ be such that $a_{ii} \neq 0$ and let $z \in \mathfrak{z}$. Then $\chi_\lambda(\xi(z)) = \chi_{r_i(\lambda)}(\xi(z)) = \chi_\lambda(r_i(\xi(z)))$ for any $\lambda \in \mathfrak{h}^*$ such that $\lambda(h_i) \in \frac{a_{ii}}{2}\mathbf{Z}_{\geq 0}$. Hence $\chi_\lambda(\xi(z) - r_i\xi(z)) = 0$ for any $\lambda \in \mathfrak{h}^*$ such that $\lambda(h_i) \in \frac{a_{ii}}{2}\mathbf{Z}_{\geq 0}$, which implies $\xi(z) = r_i(\xi(z))$ for all $i \in I$ with $a_{ii} \neq 0$. $\square$

For $J \subset \{(i, k) | i \in I, k = 1, 2, \cdots, m_i\}$, let $U_J = \langle e_{ik}, f_{ik}, U^0 | (i, k) \in J\rangle$ be the subalgebra of U generated by U^0 and e_{ik}, f_{ik} with $(i, k) \in J$. We denote by $\mathfrak{z}_J$ the center of U_J and $\xi_J\colon \mathfrak{z}_J \to U^0$ the Harish-Chandra homomorphism for U_J. We would like to show $\mathrm{Im}(\xi) \subset \mathrm{Im}(\xi_J)$. Let U_J^+ (resp. U_J^-) be the subalgebra of U_J generated by e_{ik} (resp. f_{ik}) with $(i, k) \in J$, and set

$$(4.3)\qquad \begin{aligned} R_J^+ &= \{x \in U^+ | (x | U_J^-) = 0\} = \{x \in U^+ | (x | U_J^- U^0) = 0\}, \\ R_J^- &= \{y \in U^- | (U_J^+ | y) = 0\} = \{y \in U^- | (U^0 U_J^+ | y) = 0\}, \\ R_J &= R_J^- U^0 U^+ + U^- U^0 R_J^+. \end{aligned}$$

Then we have:

LEMMA 4.2.
(a) $U = U_J \oplus R_J$,
(b) $U_J R_J U_J \subset R_J$,
(c) $(\varepsilon \otimes 1 \otimes \varepsilon)(R_J) = 0$.

PROOF. (a) It suffices to show $U_\gamma^+ = U_{J,\gamma}^+ \oplus R_{J,\gamma}^+$ for any $\gamma \in Q_+$. Since

$$R_{J,\gamma}^+ = \mathrm{Ker}(U_\gamma^+ \tilde{\to} (U_{-\gamma}^-)^* \to (U_{J,-\gamma}^-)^*),$$

$$\dim R_{J,\gamma}^+ = \dim U_\gamma^+ - \dim U_{J,\gamma}^- = \dim U_\gamma^+ - \dim U_{J,\gamma}^+.$$

Since $(\ |\ )$ is nondegenerate on $U_{J,\gamma}^+ \times U_{J,-\gamma}^-$, we have $R_{J,\gamma}^+ \cap U_{J,\gamma}^+ = \{0\}$.

(b) First, note that R_J^+ (resp. R_J^-) is a two-sided ideal of U^+ (resp. U^-), and that $U^0 R_J^\pm = R_J^\pm U^0$. Hence it suffices to show

$$(4.4)\qquad U_J^+ R_J^- \subset R_J^- U, \qquad R_J^+ U_J^- \subset U R_J^+.$$

Let $y \in R^-_{J,-\gamma}$. For $(i, k) \in J$, by Lemma 2.3, we have

$$e_{ik}y = \sum_{(y)_2} (e_{ik}|y_{(0)})(1|S(y_{(2)}))y_{(1)} + \sum_{(y)_2} (K_i|y_{(0)})(1|S(y_{(2)}))y_{(1)}e_{ik} + \sum_{(y)_2} (K_i|y_{(0)})(e_{ik}|S(y_{(2)}))y_{(1)}K_i.$$

Hence it suffices to show

$$\left(x \middle| \sum_{(y)_2} (e_{ik}|y_{(0)})(1|S(y_{(2)}))y_{(1)}\right) = 0,$$

$$\left(x \middle| \sum_{(y)_2} (K_i|y_{(0)})(1|S(y_{(2)}))y_{(1)}\right) = 0,$$

$$\left(x \middle| \sum_{(y)_2} (K_i|y_{(0)})(e_{ik}|S(y_{(2)}))y_{(1)}\right) = 0$$

for all $x \in U_J^+$. Indeed, we have, for example,

$$\begin{aligned}\left(x \middle| \sum_{(y)_2} (K_i|y_{(0)})(e_{ik}|S(y_{(2)}))y_{(1)}\right) &= \sum_{(y)_2} (K_i|y_{(0)})(e_{ik}|S(y_{(2)}))(x|y_{(1)}) \\ &= \sum_{(y)_2} (K_i \otimes x \otimes S^{-1}(e_{ik})|\Delta^{(2)}(y)) \\ &= (S^{-1}(e_{ik})xK_i|y) = 0.\end{aligned}$$

The other cases can be proved in a similar way.

(c) Clear. □

PROPOSITION 4.3. $\mathrm{Im}(\xi) \subset \mathrm{Im}(\xi_J)$.

PROOF. Let $z \in \mathfrak{z}$ and write $z = z_1 + z_2$ with $z_1 \in U_J$, $z_2 \in R_J$. By Lemma 4.2 (b), $z_1 \in \mathfrak{z}_J$, and hence by Lemma 4.2 (c), $\xi(z) = \xi_J(z_1) \in \mathrm{Im}(\xi_J)$. □

We now consider the special cases when $|I| = 1$ or $|I| = 2$. By a direct calculation, we have:

PROPOSITION 4.4. *Suppose $I = \{i\}$ and $m_i = 1$.*

(a) *If $a_{ii} \neq 0$, then*

$$\mathfrak{z} = \left\langle f_{i,1}e_{i,1} + \frac{1}{\xi_i(q_i - q_i^{-1})}(q_iK_i + q_i^{-1}K_i^{-1}), q^h \middle| \alpha_i(h) = 0 \right\rangle.$$

(b) *If $a_{ii} = 0$, then $\mathfrak{z} \subset U^0$.*

PROPOSITION 4.5. *Assume either*

(a) *$I = \{i\}$ with $a_{ii} < 0$, $m_i = 2$, or*

(b) $I = \{i, j\}$ *with* $a_{ii} < 0$, $a_{jj} < 0$, $a_{ij} < 0$, *and* $m_i = m_j = 1$.
Then $\mathfrak{z} \subset U^0$.

PROOF. Set $e = e_{i,1}$, $e' = e_{i,2}$, $f = f_{i,1}$, $f' = f_{i,2}$ in case (a), and $e = e_{i,1}$, $e' = e_{j,1}$, $f = f_{i,1}$, $f' = f_{j,1}$ in case (b). Then the subalgebra $U^+ = \langle e, e' \rangle = \bigoplus_{n=0}^{\infty} U_n^+$ (resp. $U^- = \langle f, f' \rangle = \bigoplus_{n=0}^{\infty} U_{-n}^-$) is the free associative algebra over $\mathbf{F}$ generated by the elements e, e' (resp. f, f'), where U_n^+ (resp. U_{-n}^-) is the homogeneous subspace of degree n (resp. $-n$). Then, for $n \geq 1$, we have $U_n^+ = U_{n-1}^+ e \oplus U_{n-1}^+ e'$.

Let $z \in \mathfrak{z} \cap (\bigoplus_{k=0}^{n} U^- U^0 U_k^+)$, and let $\{x_\lambda\}$ be a basis of U_{n-1}^+. Then

$$z = \sum_{\lambda} \sum_{h \in P^\vee} y_{\lambda,h} q^h x_\lambda e + \sum_{\lambda} \sum_{h \in P^\vee} y'_{\lambda,h} q^h x_\lambda e' + y,$$

where $y \in \sum_{k=0}^{n-1} U^- U^0 U_k^+$, $y_{\lambda,h}$, $y'_{\lambda,h} \in U^-$. Hence we have

$$ez = \sum_{\lambda} \sum_{h \in P^\vee} y_{\lambda,h} q^{-\alpha_i(h)} q^h e x_\lambda e + \sum_{\lambda} \sum_{h \in P^\vee} y'_{\lambda,h} q^{-\alpha_i(h)} q^h e x_\lambda e' + z',$$

and

$$ze = \sum_{\lambda} \sum_{h \in P^\vee} y_{\lambda,h} q^h x_\lambda e^2 + \sum_{\lambda} \sum_{h \in P^\vee} y'_{\lambda,h} q^h x_\lambda e' e + z'',$$

where z', $z'' \in \sum_{k=0}^{n} U^- U^0 U_k^+$. Hence $y'_{\lambda,h} = 0$ for all λ and h. Similarly, $y_{\lambda,h} = 0$ for all λ and h. Therefore, $z \in \mathfrak{z} \cap (\bigoplus_{k=0}^{n-1} U^- U^0 U_k^+)$, and hence, by induction, we see that $\mathfrak{z} = \mathfrak{z} \cap U^- U^0 = \mathfrak{z} \cap U^0$. □

PROPOSITION 4.6. *Assume that* $I = \{i, j\}$ *and* $a_{ii} = 2$, $a_{jj} < 0$, $a_{ij} < 0$, *and* $m_j = 1$. *Then we have* $\mathfrak{z} \subset U^0$.

PROOF. Let $V' = \mathbf{Q}h_i \oplus \mathbf{Q}h_j$ and $V = \{h \in \mathfrak{h} \mid \alpha_i(h) = \alpha_j(h) = 0\}$. Then $\mathfrak{h} = V \oplus V'$. Note that $\tilde{W}$ preserves V and V' and that

$$\det \begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix} = a_{ii} a_{jj} - a_{ij} a_{ji} < 0.$$

We would like to show $\mathrm{Im}(\xi) \subset \bigoplus_{h \in V \cap P^\vee} \mathbf{F} q^h$. Since $\mathrm{Im}(\xi) \subset (U^0)^{\tilde{W}}$, it suffices to show $h \in \mathfrak{h}$ and $|\tilde{W}(h)| < \infty$ if and only if $h \in V$. Hence we need only to show if $\bar{h} \in \mathfrak{h}/V \cong V'$, $|\tilde{W}(\bar{h})| < \infty$, then $\bar{h} = 0$. Therefore, it suffices to show that the eigenvalues of $r_i r_j|_{V'}$ are not roots of unity. Since the characteristic polynomial of $r_i r_j|_{V'}$ is $t^2 - \left(\dfrac{2a_{ij}a_{ji}}{a_{jj}} - 2\right)t + 1$, $r_i r_j|_{V'}$ has an eigenvalue that is a root of unity if and only if $\dfrac{2a_{ij}a_{ji}}{a_{jj}} = 0, 1, 2, 3, 4$, which is a contradiction to our assumption. □

LEMMA 4.7. *Assume that the Borcherds-Cartan matrix* $A = (a_{ij})_{i,j \in I}$ *is indecomposable. If there is a nonempty subset* J *of* $\{(i,k) | i \in I, k = 1, \dots, m_i\}$ *such that* $\mathfrak{z}_J \subset U^0$, *then* $\mathfrak{z}$ *is contained in* U^0.

PROOF. Let $\bar{J} = \{i \in I | (i,k) \in J \text{ for some } k\}$. Then we have

$$\mathfrak{z} \cap U^0 = \bigoplus_{\substack{h \in P^\vee \\ \alpha_i(h)=0\ (i \in I)}} \mathbf{F} q^h, \qquad \mathfrak{z}_J \cap U^0 = \bigoplus_{\substack{h \in P^\vee \\ \alpha_i(h)=0\ (i \in \bar{J})}} \mathbf{F} q^h.$$

For $i \in I$, set $T_i = \bigoplus_{\substack{h \in P^\vee \\ \alpha_i(h)=0}} \mathbf{F} q^h$. We would like to show $\mathrm{Im}(\xi) \subset \bigcap_{i \in I} T_i$. By Proposition 4.3, we have $\mathrm{Im}(\xi) \subset \mathrm{Im}(\xi_J) \subset \bigcap_{i \in \bar{J}} T_i$.

If $a_{ii} = 0$, then by Proposition 4.4 (b), $\mathrm{Im}(\xi) \subset \mathrm{Im}(\xi_{\{(i,1)\}}) \subset T_i$. Hence it suffices to show that if $a_{ji} \neq 0$, $a_{jj} \neq 0$, then $T_i \cap (U^o)^{\tilde{W}} \subset T_j$.

Let $x = \sum_{\substack{h \in P^\vee \\ \alpha_i(h)=0}} c_h q^h \in T_i \cap (U^0)^{\tilde{W}}$. Then $x = r_j(x) = \sum_{\substack{h \in P^\vee \\ \alpha_i(h)=0}} c_h q^{r_j(h)}$. Hence if $c_h \neq 0$, then $\alpha_i(r_j(h)) = \alpha_i(h) = 0$, which implies $\alpha_j(h) = 0$. □

By Proposition 4.4–Lemma 4.7, we have the following theorem.

THEOREM 4.8. *Suppose that the Borcherds-Cartan matrix* $A = (a_{ij})_{i,j \in I}$ *is indecomposable and* $I^{im} \neq \phi$. *Then*

$$\mathfrak{z}(U) = \bigoplus_{\substack{h \in P^\vee \\ \alpha_i(h)=0\ (i \in I)}} \mathbf{F} q^h \subset U^0$$

except for the case I *consists of a single element* i *with* $a_{ii} < 0$ *and* $m_i = 1$.

References

[1] R. E. Borcherds, Generalized Kac-Moody algebras, J. Algebra **115** (1988), 501–512.

[2] R. E. Borcherds, Monstrous moonshine and monstrous Lie superalgebras, Invent. Math. **109** (1992), 405–444.

[3] J. H. Conway, S. Norton, Monstrous moonshine, Bull. Lond. Math. Soc. **11** (1979), 308–339.

[4] C. De Concini, V. G. Kac, Representations of quantum groups at roots of 1, Progress in Mathematics **92**, Birkhäuser, 1994, pp. 471–506.

[5] J. Dixmier, Algébres Enveloppantes, Gauthier-Vilars, Paris, 1974.

[6] V. G. Drinfel'd, Hopf algebra and the Yang-Baxter equation, Soviet Math. Dokl. **32** (1985), 254–258.

[7] M. Jimbo, A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation, Lett. Math. Phys. **10** (1985), 63–69.

[8] A. Joseph, G. Letzter, Local finiteness for the adjoint action for quantized enveloping algebras, J. Algebra **153** (1992), 289–318.

[9] V. G. Kac, Infinite Dimensional Lie Algebras, 3rd ed., Cambridge University Press, 1990.

[10] S.-J. Kang, Quantum deformations of generalized Kac-Moody algebras and their modules, J. Algebra **175** (1995), 1041–1066.

[11] G. Lusztig, Introduction to Quantum Groups, Birkhäuser, 1993.
[12] M. Rosso, Analogues de la forme de Killing et du théorèm d'Harish-Chandra pour les groupes quantiques, Ann. Ec. Norm. Sup. **23** (1990), 445–467.
[13] T. Tanisaki, Killing forms, Harish-Chandra isomorphisms, and universal R-matrices for quantum algebras, Inter. J. Mod. Phys. **A7**, **Suppl. 1B** (1992), 941–961.

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