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Quantum deformations of certain prehomogeneous vector spaces I

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Abstract. We shall construct a quantum analogue of the prehomogeneous vector space associated to a parabolic subgroup with commutative unipotent radical.

0. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over the complex number field $\mathbf{C}$, and let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{m}^+$ be a parabolic subalgebra of $\mathfrak{g}$, where $\mathfrak{l}$ is a maximal reductive subalgebra of $\mathfrak{p}$ and $\mathfrak{m}^+$ is the nilpotent part. We denote by $\mathfrak{m}^-$ the nilpotent subalgebra of $\mathfrak{g}$ such that $\mathfrak{l} \oplus \mathfrak{m}^-$ is a parabolic subalgebra of $\mathfrak{g}$ opposite to $\mathfrak{p}$. Take an algebraic group L with Lie algebra $\mathfrak{l}$.

In this paper we shall deal with the case where $\mathfrak{m}^\pm$ is nonzero and commutative. Then $\mathfrak{m}^+$ consists of finitely many L-orbits.

Our aim is to give a quantum analogue of the prehomogeneous vector space $(L, \mathfrak{m}^+)$. More precisely, we shall construct a quantum analogue A_q of the ring $A = \mathbf{C}[\mathfrak{m}^+]$ of polynomial functions on $\mathfrak{m}^+$ as a noncommutative $\mathbf{C}(q)$-algebra endowed with the action of the quantized enveloping algebra $U_q(\mathfrak{l})$ of $\mathfrak{l}$, and show that for each L-orbit C on $\mathfrak{m}^+$ there exists a two-sided ideal $J_{C,q}$ of A_q which can be regarded as a quantum analogue of the defining ideal J_C of the closure $\bar{C}$ of C. Such an object was intensively studied in the cases $\mathfrak{g} = \mathfrak{sl}_n$ (see Hashimoto-Hayashi [3], Noumi-Yamada-Mimachi [10]) and $\mathfrak{g} = \mathfrak{so}_{2n}$ (see Strickland [13]).

Our method is as follows. Since $\mathfrak{m}^-$ is identified with the dual space of $\mathfrak{m}^+$ via the Killing form, A is isomorphic to the symmetric algebra $S(\mathfrak{m}^-)$. By the commutativity of $\mathfrak{m}^-$ the enveloping algebra $U(\mathfrak{m}^-)$ is naturally identified with the symmetric algebra $S(\mathfrak{m}^-)$. Hence we have an identification $A = U(\mathfrak{m}^-)$. Then using the Poincaré-Birkhoff-Witt type basis of the quantized enveloping algebra $U_q(\mathfrak{g})$ (Lusztig [9]) we obtain a natural quantization A_q of A as a subalgebra of $U_q(\mathfrak{g})$. The algebra A_q has a canonical generator system satisfying quadratic fundamental relations. In particular, it is a graded algebra. The adjoint action of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$ is defined using the Hopf

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algebra structure, and we can show that A_q is preserved under the adjoint action of $U_q(\mathfrak{l})$. As a $U_q(\mathfrak{l})$-module A_q is a direct sum of finite dimensional irreducible submodules.

Let C be a non-open L-orbit on $\mathfrak{m}^+$. It is known that J_C is an $\mathfrak{l}$-stable homogeneous ideal generated by the lowest degree part J_C^0. Since A is a multiplicity free $\mathfrak{l}$-module, there exist unique $U_q(\mathfrak{l})$-submodules $J_{C,q}$ and $J_{C,q}^0$ of A_q satisfying $J_{C,q}|_{q=1} = J_C$ and $J_{C,q}^0|_{q=1} = J_C^0$. We can show that $J_{C,q}$ is a two-sided ideal of A_q and that $J_{C,q}$ is generated by $J_{C,q}^0$ both as a left ideal and a right ideal. The proof uses the quantum counterpart of the results on a generalized Verma module of $\mathfrak{g}$ whose maximal proper submodule is explicitly described in terms of J_C (see Enright-Joseph [2], Tanisaki [14]).

Explicit descriptions of A_q and $J_{C,q}$ in each individual case will be given in our subsequent papers.

1. Quantized enveloping algebras

Let $\mathfrak{g}$ be a simple Lie algebra over the complex number field $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ and $W \subset GL(\mathfrak{h})$ be the root system and the Weyl group respectively. For each $\alpha \in \Delta$ we denote the corresponding root space by $\mathfrak{g}_\alpha$. We fix an ordering on Δ, and denote the set of positive roots by Δ^+ and the set of simple roots by $\{\alpha_i\}_{i\in I_0}$, where I_0 is an index set. We set

$$\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^- = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}.$$

For $i \in I_0$ let $h_i \in \mathfrak{h}$, $\varpi_i \in \mathfrak{h}^*$ and $s_i \in W$ be the simple coroot, the fundamental weight, the simple reflection corresponding to i respectively. Take $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ satisfying $[e_i, f_i] = h_i$. Let $(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the invariant symmetric bilinear form such that $(\alpha, \alpha) = 2$ for short roots α. Set

$$d_i = (\alpha_i, \alpha_i)/2 \quad (i \in I_0), \qquad a_{ij} = \alpha_j(h_i) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \quad (i, j \in I_0).$$

For a subset I of I_0 we set

$$\Delta_I = \Delta \cap \sum_{i\in I} \mathbb{Z}\alpha_i, \qquad W_I = \langle s_i \mid i \in I \rangle,$$

$$\mathfrak{l}_I = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Delta_I} \mathfrak{g}_\alpha\right), \quad \mathfrak{n}_I^+ = \bigoplus_{\alpha\in\Delta^+\setminus\Delta_I} \mathfrak{g}_\alpha, \quad \mathfrak{n}_I^- = \bigoplus_{\alpha\in-\Delta^+\setminus\Delta_I} \mathfrak{g}_\alpha.$$

For a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ the enveloping algebra of $\mathfrak{a}$.

Let us recall the definition of the quantized enveloping algebra $U_q(\mathfrak{g})$ (Drinfel'd [1], Jimbo [7]). It is an associative algebra over the rational function field $\mathbb{C}(q)$ generated by the elements $\{E_i, F_i, K_i, K_i^{-1}\}_{i\in I_0}$ satisfying the following fundamental relations:

$$K_iK_j = K_jK_i,$$
$$K_iK_i^{-1} = K_i^{-1}K_i = 1,$$
$$K_iE_jK_i^{-1} = q_i^{a_{ij}}E_j,$$
$$K_iF_jK_i^{-1} = q_i^{-a_{ij}}F_j,$$
$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q_i}E_i^{1-a_{ij}-k}E_jE_i^k = 0 \quad (i \neq j),$$
$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q_i}F_i^{1-a_{ij}-k}F_jF_i^k = 0 \quad (i \neq j),$$

where $q_i = q^{d_i}$, and

$$[m]_t = \frac{t^m - t^{-m}}{t - t^{-1}}, \quad [m]_t! = \prod_{k=1}^{m}[k]_t \quad \begin{bmatrix}m\\ n\end{bmatrix}_t = \frac{[m]_t!}{[n]_t![m-n]_t!} \quad (m \geq n \geq 0).$$

For $i \in I_0$ and $n \in \mathbb{Z}_{\geq 0}$ we set

$$E_i^{(n)} = \frac{1}{[n]_{q_i}!}E_i^n, \qquad F_i^{(n)} = \frac{1}{[n]_{q_i}!}F_i^n.$$

The algebra $U_q(\mathfrak{g})$ is endowed with a Hopf algebra structure via the following formula:

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i,$$
$$\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,$$
$$S(K_i) = K_i^{-1}, \quad S(E_i) = -E_iK_i, \quad S(F_i) = -K_i^{-1}F_i,$$

where $\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ and $\varepsilon : U_q(\mathfrak{g}) \to \mathbb{C}(q)$ are the algebra homomorphisms giving the comultiplication and the counit respectively, and $S : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is the algebra anti-automorphism giving the antipode.

We define the adjoint action of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$ as follows. For x, $y \in U_q(\mathfrak{g})$ write $\Delta(x) = \sum_k x_k^1 \otimes x_k^2$ and set $(\operatorname{ad} x)(y) = \sum_k x_k^1 y S(x_k^2)$. Then

$$\operatorname{ad} : U_q(\mathfrak{g}) \to \operatorname{End}_{\mathbb{C}(q)}(U_q(\mathfrak{g}))$$

is a homomorphism of algebras.

Define subalgebras $U_q(\mathfrak{n}^{\pm})$, $U_q(\mathfrak{h})$ and $U_q(\mathfrak{l}_I)$ for $I \subset I_0$ by

$$U_q(\mathfrak{n}^+) = \langle E_i \mid i \in I_0 \rangle, \quad U_q(\mathfrak{n}^-) = \langle F_i \mid i \in I_0 \rangle, \quad U_q(\mathfrak{h}) = \langle K_i^{\pm 1} \mid i \in I_0 \rangle,$$
$$U_q(\mathfrak{l}_I) = \langle K_i^{\pm 1}, E_j, F_j \mid i \in I_0, j \in I \rangle.$$

For $i \in I_0$ define an algebra automorphism T_i of $U_q(\mathfrak{g})$ by

$$T_i(K_j) = K_j K_i^{-a_{ij}},$$
$$T_i(E_j) = \begin{cases} -F_i K_i & (i = j) \\ \sum_{k=0}^{-a_{ij}} (-q_i)^{-k} E_i^{(-a_{ij}-k)} E_j E_i^{(k)} & (i \neq j), \end{cases}$$
$$T_i(F_j) = \begin{cases} -K_i^{-1} E_i & (i = j) \\ \sum_{k=0}^{-a_{ij}} (-q_i)^{k} F_i^{(k)} F_j F_i^{(-a_{ij}-k)} & (i \neq j). \end{cases}$$

(see Lusztig [9]). For $w \in W$ choose a reduced expression $w = s_{i_1} \cdots s_{i_k}$ and set $T_w = T_{i_1} \cdots T_{i_k}$. It is known that T_w does not depend on the choice of the reduced expression.

For $I \subset I_0$ let w_I be the longest element of W_I and define a subalgebra $U_q(\mathfrak{n}_I^-)$ by

$$U_q(\mathfrak{n}_I^-) = U_q(\mathfrak{n}^-) \cap T_{w_I}^{-1} U_q(\mathfrak{n}^-).$$

Let w_0 be the longest element of W. Take a reduced expression $w_I w_0 = s_{i_1} \cdots s_{i_m}$ of $w_I w_0$ and set

$$\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), \quad Y_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k}), \quad Y_{\beta_k}^{(n)} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k}^{(n)})$$

for $k = 1, \dots, m$. Then it is known that $\{\beta_k \mid 1 \leq k \leq m\} = \Delta^+ \backslash \Delta_I$, and that $\{Y_{\beta_1}^{(d_1)} \cdots Y_{\beta_m}^{(d_m)} \mid d_1, \dots, d_m \in \mathbb{Z}_{\geq 0}\}$ is a basis of $U_q(\mathfrak{n}_I^-)$. We note that this basis depends on the choice of the reduced expression of $w_I w_0$ in general.

Let $\tau : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ be the algebra anti-automorphism given by

$$\tau(K_i) = K_i^{-1}, \quad \tau(E_i) = E_i, \quad \tau(F_i) = F_i \quad (i \in I_0).$$

LEMMA 1.1. (i) $\tau T_{w_I}(U_q(\mathfrak{n}_I^-)) = U_q(\mathfrak{n}_I^-)$.

(ii) *Let $i, j \in I$ be such that $w_I(\alpha_i) = -\alpha_j$. Then we have*

$$(\operatorname{ad} F_i)(\tau T_{w_I}(x)) = \tau T_{w_I}((\operatorname{ad} E_j)(x)), \quad (\operatorname{ad} E_i)(\tau T_{w_I}(x)) = \tau T_{w_I}((\operatorname{ad} F_j)(x)),$$
$$(\operatorname{ad} K_i)(\tau T_{w_I}(x)) = \tau T_{w_I}((\operatorname{ad}(K_j^{-1}))(x))$$

for any $x \in U_q(\mathfrak{g})$.

PROOF. (i) We have $\tau T_k = T_k^{-1}\tau$ for any $k \in I_0$, and hence $\tau T_w = T_{w^{-1}}^{-1}\tau$ for any $w \in W$. Hence

$$\begin{aligned}\tau T_{w_I}(U_q(\mathfrak{n}_I^-)) &= \tau T_{w_I}(U_q(\mathfrak{n}^-) \cap T_{w_I}^{-1}(U_q(\mathfrak{n}^-)) \\ &= T_{w_I}^{-1}(U_q(\mathfrak{n}^-)) \cap U_q(\mathfrak{n}^-) = U_q(\mathfrak{n}_I^-).\end{aligned}$$

(ii) We have

$$\tau T_{w_I}(E_j) = \tau T_{w_I s_j} T_{s_j}(E_j) = \tau T_{w_I s_j}(-F_j K_j) = -\tau(F_i K_i) = -K_i^{-1} F_i.$$

Here we have used the formula:

$$T_y(F_k) = F_\ell, \quad T_y(K_k) = K_\ell \quad (y \in W, k, \ell \in I_0, y(\alpha_k) = \alpha_\ell)$$

(see Lusztig [9]). Hence

$$\begin{aligned}\tau T_{w_I}((\operatorname{ad} E_j)(x)) &= \tau T_{w_I}((E_j x - x E_j)K_j) = K_i(z(-K_i^{-1}F_i) - (-K_i^{-1}F_i)z) \\ &= F_i z - (K_i z K_i^{-1})F_i = (\operatorname{ad} F_i)(z)\end{aligned}$$

with $z = \tau T_{w_I}(x)$. Other formulas are proved similarly. □

PROPOSITION 1.2. $(\operatorname{ad} U_q(\mathfrak{l}_I))(U_q(\mathfrak{n}_I^-)) \subset U_q(\mathfrak{n}_I^-)$.

PROOF. We see easily that $(\operatorname{ad} U_q(\mathfrak{h}))(U_q(\mathfrak{n}_I^-)) = U_q(\mathfrak{n}_I^-)$. Hence it is sufficient to show that $U_q(\mathfrak{n}_I^-)$ is stable under $\operatorname{ad} E_i$, $\operatorname{ad} F_i$ for $i \in I$.

Let $i \in I$ and define $j \in I$ by $\alpha_j = -w_I(\alpha_i)$. By Lemma 1.1 we have

$$\begin{aligned}(\operatorname{ad} E_i)(U_q(\mathfrak{n}_I^-)) &= T_{w_I}^{-1}\tau^{-1}\tau T_{w_I}(\operatorname{ad} E_i)(U_q(\mathfrak{n}_I^-)) = T_{w_I}^{-1}\tau^{-1}(\operatorname{ad} F_j)(\tau T_{w_I} U_q(\mathfrak{n}_I^-)) \\ &\subset T_{w_I}^{-1}\tau^{-1}(\operatorname{ad} F_j)(U_q(\mathfrak{n}^-)) \subset T_{w_I}^{-1}(U_q(\mathfrak{n}^-)).\end{aligned}$$

Let us show $(\operatorname{ad} E_i)(U_q(\mathfrak{n}^-)) \subset U_q(\mathfrak{n}^-)$. For any $y \in U_q(\mathfrak{n}^-)$ we can write

$$[E_i, y] = K_i r_1(y) - r_2(y) K_i^{-1} \quad (r_1(y), r_2(y) \in U_q(\mathfrak{n}^-)),$$

and hence $(\operatorname{ad} E_i)(y) = K_i r_1(y) K_i - r_2(y)$. On the other hand by Jantzen [5] we have

$$\{y \in U_q(\mathfrak{n}^-) \mid r_1(y) = 0\} = U_q(\mathfrak{n}^-) \cap T_i^{-1} U_q(\mathfrak{n}^-).$$

Hence we have to show $U_q(\mathfrak{n}^-) \cap T_{w_I}^{-1} U_q(\mathfrak{n}^-) \subset U_q(\mathfrak{n}^-) \cap T_i^{-1} U_q(\mathfrak{n}^-)$. It is sufficient to show for any $y \in W$ and $k \in I_0$ satisfying $s_k y < y$ that $U_q(\mathfrak{n}^-) \cap T_{s_k y}^{-1} U_q(\mathfrak{n}^-) \subset U_q(\mathfrak{n}^-) \cap T_y^{-1} U_q(\mathfrak{n}^-)$. This follows from Lusztig [9]. Therefore we have $(\operatorname{ad} E_i)(U_q(\mathfrak{n}_I^-)) \subset U_q(\mathfrak{n}_I^-)$. Then we see from Lemma 1.1 that $(\operatorname{ad} F_\ell)(U_q(\mathfrak{n}_I^-)) \subset U_q(\mathfrak{n}_I^-)$. □

Let $U_q^0(\mathfrak{n}^-)$ be the $\mathbf{C}[q^{\pm 1}]$-subalgebra of $U_q(\mathfrak{n}^-)$ generated by $\{F_i^{(n)} \mid i \in I_0, n \in \mathbf{Z}_{\geq 0}\}$. We have a natural $\mathbf{C}$-algebra homomorphism $\varphi : U_q^0(\mathfrak{n}^-) \to U(\mathfrak{n}^-)$ given by $F_i^{(n)} \to f_i^n/n!$, and it induces the isomorphism $\mathbf{C} \otimes_{\mathbf{C}[q^{\pm 1}]} U_q^0(\mathfrak{n}^-) \simeq U(\mathfrak{n}^-)$ where $\mathbf{C}[q^{\pm 1}] \to \mathbf{C}$ is given by $q \mapsto 1$. For $I \subset I_0$ the restriction of φ to $U_q^0(\mathfrak{n}_I^-) = U_q^0(\mathfrak{n}^-) \cap U_q(\mathfrak{n}_I^-)$ gives a surjective $\mathbf{C}$-algebra homomorphism $\varphi_I : U_q^0(\mathfrak{n}_I^-) \to U(\mathfrak{n}_I^-)$ inducing $\mathbf{C} \otimes_{\mathbf{C}[q^{\pm 1}]} U_q^0(\mathfrak{n}_I^-) \simeq U(n_I^-)$.

For $N \in \mathbf{Z}_{>0}$ set

$$U_{q,N}(\mathfrak{g}) = \mathbf{C}(q^{1/N}) \otimes_{\mathbf{C}(q)} U_q(\mathfrak{g}),$$

and let $U_{q,N}(\mathfrak{n}^{\pm})$, $U_{q,N}(\mathfrak{h})$, $U_{q,N}(\mathfrak{l}_I)$, $U_{q,N}(\mathfrak{n}_I^-)$ be the $\mathbf{C}(q^{1/N})$-subalgebras of $U_{q,N}(\mathfrak{g})$ generated by $U_q(\mathfrak{n}^{\pm})$, $U_q(\mathfrak{h})$, $U_q(\mathfrak{l}_I)$, $U_q(\mathfrak{n}_I^-)$ respectively.

2. Highest weight modules

For a $U(\mathfrak{h})$-module M and $\mu \in \mathfrak{h}^*$ we set

$$M_\mu = \{m \in M \mid hm = \mu(h)m \quad (h \in \mathfrak{h})\}.$$

It is called a weight space of M with weight μ. A $U(\mathfrak{h})$-module M satisfying $M = \bigoplus_\mu M_\mu$ and $\dim M_\mu < \infty$ for any μ is called a weight module. We define its character $\mathrm{ch}(M)$ as the formal infinite sum

$$\mathrm{ch}(M) = \sum_\mu \dim M_\mu \, e^\mu.$$

A $U(\mathfrak{g})$-module M is called a highest weight module with highest weight $\lambda \in \mathfrak{h}^*$ if there exists $m \in M_\lambda \setminus \{0\}$ satisfying $M = U(\mathfrak{g})m$, $\mathfrak{n}^+ m = 0$. Such m is determined up to a nonzero constant multiple and is called the highest weight vector of M. For each $\lambda \in \mathfrak{h}^*$ there exists a unique (up to an isomorphism) irreducible highest weight module with highest weight λ, which we denote by $L(\lambda)$. Since highest weight modules are weight modules, their characters are defined. For $I \subset I_0$ set

$$\mathfrak{h}_I^* = \bigoplus_{i \in I_0 \setminus I} \mathbf{C}\varpi_i \subset \mathfrak{h}^*.$$

For $\lambda \in \mathfrak{h}_I^*$ we define a $U(\mathfrak{g})$-module $M_I(\lambda)$ by

$$M_I(\lambda) = U(\mathfrak{g}) \Big/ \left(\sum_{h \in \mathfrak{h}} U(\mathfrak{g})(h - \lambda(h)) + U(\mathfrak{g})\mathfrak{n}^+ + U(\mathfrak{g})(\mathfrak{l}_I \cap \mathfrak{n}^-) \right).$$

It is a highest weight module with highest weight λ and the highest weight vector $m_{I,\lambda} = \bar{1}$, where $\bar{1}$ denotes the element of $M_I(\lambda)$ corresponding to $1 \in U(\mathfrak{g})$. Moreover it is a rank one free $U(\mathfrak{n}_I^-)$-module generated by the highest weight vector $m_{I,\lambda}$, and hence we have

$$\mathrm{ch}(M_I(\lambda)) = \frac{e^\lambda}{\prod_{\alpha \in \Delta^+ \setminus \Delta_I}(1 - e^{-\alpha})}.$$

It contains a unique maximal proper submodule $K_I(\lambda)$, and we have $L(\lambda) = M_I(\lambda)/K_I(\lambda)$.

Now we define the corresponding notions for the quantized enveloping algebras. Set

$$\mathfrak{h}^*_{\mathbf{Z}} = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbf{Z} \ (i \in I_0)\} = \bigoplus_{i \in I_0} \mathbf{Z}\varpi_i \subset \mathfrak{h}^*.$$

For a $U_{q,N}(\mathfrak{h})$-module M the weight space M_μ with weight $\mu \in \mathfrak{h}^*_{\mathbf{Z}}/N$ is defined by

$$M_\mu = \{m \in M \mid K_i m = q_i^{\mu(h_i)} m \quad (i \in I_0)\}.$$

We call a $U_{q,N}(\mathfrak{h})$-module M a weight module if $M = \bigoplus_\mu M_\mu$ and $\dim M_\mu < \infty$ for any $\mu \in \mathfrak{h}^*_{\mathbf{Z}}/N$. Let M be a $U_{q,N}(\mathfrak{g})$-module. If there exists $m \in M_\lambda$ satisfying $U_{q,N}(\mathfrak{g})m = M$, $E_i m = 0$ $(i \in I_0)$, then M is called a highest weight module with highest weight λ and m is called its highest weight vector. There exists a unique irreducible highest weight module $L_{q,N}(\lambda)$ with highest weight λ. Highest weight modules are weight modules. For $I \subset I_0$ set

$$\mathfrak{h}^*_{I,\mathbf{Z}} = \bigoplus_{i \in I_0 \setminus I} \mathbf{Z}\varpi_i \subset \mathfrak{h}^*.$$

For $\lambda \in \mathfrak{h}^*_{I,\mathbf{Z}}/N$ we define a highest weight module $M_{I,q,N}(\lambda)$ by

$$M_{I,q,N}(\lambda) = U_{q,N}(\mathfrak{g}) \Big/ \left(\sum_{i \in I_0} U_{q,N}(\mathfrak{g})(K_i - q_i^{\lambda(h_i)}) + \sum_{i \in I_0} U_{q,N}(\mathfrak{g})E_i + \sum_{j \in I} U_{q,N}(\mathfrak{g})F_j \right).$$

Its highest weight vector is given by $m_{I,\lambda,q,N} = \bar{1}$. Since $M_{I,q,N}(\lambda)$ is a rank one free module generated by $m_{I,\lambda,q,N}$, we have

$$\mathrm{ch}(M_{I,q,N}(\lambda)) = \mathrm{ch}(M_I(\lambda)).$$

We have a unique maximal proper submodule $K_{I,q,N}(\lambda)$ of $M_{I,q,N}(\lambda)$, and hence $L_{q,N}(\lambda) = M_{I,q,N}(\lambda)/K_{I,q,N}(\lambda)$.

PROPOSITION 2.1. *Let $I \subset I_0$ and $\lambda \in \mathfrak{h}^*_{I,\mathbf{Z}}/N$. Let Y be a subset of $U_q^0(\mathfrak{n}_I^-)$ such that $Y m_{I,\lambda,q,N} \subset K_{I,q,N}(\lambda)$ and $U(\mathfrak{g})\varphi_I(Y)m_{I,\lambda} = K_I(\lambda)$. Then we have $U_{q,N}(\mathfrak{g})Y m_{I,\lambda,q,N} = K_{I,q,N}(\lambda)$ and $\mathrm{ch}(L_{q,N}(\lambda)) = \mathrm{ch}(L(\lambda))$.*

PROOF. Let M be any highest weight $U_{q,N}(\mathfrak{g})$-module with highest weight λ. Take a highest weight vector $m \in M$ and set

$$M^0 = U_q^0(\mathfrak{n}^-)m, \quad \overline{M}^0 = M^0|_{q=1} = \mathbf{C} \otimes_{\mathbf{C}[q^{\pm 1/N}]} M^0.$$

Then we can show as in Lusztig [8] that M^0 is stable under the actions of E_i, F_i, $(K_i - K_i^{-1})/(q_i - q_i^{-1})$ $(i \in I_0)$ and that $\overline{M}^0$ becomes a highest weight $U(\mathfrak{g})$-module with highest weight λ via the operators

$$e_i = \bar{E}_i, \quad f_i = \bar{F}_i, \quad h_i = \overline{\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}} \quad (i \in I_0).$$

In particular we have

$$\dim M_\mu = \dim(\overline{M}^0)_\mu \geq \dim L(\lambda)_\mu.$$

Now we set

$$M = M_{I,q,N}(\lambda)/U_{q,N}(\mathfrak{g})Ym_{I,\lambda,q,N}, \quad m = \overline{m_{I,\lambda,q,N}} \in M.$$

By the above argument $\overline{M}^0$ is a highest weight $U(\mathfrak{g})$-module with highest weight λ and the highest weight vector $\bar{m}$. Moreover, since $Ym = 0$, we have $\varphi_I(Y)\bar{m} = 0$. Hence we have $\overline{M}^0 \simeq L(\lambda)$. It follows that

$$\dim L_{q,N}(\lambda)_\mu \leq \dim M_\mu = \dim(\overline{M}^0)_\mu = \dim L(\lambda)_\mu \leq \dim L_{q,N}(\lambda)_\mu.$$

Therefore we have $M \simeq L_{q,N}(\lambda)$ and $\mathrm{ch}(L_{q,N}(\lambda)) = \mathrm{ch}(L(\lambda))$. □

3. Parabolic subalgebras with commutative nilpotent radicals

In the rest of this paper we fix $I \subset I_0$ satisfying $\mathfrak{n}_I^+ \neq \{0\}$ and $[\mathfrak{n}_I^+, \mathfrak{n}_I^+] = \{0\}$ (see, for example, [14] for the list of $(\mathfrak{g}, I)$'s satisfying the condition). We have $I = I_0 \backslash \{i_0\}$ for some $i_0 \in I_0$.

We set $\mathfrak{l} = \mathfrak{l}_I$, $\mathfrak{m}^\pm = \mathfrak{n}_I^\pm$ for simplicity.

PROPOSITION 3.1. *The element $Y_\beta \in U_q(\mathfrak{m}^-)$ for $\beta \in \Delta^+ \backslash \Delta_I$ does not depend on the choice of a reduced expression of $w_I w_0$.*

PROOF. For i, $j \in I_0$ set

$$r(i,j) = (\overbrace{i,j,i,j,\ldots}^{m_{ij}}),$$

where m_{ij} denotes the order of $s_i s_j \in W$. Let $s_{i_1} \cdots s_{i_r}$ be a reduced expression of $w \in W$. Then $s_{j_1} \cdots s_{j_r}$ is a reduced expression of w if and only if $(j_1, \ldots, j_r)$ can be obtained from $(i_1, \ldots, i_r)$ by successively exchanging a subsequence of the form $r(i,j)$ to $r(j,i)$.

We first show that for any reduced expression $s_{i_1} \cdots s_{i_r}$ of $w_I w_0$ the sequence $(i_1, \ldots, i_r)$ does not contain a subsequence of the form $r(i,j)$ with $m_{ij} \geq 3$. Assume that there exists a subsequence $r(i,j)$ with $m_{ij} = 3$ in $(i_1, \ldots, i_r)$. We have $(i_p, i_{p+1}, i_{p+2}) = (i,j,i)$ for some p. Set $y = s_{i_1} \cdots s_{i_{p-1}}$.

Then we have

$$\beta_p = y(\alpha_i), \quad \beta_{p+1} = ys_i(\alpha_j) = y(\alpha_i + \alpha_j), \quad \beta_{p+2} = ys_is_j(\alpha_i) = y(\alpha_j),$$

and hence $\beta_p + \beta_{p+2} = \beta_{p+1}$. This contradicts the commutativity of $\mathfrak{m}^-$. Thus the sequence $(i_1, \dots, i_r)$ does not contain a subsequence of the form $r(i,j)$ with $m_{ij} = 3$. Similarly we can show that there does not exist a subsequence of the form $r(i,j)$ with $m_{ij} = 4, 6$.

Therefore it is sufficient to show that for two reduced expressions

$$s_{i_1} \cdots s_{i_p} s_i s_j s_{j_1} \cdots s_{j_q}, \quad s_{i_1} \cdots s_{i_p} s_j s_i s_{j_1} \cdots s_{j_q}, \quad (s_i s_j = s_j s_i)$$

of $w_I w_0$ the resulting Y_β's are the same. This follows from $T_i(F_j) = F_j$, $T_j(F_i) = F_i$, and $T_i T_j = T_j T_i$. $\square$

We fix a reduced expression $w_I w_0 = s_{i_1} \cdots s_{i_r}$ and set $\beta_p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p})$. Set

$$Q^+ = \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \alpha_i, \quad Q_I^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i,$$

$$U_q(\mathfrak{m}^-)^m = \sum_{p_1, \dots, p_m = 1}^{r} \mathbb{C}(q) Y_{\beta_{p_1}} \cdots Y_{\beta_{p_m}} \quad (m \geq 0).$$

LEMMA 3.2. *We have*

$$U_q(\mathfrak{m}^-) = \bigoplus_{m=0}^{\infty} U_q(\mathfrak{m}^-)^m.$$

$$U_q(\mathfrak{m}^-)^m = \bigoplus_{\sum_p m_p = m} \mathbb{C}(q) Y_{\beta_1}^{(m_1)} \cdots Y_{\beta_r}^{(m_r)} = \bigoplus_{\gamma \in m\alpha_{i_0} + Q_I^+} U_q(\mathfrak{m}^-)_{-\gamma}.$$

Here $U_q(\mathfrak{m}^-)_{-\gamma}$ is the weight space with respect to the adjoint action of $U_q(\mathfrak{h})$ on $U_q(\mathfrak{m}^-)$.

PROOF. Set

$$V_0^m = \bigoplus_{\sum_p m_p = m} \mathbb{C}(q) Y_{\beta_1}^{(m_1)} \cdots Y_{\beta_r}^{(m_r)}, \quad V_1^m = \bigoplus_{\gamma \in m\alpha_{i_0} + Q_I^+} U_q(\mathfrak{m}^-)_{-\gamma}.$$

By $\beta_p \in \alpha_{i_0} + Q_I^+$ we have $V_0^m \subset U_q(\mathfrak{m}^-)^m \subset V_1^m$. Since $U_q(\mathfrak{m}^-) = \bigoplus_m V_0^m$, we obtain $V_0^m = U_q(\mathfrak{m}^-)^m = V_1^m$ and $U_q(\mathfrak{m}^-) = \bigoplus_{m=0}^{\infty} U_q(\mathfrak{m}^-)^m$. $\square$

By Lemma 3.2 we can write

$$(3.1) \qquad Y_{\beta_{p_1}} Y_{\beta_{p_2}} = \sum_{\substack{s_1 \leq s_2 \\ \beta_{p_1} + \beta_{p_2} = \beta_{s_1} + \beta_{s_2}}} a_{s_1, s_2}^{p_1, p_2} Y_{\beta_{s_1}} Y_{\beta_{s_2}} \quad (a_{s_1, s_2}^{p_1, p_2} \in \mathbb{C}(q))$$

for $p_1 > p_2$.

PROPOSITION 3.3. *The $\mathbf{C}(q)$-algebra $U_q(\mathfrak{m}^-)$ is generated by the elements $\{Y_{\beta_p} \mid 1 \leq p \leq r\}$ satisfying the fundamental relations* (3.1) *for $p_1 > p_2$.*

PROOF. It is sufficient to show that any element of the form $Y_{\beta_{t_1}} \cdots Y_{\beta_{t_n}}$ $(1 \leq t_i \leq r)$ can be rewritten as a linear combination of the elements of the form $Y_{\beta_{s_1}} \cdots Y_{\beta_{s_n}}$ $(1 \leq s_1 \leq \cdots \leq s_n \leq r)$ by a successive use of the relations (3.1) for $p_1 > p_2$. For $1 \leq k \leq r$ let V_k be the subalgebra of $U_q(\mathfrak{m}^-)$ generated by $\{Y_{\beta_p} \mid 1 \leq p \leq k\}$. By Lusztig [9] we have

$$V_k = \bigoplus_{m_1,\dots,m_k} \mathbf{C}(q)\, Y_{\beta_1}^{(m_1)} \cdots Y_{\beta_k}^{(m_k)}.$$

We shall show by the induction on k that any element of the form $Y_{\beta_{t_1}} \cdots Y_{\beta_{t_n}}$ $(1 \leq t_i \leq k)$ can be rewritten as a linear combination of the elements of the form $Y_{\beta_{s_1}} \cdots Y_{\beta_{s_n}}$ $(1 \leq s_1 \leq \cdots \leq s_n \leq k)$ by a successive use of the relations (3.1) for $k \geq p_1 > p_2$. It is trivial for $k = 1$. Assume that $k \geq 2$ and the assertion is proved up to $k-1$. We shall show the statement by induction on n. It is obvious for $n = 0$. Assume that $n > 0$ and the statement is already proved up to $n-1$. Take j such that $t_1 = \cdots = t_j = k$, $t_{j+1} \neq k$. We use induction on j. Assume that $j = 0$. Then we have $t_1 \neq k$. By using the inductive hypothesis on n we may assume that $t_2 \leq \cdots \leq t_n \leq k$. If $t_n < k$, then we have $t_i \leq k-1$ for any i, and hence the statement holds by the inductive hypothesis on k. If $t_n = k$, then we can apply the inductive hypothesis on n to $Y_{\beta_{t_1}} \cdots Y_{\beta_{t_{n-1}}}$, and hence the statement also holds. Assume $0 < j < n$. Then we have

$$Y_{\beta_{t_1}} \cdots Y_{\beta_{t_n}} = Y_{\beta_k}^j Y_{\beta_{t_{j+1}}} \cdots Y_{\beta_{t_n}}$$

with $t_{j+1} \neq k$. Applying (3.1) for $(p_1, p_2) = (k, t_{j+1})$ we obtain

$$Y_{\beta_k} Y_{\beta_{t_{j+1}}} = \sum_{\substack{s_1 \leq s_2 \leq k \\ \beta_k + \beta_{t_{j+1}} = \beta_{s_1} + \beta_{s_2}}} a_{s_1,s_2}^{k,t_{j+1}} Y_{\beta_{s_1}} Y_{\beta_{s_2}}.$$

Since $s_1 < k$ by the condition $\beta_k + \beta_{t_{j+1}} = \beta_{s_1} + \beta_{s_2}$, we can apply the inductive hypothesis on j to $Y_{\beta_k}^{j-1} Y_{\beta_{s_1}} Y_{\beta_{s_2}} Y_{\beta_{t_{j+2}}} \cdots Y_{\beta_{t_n}}$, and the statement holds. If $j = n$, then we have $Y_{\beta_{t_1}} \cdots Y_{\beta_{t_n}} = Y_{\beta_k}^n$, and the statement is obvious. $\square$

Since $\mathfrak{m}^-$ is commutative, $U(\mathfrak{m}^-)$ is isomorphic to the symmetric algebra $S(\mathfrak{m}^-)$. By identifying $\mathfrak{m}^-$ with $(\mathfrak{m}^+)^*$ via the Killing form of $\mathfrak{g}$, $S(\mathfrak{m}^-)$ is naturally identified with the algebra $\mathbf{C}[\mathfrak{m}^+]$ of polynomial functions on $\mathfrak{m}^+$. Hence we have an identification $U(\mathfrak{m}^-) = \mathbf{C}[\mathfrak{m}^+]$. We denote by $\mathbf{C}[\mathfrak{m}^+]^m$ $(m \in \mathbf{Z}_{\geq 0})$ the subspace of $\mathbf{C}[\mathfrak{m}^+]$ consisting of homogeneous polynomials with degree m.

Set

$$\mathfrak{h}_{\mathbf{Z}}^*(I,+) = \{\lambda \in \mathfrak{h}_{\mathbf{Z}}^* \mid \lambda(h_i) \geq 0 \ (i \in I)\}.$$

For $\lambda \in \mathfrak{h}_{\mathbf{Z}}^*(I,+)$ we denote the finite dimensional irreducible $U(\mathfrak{l})$-module (resp. $U_q(\mathfrak{l})$-module) with highest weight λ by $V(\lambda)$ (resp. $V_q(\lambda)$). We can decompose the finite dimensional $\mathfrak{l}$-module $\mathbf{C}[\mathfrak{m}^+]^m$ into a direct sum of submodules isomorphic to $V(\lambda)$ for some $\lambda \in \mathfrak{h}_{\mathbf{Z}}^*(I,+)$. Moreover, it is known that

$$\dim \mathrm{Hom}_{\mathfrak{l}}(V(\lambda), \mathbf{C}[\mathfrak{m}^+]) \geq 1 \quad (\lambda \in \mathfrak{h}_{\mathbf{Z}}^*(I,+)),$$

and hence we have

$$\mathbf{C}[\mathfrak{m}^+]^m \simeq \bigoplus_{\lambda \in \Gamma^m} V(\lambda)$$

for finite subsets Γ^m of $\mathfrak{h}_{\mathbf{Z}}^*(I,+)$ satisfying $\Gamma^m \cap \Gamma^{m'} = \varnothing$ for $m \neq m'$ (see Schmid [11], Takeuchi [12], Johnson [6] for the explicit description of Γ^m). On the other hand, since $U_q(\mathfrak{m}^-)^m$ is a finite dimensional $U_q(\mathfrak{l})$-module whose character is the same as that of $\mathbf{C}[\mathfrak{m}^+]^m$, we have

$$U_q(\mathfrak{m}^-)^m \simeq \bigoplus_{\lambda \in \Gamma^m} V_q(\lambda).$$

Let L be the algebraic group corresponding to $\mathfrak{l}$. It is known that the set of L-orbits on $\mathfrak{m}^+$ is a finite totally ordered set with respect to the closure relation. Hence we can label the orbits by

$$\{L\text{-orbits on } \mathfrak{m}^+\} = \{C_0, C_1, \ldots, C_t\}, \quad \{0\} = C_0 \subset \bar{C}_1 \subset \cdots \subset \bar{C}_t = \mathfrak{m}^+.$$

Set

$$\mathscr{I}(\bar{C}_p) = \{f \in \mathbf{C}[\mathfrak{m}^+] \mid f(\bar{C}_p) = 0\}.$$

Since $\mathscr{I}(\bar{C}_p)$ is an $\mathfrak{l}$-submodule of $\mathbf{C}[\mathfrak{m}^+]$, we have

$$\mathscr{I}(\bar{C}_p) = \bigoplus_m \mathscr{I}^m(\bar{C}_p), \quad \mathscr{I}^m(\bar{C}_p) = \mathscr{I}(\bar{C}_p) \cap \mathbf{C}[\mathfrak{m}^+]^m \simeq \bigoplus_{\lambda \in \Gamma_p^m} V(\lambda)$$

for a subset Γ_p^m of Γ^m. Moreover the following fact is known (see, for example, [14]):

PROPOSITION 3.4. *Let* $p = 0, \ldots, t-1$.

(i) $\mathscr{I}^m(\bar{C}_p) = 0$ *for* $m \leq p$.

(ii) $\mathscr{I}^{p+1}(\bar{C}_p)$ *is an irreducible* $\mathfrak{l}$*-module, i.e.* Γ_p^{p+1} *consists of a single element* ν_p.

(iii) $\mathscr{I}(\bar{C}_p)$ *is generated by* $\mathscr{I}^{p+1}(\bar{C}_p)$ *as an ideal of* $\mathbf{C}[\mathfrak{m}^+]$.

PROPOSITION 3.5. *For* $p = 0, \ldots, t-1$ *there exists a unique* $\lambda_p \in \mathfrak{h}_I^*$ *such that* $K_I(\lambda_p) = \mathscr{I}(\bar{C}_p) m_{I,\lambda_p}$. *Moreover, we have* $\lambda_p \in \mathfrak{h}_{I,\mathbf{Z}}^*/2$.

Let v^p be the highest weight vector of the $\mathfrak{l}$-module $\mathscr{I}^{p+1}(\bar{C}_p)(\simeq V(\nu_p))$. Then we have

$$\begin{aligned} K_I(\lambda_p) &= \mathscr{I}(\bar{C}_p)m_{I,\lambda_p} = U(\mathfrak{m}^-)\mathscr{I}^{p+1}(\bar{C}_p)m_{I,\lambda_p} \\ &= U(\mathfrak{m}^-)((\operatorname{ad} U(\mathfrak{l}\cap\mathfrak{n}^-))(v^p))m_{I,\lambda_p} \\ &= U(\mathfrak{m}^-)(U(\mathfrak{l}\cap\mathfrak{n}^-))v^p m_{I,\lambda_p} = U(\mathfrak{n}^-)v^p m_{I,\lambda_p} \end{aligned}$$

and hence $K_I(\lambda_p)$ is a highest weight module with highest weight $\lambda_p + \nu_p$.

We set

$$\mathscr{I}_q^m(\bar{C}_p) = \bigoplus_{\lambda\in\Gamma_p^m} V_q(\lambda) \subset U_q(\mathfrak{m}^-)^m, \quad \mathscr{I}_q(\bar{C}_p) = \bigoplus_m \mathscr{I}_q^m(\bar{C}_p) \subset U_q(\mathfrak{m}^-),$$

$$\mathscr{I}_{q,N}^m(\bar{C}_p) = \mathbf{C}(q^{1/N})\otimes_{\mathbf{C}(q)}\mathscr{I}_q^m(\bar{C}_p) \subset U_{q,N}(\mathfrak{m}^-)^m,$$

$$\mathscr{I}_{q,N}(\bar{C}_p) = \bigoplus_m \mathscr{I}_{q,N}^m(\bar{C}_p) \subset U_{q,N}(\mathfrak{m}^-).$$

Here we identify $U_q(\mathfrak{m}^-)^m$ with $\bigoplus_{\lambda\in\Gamma^m} V_q(\lambda)$.

PROPOSITION 3.6. *For $p = 0,\dots,t-1$ we have*

$$\operatorname{ch}(L_{q,2}(\lambda_p)) = \operatorname{ch}(L(\lambda_p)), \quad K_{I,q,2}(\lambda_p) = U_{q,2}(\mathfrak{m}^-)\mathscr{I}_{q,2}^{p+1}(\bar{C}_p)m_{I,\lambda_p,q,2}.$$

PROOF. We shall only give a sketch of the proof. We can prove a quantum analogue of the determinant formula for the contravariant forms on generalized Verma modules given by Jantzen [4]. It implies that $K_{I,q,N}(\lambda)_\mu = 0$ if and only if $K_I(\lambda)_\mu = 0$. In particular, we have $K_{I,q,2}(\lambda_p)_{\lambda_p+\nu_p} \neq 0$ and $K_{I,q,2}(\lambda_p)_{\lambda_p+\nu_p+\alpha_i} = 0$ for any $i\in I_0$. Let $vm_{I,\lambda_p,q,2}$ ($v\in U_{q,2}(\mathfrak{m}^-)_{\nu_p}$) be a nonzero element of $K_{I,q,2}(\lambda_p)_{\lambda_p+\nu_p}$. Then for $i\in I$ we have

$$\begin{aligned} ((\operatorname{ad} E_i)(v))m_{I,\lambda_p,q,2} &= (E_i v - vE_i)K_i m_{I,\lambda_p,q,2} \\ &\in \mathbf{C}(q^{1/2})E_i v m_{I,\lambda_p,q,2} \subset K_{I,q,2}(\lambda_p)_{\lambda_p+\nu_p+\alpha_i} = \{0\}. \end{aligned}$$

Hence $(\operatorname{ad} E_i)(v) = 0$ for any $i\in I$. It follows that v is a highest weight vector of the $U_{q,2}(\mathfrak{l})$-module $V_{q,2}(\nu_p)$. We may assume $v\in U_q^0(\mathfrak{m}^-)$ and $\varphi_I(v)\neq 0$. By Proposition 2.1 we conclude that $\operatorname{ch}(L_{q,2}(\lambda_p)) = \operatorname{ch}(L(\lambda_p))$ and $K_{I,q,2}(\lambda_p) = U_{q,2}(\mathfrak{g})vm_{I,\lambda_p,q,2}$. Then we have

$$\begin{aligned} K_{I,q,2}(\lambda_p) &= U_{q,2}(\mathfrak{g})vm_{I,\lambda_p,q,2} \\ &= U_{q,2}(\mathfrak{m}^-)(U_{q,2}(\mathfrak{l})\cap U_{q,2}(\mathfrak{n}^-))U_{q,2}(\mathfrak{h})U_{q,2}(\mathfrak{n}^+)vm_{I,\lambda_p,q,2} \\ &= U_{q,2}(\mathfrak{m}^-)(U_{q,2}(\mathfrak{l})\cap U_{q,2}(\mathfrak{n}^-))vm_{I,\lambda_p,q,2} \\ &= U_{q,2}(\mathfrak{m}^-)((\operatorname{ad}(U_{q,2}(\mathfrak{l})\cap U_{q,2}(\mathfrak{n}^-))(v))m_{I,\lambda_p,q,2} \\ &= U_{q,2}(\mathfrak{m}^-)\mathscr{I}_{q,2}^{p+1}(\bar{C}_p)m_{I,\lambda_p,q,2}. \end{aligned}$$

□

Theorem 3.7. *We have*

$$\mathscr{I}_q(\bar{C}_p) = U_q(\mathfrak{m}^-)\mathscr{I}_q^{p+1}(\bar{C}_p) = \mathscr{I}_q^{p+1}(\bar{C}_p)U_q(\mathfrak{m}^-).$$

Proof. By Proposition 3.6 we have

$$\mathrm{ch}(U_q(\mathfrak{m}^-)\mathscr{I}_q^{p+1}(\bar{C}_p)) = \mathrm{ch}(U_{q,2}(\mathfrak{m}^-)\mathscr{I}_{q,2}^{p+1}(\bar{C}_p)) = \mathrm{ch}(\mathscr{I}(\bar{C}_p)),$$

and hence $\mathscr{I}_q(\bar{C}_p) = U_q(\mathfrak{m}^-)\mathscr{I}_q^{p+1}(\bar{C}_p)$. Let us show $U_q(\mathfrak{m}^-)\mathscr{I}_q^{p+1}(\bar{C}_p) = \mathscr{I}_q^{p+1}(\bar{C}_p)U_q(\mathfrak{m}^-)$. Since τT_{w_I} is an anti-automorphism of the algebra $U_q(\mathfrak{m}^-)$ (see Lemma 1.1), it is sufficient to show that τT_{w_I} preserves $\mathscr{I}_q^{p+1}(\bar{C}_p)$. Since $U_q(\mathfrak{m}^-)$ is a multiplicity free $U_q(\mathfrak{l})$-module, we have only to show that $\tau T_{w_I}(V_q(\lambda))$ is a $U_q(\mathfrak{l})$-submodule isomorphic to $V_q(\lambda)$ for any $\lambda \in \bigcup_m \Gamma^m$. By Lemma 1.1 we see easily that $\tau T_{w_I}(V_q(\lambda))$ is an irreducible $U_q(\mathfrak{l})$-module with lowest weight $w_I(\lambda)$. Hence we have $\tau T_{w_I}(V_q(\lambda)) \simeq V_q(\lambda)$. $\square$

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