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Tail asymptotics for the extremes of bivariate Gaussian random fields

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Let $\{X(t) = (X_1(t), X_2(t))^T, t \in \mathbb{R}^N\}$ be an $\mathbb{R}^2$-valued continuous locally stationary Gaussian random field with $\mathbb{E}[X(t)] = \mathbf{0}$. For any compact sets $A_1, A_2 \subset \mathbb{R}^N$, precise asymptotic behavior of the excursion probability

$$\mathbb{P}\Big(\max_{s\in A_1} X_1(s) > u, \max_{t\in A_2} X_2(t) > u\Big) \qquad \text{as } u \to \infty$$

is investigated by applying the double sum method. The explicit results depend not only on the smoothness parameters of the coordinate fields X_1 and X_2, but also on their maximum correlation ρ.

Keywords: bivariate Gaussian field; bivariate Matérn field; double extremes; double sum method; excursion probability

1. Introduction

For a real-valued Gaussian random field $X = \{X(t), t \in T\}$, where T is the parameter set, defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the excursion probability $\mathbb{P}\{\sup_{t\in T} X(t) > u\}$ has been studied extensively. Extending the seminal work of Pickands [22], Piterbarg [23] developed a systematic theory on asymptotics of the aforementioned excursion probability for a broad class of Gaussian random fields. Their method, which is called the double sum method, has been further extended by Chan and Lai [9] to non-Gaussian random fields and, recently, by Dębicki *et al.* [11] to a non-stationary Gaussian random field $\{X(s,t), (s,t) \in \mathbb{R}^2\}$ whose variance function attains its maximum on a finite number of disjoint line segments. For smooth Gaussian random fields, more accurate approximation results have been established by using integral and differential-geometric methods (see, e.g., Adler [3], Adler and Taylor [4], Azaïs and Wschebor [7] and the references therein). For Gaussian and asymptotically Gaussian random fields, the change of measure method was developed by Nardi, Siegmund and Yakir [21] and Yakir [27]. Many of the results in the aforementioned references have found important applications in statistics and other scientific areas. We refer to Adler, Taylor and Worsley [2] and Yakir [27] for further information.

However, only a few authors have studied the excursion probability of multivariate random fields. Piterbarg and Stamatovich [24] and Dębicki *et al.* [12] established large deviation results for the excursion probability in multivariate case. Anshin [5] obtained precise asymptotics for a special class of nonstationary bivariate Gaussian processes, under quite restrictive conditions. Hashorva and Ji [16] recently derived precise asymptotics for the excursion probability of a bivariate fractional Brownian motion with constant cross correlation. The last two papers only consider multivariate processes on the real line $\mathbb{R}$ with specific cross dependence structures. Cheng and Xiao [10] established a precise approximation to the excursion probability by using the mean Euler characteristics of the excursion set for a broad class of smooth bivariate Gaussian random fields on $\mathbb{R}^N$. In the present paper, we investigate asymptotics of the excursion probability of non-smooth bivariate Gaussian random fields on $\mathbb{R}^N$, where the methods are totally different from the smooth case.

Our work is also motivated by the recent increasing interest in using multivariate random fields for modeling multivariate measurements obtained at spatial locations (see, e.g., Gelfand *et al.* [14], Wackernagel [26]). Several classes of multivariate spatial models have been introduced by Gneiting, Kleiber and Schlather [15], Apanasovich, Genton and Sun [6] and Kleiber and Nychka [17]. We will show in Section 2 that the main results of this paper are applicable to bivariate Gaussian random fields with Matérn cross-covariances introduced by Gneiting, Kleiber and Schlather [15]. Furthermore, we expect that the excursion probabilities considered in this paper will have interesting statistical applications.

Let $\{X(t), t \in \mathbb{R}^N\}$ be an $\mathbb{R}^2$-valued (not-necessarily stationary) Gaussian random field with $\mathbb{E}[X(t)] = \mathbf{0}$. We write $X(t) \triangleq (X_1(t), X_2(t))^T$ and define

$$r_{ij}(s,t) := \mathbb{E}\big[X_i(s)X_j(t)\big], \qquad i, j = 1, 2. \tag{1.1}$$

Let $|t| := \sqrt{\sum_{j=1}^N t_j^2}$ be the l^2-norm of a vector $t \in \mathbb{R}^N$. Throughout this paper, we impose the following assumptions.

(i) $r_{ii}(s,t) = 1 - c_i|t-s|^{\alpha_i} + o(|t-s|^{\alpha_i})$, where $\alpha_i \in (0,2)$ and $c_i > 0$ $(i = 1, 2)$ are constants.

(ii) $|r_{ii}(s,t)| < 1$ for all $|t-s| > 0$, $i = 1, 2$.

(iii) $r_{12}(s,t) = r_{21}(s,t) := r(|t-s|)$. Namely, the cross correlation is isotropic.

(iv) The function $r(\cdot) : [0,\infty) \to \mathbb{R}$ attains maximum only at zero with $r(0) = \rho \in (0,1)$, i.e., $|r(t)| < \rho$ for all $t > 0$. Moreover, we assume $r'(0) = 0$, $r''(0) < 0$ and there exists $\eta > 0$, for any $s \in [0,\eta]$, $r''(s)$ exists and continuous.

The cross correlation defined here is meaningful and common in spatial statistics where it is usually assumed that the correlation decreases as the distance between two observations increases (see, e.g., Gelfand *et al.* [14], Gneiting, Kleiber and Schlather [15]). We only assume that the cross correlation is twice continuously differentiable around the area where the maximum correlation is attained, which is a weaker assumption than that in Cheng and Xiao [10] who considered smooth bivariate Gaussian fields.

For any compact sets $A_1, A_2 \subset \mathbb{R}^N$, we investigate the asymptotic behavior of the following excursion probability

$$\mathbb{P}\Big(\max_{s\in A_1} X_1(s) > u, \max_{t\in A_2} X_2(t) > u\Big) \qquad \text{as } u \to \infty. \tag{1.2}$$

The main results of this paper are Theorems 2.1 and 2.2 below, which demonstrate that the excursion probability (1.2) depends not only on the smoothness parameters of the coordinate fields X_1 and X_2, but also on their maximum correlation ρ. The proofs of our Theorems 2.1 and 2.2 will be based on the double sum method. Compared with the earlier works of Ladneva and Piterbarg [18], Anshin [5] and Hashorva and Ji [16], the main difficulty in the present paper is that the correlation function of X_1 and X_2 attains its maximum over the set $D := \{(s,s) : s \in A_1 \cap A_2\}$ which may have different geometric configurations. Several non-trivial modifications for carrying out the arguments in the double sum method have to be made.

This paper raises several open questions. First, the cases of $\alpha_1 = 2$ or $\alpha_2 = 2$ have not been considered in this paper. The main difficulty is that, when $\alpha_1 = 2$, the sample functions of X_1 may either be differentiable or non-differentiable. In view of the method in this paper, the proof of Lemma 4.1 on the uniform convergence of finite dimensional distributions for bivariate process breaks down when $\alpha_1 = 2$ or $\alpha_2 = 2$. Studying the asymptotics of (1.2) when $\alpha_1 = 2$ or/and $\alpha_2 = 2$ requires different methods for dealing with differentiable or non-differentiable cases. When both X_1 and X_2 have twice continuously differentiable sample functions, this problem has been studied by Cheng and Xiao [10]. The authors plan to study the remaining cases in their future work. Second, it would be interesting to study the excursion probabilities when $\{X(t), t \in \mathbb{R}^N\}$ is anisotropic or non-stationary, or taking values in $\mathbb{R}^d$ with $d \geq 3$. In the last problem, the covariance and cross-covariance structures become more complicated. We expect that the pairwise maximum cross correlations and the size (e.g., the Lebesgue measure) of the set where all the pairwise cross correlations attain their maximum values (if not empty) will play an important role.

The rest of the paper is organized as follows. Section 2 states the main theorems with some discussions and provides an application of the main theorems to the bivariate Gaussian fields with Matérn cross-covariances introduced by Gneiting, Kleiber and Schlather [15]. We state the key lemmas and provide proofs of our main theorems in Section 3. The proofs of the lemmas are given in Section 4.

We end the Introduction with some notation. For any $t \in \mathbb{R}^N$, $|t|$ denotes its l^2-norm. An integer vector $\mathbf{k} \in \mathbb{Z}^N$ is written as $\mathbf{k} = (k_1, \ldots, k_N)$. For $\mathbf{k} \in \mathbb{Z}^N$ and $T \in \mathbb{R}_+ = [0, \infty)$, we define the cube $[\mathbf{k}T, (\mathbf{k}+1)T] := \prod_{i=1}^N [k_i T, (k_i+1)T]$. For any integer n, $mes_n(\cdot)$ denotes the n-dimensional Lebesgue measure. An unspecified positive and finite constant will be denoted by C_0. More specific constants are numbered by $C_1, C_2, \ldots$.

2. Main results and discussions

We recall the Pickands constant first (see, Pickands [22], Piterbarg [23], Dieker and Yakir [13]). Let $\chi = \{\chi(t), t \in \mathbb{R}^N\}$ be a (rescaled) fractional Brownian motion with Hurst index $\alpha/2 \in (0,1)$, which is a centered Gaussian field with covariance function $\mathbb{E}[\chi(t)\chi(s)] = |t|^\alpha + |s|^\alpha - |t-s|^\alpha$.

As in Ladneva and Piterbarg [18] and Anshin [5], we define for any compact sets $\mathbb{S}, \mathbb{T} \subset \mathbb{R}^N$,

$$H_\alpha(\mathbb{S}, \mathbb{T}) := \int_0^\infty e^x \cdot \mathbb{P}\Big(\sup_{s \in \mathbb{S}}\big(\chi(s) - |s|^\alpha\big) > x, \sup_{t \in \mathbb{T}}\big(\chi(t) - |t|^\alpha\big) > x\Big)\, dx. \tag{2.1}$$

Let $H_\alpha(\mathbb{T}) = H_\alpha(\mathbb{T}, \mathbb{T})$. Then, the Pickands constant is defined as

$$H_\alpha := \lim_{T\to\infty} \frac{H_\alpha([0,T]^N)}{T^N}, \tag{2.2}$$

which is positive and finite (cf. Piterbarg [23]).

Before moving to the tail probability of extremes of a bivariate Gaussian random field, let us consider the tail probability of a standard bivariate Gaussian vector (ξ, η) with correlation ρ. It is known that (see, e.g., Ladneva and Piterbarg [18])

$$\mathbb{P}(\xi > u, \eta > u) = \Psi(u, \rho)\bigl(1 + o(1)\bigr) \qquad \text{as } u \to \infty,$$

where

$$\Psi(u, \rho) := \frac{(1+\rho)^2}{2\pi u^2 \sqrt{1-\rho^2}} \exp\biggl(-\frac{u^2}{1+\rho}\biggr).$$

The exponential part of the tail probability above is determined by the correlation ρ. As shown by Theorems 2.1 and 2.2 below, similar phenomenon also happens for the tail probability of double extremes of $\{X(t), t \in \mathbb{R}^N\}$, where the exponential part is determined by the maximum cross correlation of the coordinate fields X_1 and X_2.

We will study double extremes of X on the domain $A_1 \times A_2$ where A_1, A_2 are bounded Jordan measurable sets in $\mathbb{R}^N$. That is, the boundaries of A_1 and A_2 have N-dimensional Lebesgue measure 0 (see, e.g., Piterbarg [23], page 105). We only consider the case when $A_1 \cap A_2 \neq \varnothing$, in which the maximum cross correlation ρ can be attained.

If $mes_N(A_1 \cap A_2) \neq 0$, we have the following theorem.

Theorem 2.1. *Let $\{X(t), t \in \mathbb{R}^N\}$ be a bivariate Gaussian random field that satisfies the assumptions in Section 1. If $mes_N(A_1 \cap A_2) \neq 0$, then as $u \to \infty$,*

$$\begin{aligned}
&\mathbb{P}\Bigl(\max_{s\in A_1} X_1(s) > u, \max_{t\in A_2} X_2(t) > u\Bigr) \\
&\quad = (2\pi)^{N/2}\bigl(-r''(0)\bigr)^{-N/2} c_1^{N/\alpha_1} c_2^{N/\alpha_2} mes_N(A_1 \cap A_2) H_{\alpha_1} H_{\alpha_2} \\
&\qquad \times (1+\rho)^{-N(2/\alpha_1 + 2/\alpha_2 - 1)} u^{N(2/\alpha_1 + 2/\alpha_2 - 1)} \Psi(u, \rho)\bigl(1 + o(1)\bigr).
\end{aligned} \tag{2.3}$$

If $mes_N(A_1 \cap A_2) = 0$, the above theorem is not informative. We have not been able to obtain a general explicit formula. Instead, we consider the special cases

$$A_1 = A_{1,M} \times \prod_{j=M+1}^{N} [S_j, T_j] \quad \text{and} \quad A_2 = A_{2,M} \times \prod_{M+1}^{N} [T_j, R_j], \tag{2.4}$$

where $A_{1,M}$ and $A_{2,M}$ are M dimensional Jordan sets with $mes_M(A_{1,M} \cap A_{2,M}) \neq 0$ and $S_j \le T_j \le R_j$, $j = M+1, \ldots, N$, $0 \le M \le N-1$. For simplicity of notation, let $mes_0(\cdot) \equiv 1$. Our next theorem shows that the excursion probability is smaller than that in (2.3) by a factor of u^{M-N}.

Theorem 2.2. *Let $\{X(t), t \in \mathbb{R}^N\}$ be a bivariate Gaussian random field that satisfies the assumptions in Section 1, and let A_1, A_2 be as in (2.4) such that $mes_M(A_{1,M} \cap A_{2,M}) > 0$. Then as $u \to \infty$,*

$$\begin{aligned} &\mathbb{P}\Big(\max_{s\in A_1} X_1(s) > u, \max_{t\in A_2} X_2(t) > u\Big) \\ &\quad = (2\pi)^{M/2}\big(-r''(0)\big)^{-(2N-M)/2} c_1^{N/\alpha_1} c_2^{N/\alpha_2} H_{\alpha_1} H_{\alpha_2} mes_M(A_{1,M} \cap A_{2,M}) \\ &\qquad \times (1+\rho)^{2N-M-2N/\alpha_1-2N/\alpha_2} u^{M+N(2/\alpha_1+2/\alpha_2-2)} \Psi(u,\rho)\big(1+o(1)\big). \end{aligned} \tag{2.5}$$

Remark 2.3. The following are some additional remarks about Theorems 2.1 and 2.2.

- The excursion probability in (1.2) depends on the region where the maximum cross correlation is attained. In our setting, the maximum cross correlation ρ is attained on $D := \{(s,s) | s \in A_1 \cap A_2\}$.
- For Theorem 2.2, let us consider the extreme case when $M = 0$, i.e., $A_1 \cap A_2 = \{(T_1, \dots, T_N)\}$. The exponential part still reaches $-\frac{u^2}{1+\rho}$, although the maximum cross correlation ρ is attained at a single point.
- To compare our results with Anshin [5], we consider a centered Gaussian process $\{X(t) = (X_1(t), X_2(t)), t \in \mathbb{R}\}$ and $A_1 = A_2 = [0, T]$. In our setting, the cross correlation attains its maximum on the line $D = \{(s,s) | s \in [0,T]\}$, while in Anshin [5] it only attains at a unique point in $[0,T] \times [0,T]$ because of the assumption **C2**. This is the reason why the power of u in our settings is $\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - 3$ instead of $\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - 4$ in Anshin [5].
- Even though Theorem 2.2 only deals with a special case of A_1, A_2 with $mes_N(A_1 \cap A_2) = 0$, its method of proof can be applied to more general cases provided some information on A_1 and A_2 is specified. The key step is to reevaluate the infinite series in Lemma 3.5.

We end this section with an application of Theorems 2.1 and 2.2 to bivariate Gaussian random fields with the Matérn correlation functions introduced by Gneiting, Kleiber and Schlather [15].

The Matérn correlation function $M(h|\nu, a)$, where $a > 0, \nu > 0$ are scale and smoothness parameters, is widely used to model covariance structures in spatial statistics. It is defined as

$$M(h|\nu,a) := \frac{2^{1-\nu}}{\Gamma(\nu)}\big(a|h|\big)^{\nu} K_\nu\big(a|h|\big), \tag{2.6}$$

where K_ν is a modified Bessel function of the second kind. In Gneiting, Kleiber and Schlather [15], the authors introduce the full bivariate Matérn field $X(s) = (X_1(s), X_2(s))$, that is, an $\mathbb{R}^2$-valued Gaussian random field on $\mathbb{R}^N$ with zero mean and matrix-valued covariance functions:

$$C(h) = \begin{pmatrix} C_{11}(h) & C_{12}(h) \\ C_{21}(h) & C_{22}(h) \end{pmatrix}, \tag{2.7}$$

where $C_{ij}(h) := \mathbb{E}[X_i(s+h)X_j(s)]$ are specified by

$$C_{11}(h) = \sigma_1^2 M(h|\nu_1, a_1), \tag{2.8}$$

$$C_{22}(h) = \sigma_2^2 M(h|\nu_2, a_2), \tag{2.9}$$

$$C_{12}(h) = C_{21}(h) = \rho\sigma_1\sigma_2 M(h|\nu_{12}, a_{12}). \tag{2.10}$$

According to Gneiting, Kleiber and Schlather [15], the above model is valid if and only if

$$\begin{aligned}\rho^2 \le{}& \frac{\Gamma(\nu_1 + N/2)\Gamma(\nu_2 + N/2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \frac{\Gamma(\nu_{12})^2}{\Gamma(\nu_{12} + N/2)^2} \frac{a_1^{2\nu_1} a_2^{2\nu_2}}{a_{12}^{4\nu_{12}}} \\ &\times \inf_{t\ge 0} \frac{(a_{12}^2 + t^2)^{2\nu_{12}+N}}{(a_1^2 + t^2)^{\nu_1+N/2}(a_2^2 + t^2)^{\nu_2+N/2}}.\end{aligned} \tag{2.11}$$

Especially, when $a_1 = a_2 = a_{12}$, condition (2.11) is reduced to

$$\rho^2 \le \frac{\Gamma(\nu_1 + N/2)\Gamma(\nu_2 + N/2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \frac{\Gamma(\nu_{12})^2}{\Gamma(\nu_{12} + N/2)^2}, \tag{2.12}$$

in which case the choice of ρ is fairly flexible.

Here we focus on a standardized bivariate Matérn field, that is, we assume $\sigma_1 = \sigma_2 = 1$, $a_1 = a_2 = a_{12} = 1$ and $\rho > 0$. Moreover, we assume $\nu_1, \nu_2 \in (0, 1)$ and $\nu_{12} > 1$. In this case, the bivariate Matérn field $\{X(t), t \in \mathbb{R}^N\}$ satisfies the assumptions in Section 1.

Indeed, assumption (i) in Section 1 is satisfied since

$$M(h|\nu_i, a) = 1 - c_i|t|^{2\nu_i} + o\big(|t|^{2\nu_i}\big),$$

where $c_i = \frac{\Gamma(1-\nu_i)}{2^{2\nu_i}\Gamma(1+\nu_i)}$, $i = 1, 2$ (see, e.g., Stein [25], page 32). Assumption (ii) holds immediately if we use the following integral representation of $M(h|\nu, a)$ (see, e.g., Abramowitz and Stegun [1], Section 9.6)

$$M(h|\nu, a) = \frac{2\Gamma(\nu + 1/2)}{\sqrt{\pi}\Gamma(\nu)} \int_0^\infty \frac{\cos(a|h|r)}{(1 + r^2)^{\nu+1/2}}\, dr. \tag{2.13}$$

Assumption (iii) holds by the definition of cross correlation in (2.10). For assumption (iv), we only need to check the smoothness of $M(h|\nu, a)$. By another integral representation of $M(h|\nu, a)$ (see, e.g., Abramowitz and Stegun [1], Section 9.6), that is,

$$M(h|\nu, a) = \frac{2^{1-2\nu}(a|h|)^{2\nu}}{\Gamma(\nu + 1/2)\Gamma(\nu)} \int_1^\infty e^{-a|h|r}\big(r^2 - 1\big)^{\nu-1/2}\, dr,$$

one can verify that $M(h|\nu, a)$ is infinitely differentiable when $|h| \ne 0$. Meanwhile, $M''(0|\nu, a)$ exists and is continuous when $\nu > 1$ which can be proven by taking twice derivatives to the integral representation in (2.13) w.r.t. $|h|$. So assumption (iv) holds.

Applying Theorem 2.1 to the double excursion probability of $X(s)$ over $[0,1]^N$, we have

$$
\begin{aligned}
&\mathbb{P}\Big(\max_{s\in[0,1]^N} X_1(s) > u, \max_{t\in[0,1]^N} X_2(t) > u\Big) \\
&\quad = (2\pi)^{N/2}\big(-C_{12}''(0)\big)^{-N/2} c_1^{N/(2\nu_1)} c_2^{N/(2\nu_2)} (1+\rho)^{-N(1/\nu_1+1/\nu_2-1)} H_{2\nu_1} H_{2\nu_2} \\
&\qquad \times u^{N(1/\nu_1+1/\nu_2-1)} \Psi(u,\rho)\big(1+o(1)\big) \qquad \text{as } u\to\infty.
\end{aligned}
$$

Secondly, when the two measurements are observed on two regions which only share part of boundaries, we use Theorem 2.2 to obtain the excursion probability. For example, if $X_1(s)$ are observed on the region $[0,1]^N$ and $X_2(s)$ on $[0,1]^{N-1}\times[1,2]$, then as $u\to\infty$,

$$
\begin{aligned}
&\mathbb{P}\Big(\max_{s\in[0,1]^N} X_1(s) > u, \max_{t\in[0,1]^{N-1}\times[1,2]} X_2(t) > u\Big) \\
&\quad = (2\pi)^{(N-1)/2}\big(-C_{12}''(0)\big)^{-(N+1)/2} c_1^{N/(2\nu_1)} c_2^{N/(2\nu_2)} (1+\rho)^{1-N(1/\nu_1+1/\nu_2-1)} H_{2\nu_1} H_{2\nu_2} \\
&\qquad \times u^{N(1/\nu_1+1/\nu_2-1)-1} \Psi(u,\rho)\big(1+o(1)\big).
\end{aligned}
$$

3. Proofs of the main results

The proofs of Theorems 2.1 and 2.2 are based on the double sum method (Piterbarg [23]) and the work of Ladneva and Piterbarg [18]. Since the latter deals with the tail probability $\mathbb{P}(\max_{t\in[T_1,T_2]} X(t) > u, \max_{t\in[T_3,T_4]} X(t) > u)$ of a univariate Gaussian process $\{X(t), t\in\mathbb{R}\}$, their method is not sufficient for carrying out the double sum method for a bivariate random field.

Lemmas 3.1 and 3.2 below extend Lemmas 1 and 9 in Ladneva and Piterbarg [18] to the bivariate random field $\{(X_1(t), X_2(t)), t\in\mathbb{R}^N\}$. Moreover, we have strengthened the conclusions by showing that the convergence is uniform in certain sense. This will be useful for dealing with sums of local approximations around the regions where the maximum cross correlation is attained. The details will be illustrated in the proof of Theorem 2.1 (see, e.g., (3.10), (3.21)). In the following lemmas, $\{X(t), t\in\mathbb{R}^N\}$ is a bivariate Gaussian random field as defined in Section 1.

Lemma 3.1. *Let s_u and t_u be two $\mathbb{R}^N$-valued functions of u and let $\tau_u := t_u - s_u$. For any compact sets $\mathbb{S}$ and $\mathbb{T}$ in $\mathbb{R}^N$, we have*

$$
\begin{aligned}
&\mathbb{P}\Big(\max_{s\in s_u+u^{-2/\alpha_1}\mathbb{S}} X_1(s) > u, \max_{t\in t_u+u^{-2/\alpha_2}\mathbb{T}} X_2(t) > u\Big) \\
&\quad = \frac{(1+\rho)^2}{2\pi\sqrt{1-\rho^2}} H_{\alpha_1}\Big(\frac{c_1^{1/\alpha_1}\mathbb{S}}{(1+\rho)^{2/\alpha_1}}\Big) H_{\alpha_2}\Big(\frac{c_2^{1/\alpha_2}\mathbb{T}}{(1+\rho)^{2/\alpha_2}}\Big) \\
&\qquad \times u^{-2}\exp\Big(-\frac{u^2}{1+r(|\tau_u|)}\Big)\big(1+o(1)\big),
\end{aligned} \tag{3.1}
$$

where $o(1)\to 0$ uniformly w.r.t. τ_u satisfying $|\tau_u| \le C_0\sqrt{\log u}/u$ as $u\to\infty$.

Lemma 3.2. *Let s_u, t_u and τ_u be the same as in Lemma 3.1. For all $T > 0$, $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^N$, we have*

$$\begin{aligned}
&\mathbb{P}\Big(\max_{s \in s_u + u^{-2/\alpha_1}[0,T]^N} X_1(s) > u, \max_{t \in t_u + u^{-2/\alpha_2}[0,T]^N} X_2(t) > u, \\
&\quad \max_{s \in s_u + u^{-2/\alpha_1}[\mathbf{m}T,(\mathbf{m}+1)T]} X_1(s) > u, \max_{t \in t_u + u^{-2/\alpha_2}[\mathbf{n}T,(\mathbf{n}+1)T]} X_2(t) > u\Big) \\
&\quad = \frac{(1+\rho)^2}{2\pi\sqrt{1-\rho^2}u^2} e^{-u^2/(1+r(|\tau_u|))} H_{\alpha_1}\bigg(\frac{c_1^{1/\alpha_1}[0,T]^N}{(1+\rho)^{2/\alpha_1}}, \frac{c_1^{1/\alpha_1}[\mathbf{m}T,(\mathbf{m}+1)T]}{(1+\rho)^{2/\alpha_1}}\bigg) \\
&\qquad \times H_{\alpha_2}\bigg(\frac{c_2^{1/\alpha_2}[0,T]^N}{(1+\rho)^{2/\alpha_2}}, \frac{c_2^{1/\alpha_2}[\mathbf{n}T,(\mathbf{n}+1)T]}{(1+\rho)^{2/\alpha_2}}\bigg)\big(1+o(1)\big),
\end{aligned} \tag{3.2}$$

where $H_\alpha(\cdot,\cdot)$ is defined in (2.1) and $o(1) \to 0$ uniformly for all s_u and t_u that satisfy $|\tau_u| \le C_0\sqrt{\log u}/u$ as $u \to \infty$.

Proofs of Lemmas 3.1 and 3.2 will be given in Section 4. Now we proceed to prove our main theorems.

Proof of Theorem 2.1. Let $\Pi = A_1 \times A_2, \delta(u) = C\sqrt{\log u}/u$, where C is a constant whose value will be determined later. Let

$$\mathcal{D} = \big\{(s,t) \in \Pi : |t-s| \le \delta(u)\big\}. \tag{3.3}$$

Since

$$\begin{aligned}
&\mathbb{P}\Big(\bigcup_{(s,t)\in\mathcal{D}} \big\{X_1(s) > u, X_2(t) > u\big\}\Big) \\
&\quad \le \mathbb{P}\Big(\max_{s\in A_1} X_1(s) > u, \max_{t\in A_2} X_2(t) > u\Big) \\
&\quad \le \mathbb{P}\Big(\bigcup_{(s,t)\in\mathcal{D}} \big\{X_1(s) > u, X_2(t) > u\big\}\Big) + \mathbb{P}\Big(\bigcup_{(s,t)\in\Pi\setminus\mathcal{D}} \big\{X_1(s) > u, X_2(t) > u\big\}\Big),
\end{aligned}$$

it is sufficient to prove that, by choosing appropriate constant C, we have

$$\begin{aligned}
&\mathbb{P}\Big(\bigcup_{(s,t)\in\mathcal{D}} \big\{X_1(s) > u, X_2(t) > u\big\}\Big) \\
&\quad = (2\pi)^{N/2}\big(-r''(0)\big)^{-N/2} c_1^{N/\alpha_1} c_2^{N/\alpha_2} (1+\rho)^{-N(2/\alpha_1+2/\alpha_2-1)} mes_N(A_1 \cap A_2) \\
&\qquad \times H_{\alpha_1} H_{\alpha_2} u^{N(2/\alpha_1+2/\alpha_2-1)} \Psi(u,\rho)\big(1+o(1)\big) \qquad \text{as } u \to \infty
\end{aligned} \tag{3.4}$$

and

$$\lim_{u\to\infty} \frac{\mathbb{P}(\bigcup_{(s,t)\in\Pi\setminus\mathcal{D}}\{X_1(s) > u, X_2(t) > u\})}{\mathbb{P}(\bigcup_{(s,t)\in\mathcal{D}}\{X_1(s) > u, X_2(t) > u\})} = 0. \tag{3.5}$$

We prove (3.4) first. For any fixed $T > 0$ and $i = 1, 2$, let $d_i(u) = Tu^{-2/\alpha_i}$ and, for any $\mathbf{k} = (k_1, \dots, k_N) \in \mathbb{Z}^N$, define

$$\Delta_{\mathbf{k}}^{(i)} \triangleq \prod_{j=1}^{N} \big[k_j d_i(u), (k_j + 1)d_i(u)\big] = \big[\mathbf{k}d_i(u), (\mathbf{k} + 1)d_i(u)\big]. \tag{3.6}$$

Let

$$\mathcal{C} = \big\{(\mathbf{k}, \mathbf{l}) : \Delta_{\mathbf{k}}^{(1)} \times \Delta_{\mathbf{l}}^{(2)} \cap \mathcal{D} \neq \varnothing\big\} \quad \text{and} \quad \mathcal{C}^\circ = \big\{(\mathbf{k}, \mathbf{l}) : \Delta_{\mathbf{k}}^{(1)} \times \Delta_{\mathbf{l}}^{(2)} \subseteq \mathcal{D}\big\}. \tag{3.7}$$

It is easy to see that

$$\bigcup_{(\mathbf{k},\mathbf{l})\in\mathcal{C}^\circ} \Delta_{\mathbf{k}}^{(1)} \times \Delta_{\mathbf{l}}^{(2)} \subseteq \mathcal{D} \subseteq \bigcup_{(\mathbf{k},\mathbf{l})\in\mathcal{C}} \Delta_{\mathbf{k}}^{(1)} \times \Delta_{\mathbf{l}}^{(2)}.$$

Thus, the LHS of (3.4) is bounded above by

$$\begin{aligned}
&\mathbb{P}\Big(\bigcup_{(s,t)\in\mathcal{D}} \big\{X_1(s) > u, X_2(t) > u\big\}\Big) \\
&\quad\leq \sum_{(\mathbf{k},\mathbf{l})\in\mathcal{C}} \mathbb{P}\Big(\max_{s\in\Delta_{\mathbf{k}}^{(1)}} X_1(s) > u, \max_{t\in\Delta_{\mathbf{l}}^{(2)}} X_2(t) > u\Big) \qquad (3.8)\\
&\quad= \sum_{(\mathbf{k},\mathbf{l})\in\mathcal{C}} \mathbb{P}\Big(\max_{s\in\mathbf{k}d_1(u)+\Delta_0^{(1)}} X_1(s) > u, \max_{t\in\mathbf{l}d_2(u)+\Delta_0^{(2)}} X_2(t) > u\Big).
\end{aligned}$$

Let

$$\tau_{\mathbf{kl}} := \mathbf{l}d_2(u) - \mathbf{k}d_1(u) = \big(l_1 d_2(u) - k_1 d_1(u), \dots, l_N d_2(u) - k_N d_1(u)\big). \tag{3.9}$$

For $(\mathbf{k}, \mathbf{l}) \in \mathcal{C}$, $|\tau_{\mathbf{kl}}| \leq \delta(u) + \sqrt{N}(d_1(u) + d_2(u)) \leq 2\delta(u)$ for all u large enough, since $d_1(u) = o(\delta(u))$ and $d_2(u) = o(\delta(u))$, as $u \to \infty$. By applying Lemma 3.1 to the RHS of (3.8), we obtain

$$\begin{aligned}
&\mathbb{P}\Big(\bigcup_{(s,t)\in\mathcal{D}} \big\{X_1(s) > u, X_2(t) > u\big\}\Big) \\
&\quad\leq \frac{(1+\rho)^2(1+\gamma(u))}{2\pi\sqrt{1-\rho^2}u^2} H_{\alpha_1}\bigg(\frac{c_1^{1/\alpha_1}[0,T]^N}{(1+\rho)^{2/\alpha_1}}\bigg) H_{\alpha_2}\bigg(\frac{c_2^{1/\alpha_2}[0,T]^N}{(1+\rho)^{2/\alpha_2}}\bigg) \\
&\qquad\times \sum_{(\mathbf{k},\mathbf{l})\in\mathcal{C}} \exp\bigg(-\frac{u^2}{1+r(|\tau_{\mathbf{kl}}|)}\bigg) \qquad (3.10)\\
&\quad= H_{\alpha_1}\bigg(\frac{c_1^{1/\alpha_1}[0,T]^N}{(1+\rho)^{2/\alpha_1}}\bigg) H_{\alpha_2}\bigg(\frac{c_2^{1/\alpha_2}[0,T]^N}{(1+\rho)^{2/\alpha_2}}\bigg) \Psi(u,\rho)\big(1+\gamma(u)\big) \\
&\qquad\times \sum_{(\mathbf{k},\mathbf{l})\in\mathcal{C}} \exp\bigg\{-u^2\bigg(\frac{1}{1+r(|\tau_{\mathbf{kl}}|)} - \frac{1}{1+\rho}\bigg)\bigg\},
\end{aligned}$$

where the global error function $\gamma(u) \to 0$, as $u \to \infty$. The uniform convergence of (3.1) in Lemma 3.1 guarantees that the local error term $o(1)$ for each pair $(\mathbf{k}, \mathbf{l}) \in \mathcal{C}$ is uniformly bounded by $\gamma(u)$.

The series in the last equality of (3.10) is dealt by the following key lemma, which gives the power of the threshold u in (3.4).

Lemma 3.3. *Recall the set $\mathcal{C}$ defined in (3.7). Let*

$$h(u) := \sum_{(\mathbf{k},\mathbf{l})\in\mathcal{C}} \exp\left\{-u^2\left(\frac{1}{1+r(|\tau_{\mathbf{kl}}|)} - \frac{1}{1+\rho}\right)\right\}. \tag{3.11}$$

Then, under the assumptions of Theorem 2.1, we have

$$\begin{aligned} h(u) = (2\pi)^{N/2}\big(-r''(0)\big)^{-N/2}(1+\rho)^N T^{-2N} mes_N(A_1\cap A_2) \\ \times u^{N(2/\alpha_1+2/\alpha_2-1)}\big(1+o(1)\big) \qquad \text{as } u\to\infty. \end{aligned} \tag{3.12}$$

Moreover, if we replace $\mathcal{C}$ in (3.11) by $\mathcal{C}^\circ$ defined in (3.7), then (3.12) still holds.

We defer the proof of Lemma 3.3 to Section 4 and continue with the proof of Theorem 2.1. Applying (3.12) to (3.10), we obtain

$$\begin{aligned} &\mathbb{P}\Bigg(\bigcup_{(s,t)\in\mathcal{D}}\big\{X_1(s)>u, X_2(t)>u\big\}\Bigg) \\ &\le (2\pi)^{N/2}\big(-r''(0)\big)^{-N/2}(1+\rho)^N T^{-2N} mes_N(A_1\cap A_2) H_{\alpha_1}\left(\frac{c_1^{1/\alpha_1}[0,T]^N}{(1+\rho)^{2/\alpha_1}}\right) \\ &\quad\times H_{\alpha_2}\left(\frac{c_2^{1/\alpha_2}[0,T]^N}{(1+\rho)^{2/\alpha_2}}\right) u^{N(2/\alpha_1+2/\alpha_2-1)}\Psi(u,\rho)\big(1+\gamma_1(u)\big), \end{aligned} \tag{3.13}$$

where $\gamma_1(u)\to 0$, as $u\to\infty$. Hence,

$$\begin{aligned} &\limsup_{u\to\infty}\frac{\mathbb{P}(\bigcup_{(s,t)\in\mathcal{D}}\{X_1(s)>u, X_2(t)>u\})}{u^{N(2/\alpha_1+2/\alpha_2-1)}\Psi(u,\rho)} \\ &\le (2\pi)^{N/2}\big(-r''(0)\big)^{-N/2}(1+\rho)^N mes_N(A_1\cap A_2) \\ &\quad\times T^{-2N} H_{\alpha_1}\left(\frac{c_1^{1/\alpha_1}[0,T]^N}{(1+\rho)^{2/\alpha_1}}\right) H_{\alpha_2}\left(\frac{c_2^{1/\alpha_2}[0,T]^N}{(1+\rho)^{2/\alpha_2}}\right). \end{aligned} \tag{3.14}$$

The above inequality holds for every $T>0$. Therefore, letting $T\to\infty$, we have

$$\begin{aligned} &\limsup_{u\to\infty}\frac{\mathbb{P}(\bigcup_{(s,t)\in\mathcal{D}}\{X_1(s)>u, X_2(t)>u\})}{u^{N(2/\alpha_1+2/\alpha_2-1)}\Psi(u,\rho)} \\ &\le (2\pi)^{N/2}\big(-r''(0)\big)^{-N/2} c_1^{N/\alpha_1} c_2^{N/\alpha_2}(1+\rho)^{-N(2/\alpha_1+2/\alpha_2-1)} mes_N(A_1\cap A_2) H_{\alpha_1} H_{\alpha_2}. \end{aligned} \tag{3.15}$$

On the other hand, the lower bound for LHS of (3.4) can be derived as follows. Let

$$\mathcal{B} = \big\{(\mathbf{k}, \mathbf{l}, \mathbf{k}', \mathbf{l}') : (\mathbf{k}, \mathbf{l}) \neq (\mathbf{k}', \mathbf{l}'), (\mathbf{k}, \mathbf{l}), (\mathbf{k}', \mathbf{l}') \in \mathcal{C}\big\}. \tag{3.16}$$

By Bonferroni's inequality and symmetric property of $\mathcal{B}$, the LHS of (3.4) is bounded below by

$$\begin{aligned}
&\mathbb{P}\Big(\bigcup_{(s,t)\in\mathcal{D}} \big\{X_1(s) > u, X_2(t) > u\big\}\Big) \\
&\geq \sum_{(\mathbf{k},\mathbf{l})\in\mathcal{C}^\circ} \mathbb{P}\Big(\max_{s\in\Delta_{\mathbf{k}}^{(1)}} X_1(s) > u, \max_{t\in\Delta_{\mathbf{l}}^{(2)}} X_2(t) > u\Big) \\
&\quad - \frac{1}{2} \sum_{(\mathbf{k},\mathbf{l},\mathbf{k}',\mathbf{l}')\in\mathcal{B}} \mathbb{P}\Big(\max_{s\in\Delta_{\mathbf{k}}^{(1)}} X_1(s) > u, \max_{t\in\Delta_{\mathbf{l}}^{(2)}} X_2(t) > u, \\
&\quad \max_{s\in\Delta_{\mathbf{k}'}^{(1)}} X_1(s) > u, \max_{t\in\Delta_{\mathbf{l}'}^{(2)}} X_2(t) > u\Big) \\
&\triangleq \Sigma_1 - \Sigma_2.
\end{aligned} \tag{3.17}$$

Since $\mathcal{C}^\circ$ and $\mathcal{C}$ are almost the same, a similar argument as in (3.10)–(3.15) shows that Σ_1 is bounded below by

$$\begin{aligned}
\Sigma_1 &\geq (2\pi)^{N/2}\big(-r''(0)\big)^{-N/2}(1+\rho)^N mes_N(A_1 \cap A_2) T^{-2N} H_{\alpha_1}\bigg(\frac{c_1^{1/\alpha_1}[0,T]^N}{(1+\rho)^{2/\alpha_1}}\bigg) \\
&\quad \times H_{\alpha_2}\bigg(\frac{c_2^{1/\alpha_2}[0,T]^N}{(1+\rho)^{2/\alpha_2}}\bigg) u^{N(2/\alpha_1+2/\alpha_2-1)} \Psi(u,\rho)\big(1-\gamma_2(u)\big),
\end{aligned} \tag{3.18}$$

where $\gamma_2(u) \to 0$, as $u \to \infty$. Hence, letting $T \to \infty$, we have

$$\begin{aligned}
\liminf_{u\to\infty} \frac{\Sigma_1}{u^{N(2/\alpha_1+2/\alpha_2-1)}\Psi(u,\rho)} &\geq (2\pi)^{N/2}\big(-r''(0)\big)^{-N/2} c_1^{N/\alpha_1} c_2^{N/\alpha_2} \\
&\quad \times (1+\rho)^{-N(2/\alpha_1+2/\alpha_2-1)} mes_N(A_1 \cap A_2) H_{\alpha_1} H_{\alpha_2}.
\end{aligned} \tag{3.19}$$

Next, we consider Σ_2 in (3.17). To simplify the notation, we let

$$\begin{aligned}
I\big(\mathbf{k}, \mathbf{l}, \mathbf{k}', \mathbf{l}'\big) := \mathbb{P}\Big(&\max_{s\in\Delta_{\mathbf{k}}^{(1)}} X_1(s) > u, \max_{t\in\Delta_{\mathbf{l}}^{(2)}} X_2(t) > u, \\
&\max_{s\in\Delta_{\mathbf{k}'}^{(1)}} X_1(s) > u, \max_{t\in\Delta_{\mathbf{l}'}^{(2)}} X_2(t) > u\Big).
\end{aligned}$$

For $\mathbf{m} = (m_1, \ldots, m_N) \in \mathbb{Z}^N$, let

$$\mathcal{H}_{\alpha,c}(\mathbf{m}) \triangleq H_\alpha\bigg(\frac{c^{1/\alpha}[0,T]^N}{(1+\rho)^{2/\alpha}}, \frac{c^{1/\alpha}[\mathbf{m}T, (\mathbf{m}+1)T]}{(1+\rho)^{2/\alpha}}\bigg). \tag{3.20}$$

Rewriting Σ_2 and applying Lemma 3.2, we obtain

$$
\begin{aligned}
\Sigma_2 &= \frac{1}{2} \sum_{(\mathbf{k},\mathbf{l})\in\mathcal{C}} \Bigg(\sum_{\substack{(\mathbf{k}',\mathbf{l}')\in\mathcal{C} \\ \mathbf{k}'=\mathbf{k},\mathbf{l}'\neq\mathbf{l}}} + \sum_{\substack{(\mathbf{k}',\mathbf{l}')\in\mathcal{C} \\ \mathbf{k}'\neq\mathbf{k},\mathbf{l}'=\mathbf{l}}} + \sum_{\substack{(\mathbf{k}',\mathbf{l}')\in\mathcal{C} \\ \mathbf{k}'\neq\mathbf{k},\mathbf{l}'\neq\mathbf{l}}} \Bigg) I\big(\mathbf{k},\mathbf{l},\mathbf{k}',\mathbf{l}'\big) \\
&= \frac{(1+\rho)^2(1+\gamma_3(u))}{4\pi\sqrt{1-\rho^2}u^2} \sum_{(\mathbf{k},\mathbf{l})\in\mathcal{C}} e^{-u^2/(1+r(|\tau_{\mathbf{kl}}|))} \Bigg(\mathcal{H}_{\alpha_1,c_1}(\mathbf{0}) \sum_{\substack{(\mathbf{k}',\mathbf{l}')\in\mathcal{C} \\ \mathbf{k}'=\mathbf{k},\mathbf{l}'\neq\mathbf{l}}} \mathcal{H}_{\alpha_2,c_2}\big(\mathbf{l}'-\mathbf{l}\big) \\
&\quad + \mathcal{H}_{\alpha_2,c_2}(\mathbf{0}) \sum_{\substack{(\mathbf{k}',\mathbf{l}')\in\mathcal{C} \\ \mathbf{k}'\neq\mathbf{k},\mathbf{l}'=\mathbf{l}}} \mathcal{H}_{\alpha_1,c_1}\big(\mathbf{k}'-\mathbf{k}\big) + \sum_{\substack{(\mathbf{k}',\mathbf{l}')\in\mathcal{C} \\ \mathbf{k}'\neq\mathbf{k},\mathbf{l}'\neq\mathbf{l}}} \mathcal{H}_{\alpha_1,c_1}\big(\mathbf{k}'-\mathbf{k}\big)\mathcal{H}_{\alpha_2,c_2}\big(\mathbf{l}'-\mathbf{l}\big) \Bigg) \qquad (3.21) \\
&\le \frac{(1+\rho)^2(1+\gamma_3(u))}{4\pi\sqrt{1-\rho^2}u^2} \sum_{(\mathbf{k},\mathbf{l})\in\mathcal{C}} e^{-u^2/(1+r(|\tau_{\mathbf{kl}}|))} \Bigg(\mathcal{H}_{\alpha_1,c_1}(\mathbf{0}) \sum_{\mathbf{n}\neq\mathbf{0}} \mathcal{H}_{\alpha_2,c_2}(\mathbf{n}) \\
&\quad + \mathcal{H}_{\alpha_2,c_2}(\mathbf{0}) \sum_{\mathbf{m}\neq\mathbf{0}} \mathcal{H}_{\alpha_1,c_1}(\mathbf{m}) + \sum_{\mathbf{m}\neq\mathbf{0},\mathbf{n}\neq\mathbf{0}} \mathcal{H}_{\alpha_1,c_1}(\mathbf{m})\mathcal{H}_{\alpha_2,c_2}(\mathbf{n}) \Bigg),
\end{aligned}
$$

where $\gamma_3(u) \to 0$, as $u \to \infty$. According to the uniform convergence of (3.2), the local error term $o(1)$ for each pair $(\mathbf{k}',\mathbf{l}') \in \mathcal{C}$ is bounded above by $\gamma_3(u)$. To estimate $\mathcal{H}_{\alpha,c}(\cdot)$, we make use of the following lemma, whose proof is again postponed to Section 4.

Lemma 3.4. *Recall $\mathcal{H}_{\alpha,c}(\cdot)$ defined in (3.20). Let $i_0 = \operatorname{argmax}_{1\le i\le N} |m_i|$. Then there exist positive constants C_1 and T_0 such that for all $T \ge T_0$,*

$$\mathcal{H}_{\alpha,c}(\mathbf{0}) \le C_1 T^N; \tag{3.22}$$

$$\mathcal{H}_{\alpha,c}(\mathbf{m}) \le C_1 T^{N-1/2} \qquad \text{when } |m_{i_0}| = 1; \tag{3.23}$$

$$\mathcal{H}_{\alpha,c}(\mathbf{m}) \le C_1 T^{2N} e^{-c/(8(1+\rho)^2)(|m_{i_0}|-1)^\alpha T^\alpha} \qquad \text{when } |m_{i_0}| \ge 2. \tag{3.24}$$

Consequently,

$$\sum_{\mathbf{m}\in\mathbb{Z}^N\setminus\{\mathbf{0}\}} \mathcal{H}_{\alpha,c}(\mathbf{m}) \le C_1 T^{N-1/2}. \tag{3.25}$$

Applying Lemmas 3.3 and 3.4 to the RHS of (3.21), we obtain

$$
\begin{aligned}
\Sigma_2 &\le \frac{C_0(1+\rho)^2(1+\gamma_3(u))}{4\pi\sqrt{1-\rho^2}u^2} T^{2N-1/2} \sum_{(\mathbf{k},\mathbf{l})\in\mathcal{C}} \exp\bigg(-\frac{u^2}{1+r(|\tau_{\mathbf{kl}}|)}\bigg) \\
&\le C_0(2\pi)^{N/2}\big(-r''(0)\big)^{-N/2}(1+\rho)^N mes_N(A_1\cap A_2)T^{-1/2} \qquad (3.26) \\
&\quad \times u^{N(2/\alpha_1+2/\alpha_2-1)}\Psi(u,\rho)\big(1+\gamma_4(u)\big),
\end{aligned}
$$

where $\gamma_4(u) \to 0$, as $u \to \infty$. By letting $u \to \infty$ and $T \to \infty$ successively, we have

$$\limsup_{u\to\infty} \frac{\Sigma_2}{u^{N(2/\alpha_1+2/\alpha_2-1)}\Psi(u,\rho)} = 0. \tag{3.27}$$

By combining (3.17), (3.19) and (3.27), we have

$$\begin{aligned}
&\liminf_{u\to\infty} \frac{\mathbb{P}(\bigcup_{(s,t)\in\mathcal{D}}\{X_1(s) > u, X_2(t) > u\})}{u^{N(2/\alpha_1+2/\alpha_2-1)}\Psi(u,\rho)} \\
&\quad \ge \liminf_{u\to\infty} \frac{\Sigma_1}{u^{N(2/\alpha_1+2/\alpha_2-1)}\Psi(u,\rho)} - \limsup_{u\to\infty} \frac{\Sigma_2}{u^{N(2/\alpha_1+2/\alpha_2-1)}\Psi(u,\rho)} \qquad (3.28) \\
&\quad \ge (2\pi)^{N/2}\big(-r''(0)\big)^{-N/2} c_1^{N/\alpha_1} c_2^{N/\alpha_2} (1+\rho)^{-N(2/\alpha_1+2/\alpha_2-1)} mes_N(A_1\cap A_2) H_{\alpha_1} H_{\alpha_2}.
\end{aligned}$$

It is now clear that (3.4) follows from (3.15) and (3.28).

Now we prove (3.5). Define

$$Y(s,t) := X_1(s) + X_2(t) \qquad \text{for } (s,t) \in \Pi \setminus \mathcal{D}. \tag{3.29}$$

For $x = (s_1,t_1)$, $y = (s_2,t_2) \in \Pi \setminus \mathcal{D}$, let $|x-y| = \sqrt{|s_1-s_2|^2 + |t_1-t_2|^2}$. Then we can verify that

$$\mathbb{E}\big|Y(x) - Y(y)\big|^2 \le C_0|x-y|^{\min(\alpha_1,\alpha_2)} \qquad \forall x, y \in \Pi \setminus \mathcal{D}. \tag{3.30}$$

By applying Theorem 8.1 in Piterbarg [23], we obtain that the numerator of (3.5) is bounded above by

$$\begin{aligned}
&\mathbb{P}\Big(\bigcup_{(s,t)\in\Pi\setminus\mathcal{D}} \big\{X_1(s) > u, X_2(t) > u\big\}\Big) \qquad (3.31) \\
&\quad \le \mathbb{P}\Big(\max_{(s,t)\in\Pi\setminus\mathcal{D}} Y(s,t) > 2u\Big) \le C_0 u^{-1+2N/\min(\alpha_1,\alpha_2)} \exp\Big(-\frac{u^2}{1+\max_{(s,t)\in\Pi\setminus\mathcal{D}} r(|t-s|)}\Big).
\end{aligned}$$

Since $r(|t-s|) = \rho + \frac{1}{2}r''(0)|t-s|^2(1+o(1))$ and $r(\cdot)$ attains maximum only at zero, we have

$$\max_{(s,t)\in\Pi\setminus\mathcal{D}} r\big(|t-s|\big) \le \rho - \frac{1}{3}\big(-r''(0)\big)\delta^2(u) \tag{3.32}$$

for u large enough. So (3.31) is at most

$$\begin{aligned}
&C_0 u^{-1+2N/\min(\alpha_1,\alpha_2)} \exp\Big(-\frac{u^2}{1+\rho-(1/3)(-r''(0))\delta^2(u)}\Big) \\
&\quad \le C_0 u^{-1+2N/\min(\alpha_1,\alpha_2)} \exp\Big(-\frac{u^2}{1+\rho}\Big) \exp\Big(-\frac{(1/3)(-r''(0))\delta^2(u)u^2}{(1+\rho)^2}\Big) \qquad (3.33) \\
&\quad = \frac{2\pi\sqrt{1-\rho^2}C_0}{(1+\rho)^2} u^{1+(2N/\min(\alpha_1,\alpha_2))-(-r''(0)/3(1+\rho)^2)C^2} \Psi(u,\rho),
\end{aligned}$$

where the inequality holds since $\frac{1}{x-y} \geq \frac{1}{x} + \frac{y}{x^2}$, $\forall x > y$. Compare (3.33) with (3.4), it is easy to see (3.5) holds if and only if

$$1 + \frac{2N}{\min(\alpha_1, \alpha_2)} - \frac{-r''(0)}{3(1+\rho)^2} C^2 < N\left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - 1\right). \tag{3.34}$$

Hence, by choosing the constant C satisfying

$$C > \left[\frac{3(1+\rho)^2}{-r''(0)}\left(N\left(\frac{2}{\min(\alpha_1, \alpha_2)} + 1 - \frac{2}{\alpha_1} - \frac{2}{\alpha_2}\right) + 1\right)_+\right]^{1/2}, \tag{3.35}$$

we conclude (3.5). □

Proof of Theorem 2.2. From the proof of Theorem 2.1, we see that the exponential decaying rate of the excursion probability is only determined by the region where the maximum cross correlation is attained. In the case of $mes_N(A_1 \cap A_2) = 0$ but $A_1 \cap A_2 \neq \varnothing$, the exponential part, $e^{-u^2/(1+\rho)}$, remains the same. Yet, the dimension reduction of $A_1 \cap A_2$ does affect the polynomial power of the excursion probability, which is determined by the quantity

$$h(u) = \sum_{(\mathbf{k},\mathbf{l}) \in \mathcal{C}} \exp\left\{-u^2\left(\frac{1}{1 + r(|\tau_{\mathbf{kl}}|)} - \frac{1}{1+\rho}\right)\right\}$$

in Lemma 3.3. Under the assumptions of Theorem 2.2, the set $\mathcal{C}$ and the behavior of $h(u)$ change. We will make use of the following lemma which plays the role of Lemma 3.3.

Lemma 3.5. *Under the assumptions of Theorem 2.2, we have*

$$\begin{aligned} h(u) &= (2\pi)^{M/2}\big(-r''(0)\big)^{M/2-N}(1+\rho)^{2N-M} T^{-2N} mes_M(A_{1,M} \cap A_{2,M}) \\ &\quad \times u^{M+N(2/\alpha_1+2/\alpha_2-2)}\big(1+o(1)\big) \qquad \text{as } u \to \infty. \end{aligned} \tag{3.36}$$

Moreover, if we replace $\mathcal{C}$ with $\mathcal{C}^\circ$ defined in (3.7), then the above statement still holds.

The rest of the proof of Theorem 2.2 is the same as that of Theorem 2.1 and it is omitted here. □

4. Proof of lemmas

For proving Lemma 3.1, we will make use of the following lemma.

Lemma 4.1. *Let s_u and t_u be two $\mathbb{R}^N$-valued functions of u and let $\tau_u := t_u - s_u$. For any compact rectangles $\mathbb{S}$ and $\mathbb{T}$ in $\mathbb{R}^N$, define*

$$\begin{aligned} \xi_u(s) &:= u\big(X_1\big(s_u + u^{-2/\alpha_1}s\big) - u\big) + x \qquad \forall s \in \mathbb{S}, \\ \eta_u(t) &:= u\big(X_2\big(t_u + u^{-2/\alpha_2}t\big) - u\big) + y \qquad \forall t \in \mathbb{T} \end{aligned} \tag{4.1}$$

and for any $t \in \mathbb{R}^N$, let

$$\begin{aligned} \xi(t) &:= \sqrt{c_1}\chi_1(t) - \frac{c_1|t|^{\alpha_1}}{1+\rho}, \\ \eta(t) &:= \sqrt{c_2}\chi_2(t) - \frac{c_2|t|^{\alpha_2}}{1+\rho}, \end{aligned} \tag{4.2}$$

where $\chi_1(t), \chi_2(t)$ are two independent fractional Brownian motions with indices $\alpha_1/2$ and $\alpha_2/2$, respectively. Then, the finite dimensional distributions (abbr. f.d.d.) of $(\xi_u(\cdot), \eta_u(\cdot))$, given $X_1(s_u) = u - \frac{x}{u}$, $X_2(t_u) = u - \frac{y}{u}$, converge uniformly to the f.d.d. of $(\xi(\cdot), \eta(\cdot))$ for all s_u and t_u that satisfy $|\tau_u| \le C_0\sqrt{\log u}/u$. Furthermore, as $u \to \infty$,

$$\begin{aligned} &\mathbb{P}\Big(\max_{s\in\mathbb{S}} \xi_u(s) > x, \max_{t\in\mathbb{T}} \eta_u(t) > y \Big| X_1(s_u) = u - \frac{x}{u}, X_2(t_u) = u - \frac{y}{u}\Big) \\ &\quad \to \mathbb{P}\Big(\max_{s\in\mathbb{S}} \xi(s) > x, \max_{t\in\mathbb{T}} \eta(t) > y\Big), \end{aligned} \tag{4.3}$$

where the convergence is uniform for all s_u and t_u that satisfy $|\tau_u| \le C_0\sqrt{\log u}/u$.

Proof. First, we prove the uniform convergence of finite dimensional distributions. Given $X_1(s_u) = u - \frac{x}{u}$, $X_2(t_u) = u - \frac{y}{u}$, the distribution of the bivariate random field $(\xi_u(\cdot), \eta_u(\cdot))$ is still Gaussian. Thanks to the following lemma (whose proof will be given at the end of this section), it suffices to prove the uniform convergence of conditional mean and conditional variance.

Lemma 4.2. *Let $X(u,\tau_u) = (X_1(u,\tau_u), \dots, X_n(u,\tau_u))^T$ be a Gaussian random vector with mean $\mu(u,\tau_u) = (\mu_1(u,\tau_u), \dots, \mu_n(u,\tau_u)^T$ and covariance matrix $\Sigma(u,\tau_u)$ with entries $\sigma_{ij}(u,\tau_u) = \mathrm{Cov}(X_i(u,\tau_u), X_j(u,\tau_u))$, $i, j = 1, 2, \dots, n$. Similarly, let $X = (X_1, \dots, X_n)^T$ be a Gaussian random vector with mean $\mu = (\mu_1, \dots, \mu_n)$ and covariance matrix $\Sigma = (\sigma_{ij})_{i,j=1,\dots,n}$. Assume that Σ is non-singular. Let $F_u(\cdot)$ and $F(\cdot)$ be the distribution functions of $X(u,\tau_u)$ and X respectively. If*

$$\begin{aligned} &\lim_{u\to\infty} \max_{\tau_u} \big|\mu_j(u,\tau_u) - \mu_j\big| = 0, \\ &\lim_{u\to\infty} \max_{\tau_u} \big|\sigma_{ij}(u,\tau_u) - \sigma_{ij}\big| = 0, \qquad i, j = 1, 2, \dots, n, \end{aligned} \tag{4.4}$$

then for any $x \in \mathbb{R}^N$,

$$\lim_{u\to\infty} \max_{\tau_u} \big|F_u(x) - F(x)\big| = 0. \tag{4.5}$$

We continue with the proof of Lemma 4.1 and postpone the proof of Lemma 4.2 to the end of this section. Recall that, for two random vectors $X, Y \in \mathbb{R}^m$, their covariance is defined as $\mathrm{Cov}(X,Y) := \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)^T]$ and the variance matrix of X is defined as $\mathrm{Var}(X) :=$

$\mathrm{Cov}(X, X)$. The conditional mean of $(\xi_u(t), \eta_u(t))^T$ given $X_1(s_u) = u - \frac{x}{u}$, $X_2(t_u) = u - \frac{y}{u}$, is

$$\begin{aligned}
&\mathbb{E}\left(\begin{matrix} \xi_u(t) \\ \eta_u(t) \end{matrix} \,\middle|\, \begin{matrix} X_1(s_u) = u - \dfrac{x}{u} \\ X_2(t_u) = u - \dfrac{y}{u} \end{matrix} \right) \\
&= \mathbb{E}\begin{pmatrix} \xi_u(t) \\ \eta_u(t) \end{pmatrix} \\
&\quad + \mathrm{Cov}\left(\begin{pmatrix} \xi_u(t) \\ \eta_u(t) \end{pmatrix}, \begin{pmatrix} X_1(s_u) \\ X_2(t_u) \end{pmatrix} \right) \left(\mathrm{Var}\begin{pmatrix} X_1(s_u) \\ X_2(t_u) \end{pmatrix} \right)^{-1} \begin{pmatrix} u - \dfrac{x}{u} \\ u - \dfrac{y}{u} \end{pmatrix} \\
&= \begin{pmatrix} -u^2 + x \\ -u^2 + y \end{pmatrix} + \frac{u}{1 - r^2(|\tau_u|)} \\
&\quad \times \begin{pmatrix} r_{11}\big(s_u + u^{-2/\alpha_1} t, s_u\big) & r\big(\big|\tau_u - u^{-2/\alpha_1} t\big|\big) \\ r\big(\big|\tau_u + u^{-2/\alpha_2} t\big|\big) & r_{22}\big(t_u + u^{-2/\alpha_2} t, t_u\big) \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & -r\big(|\tau_u|\big) \\ -r\big(|\tau_u|\big) & 1 \end{pmatrix} \begin{pmatrix} u - \dfrac{x}{u} \\ u - \dfrac{y}{u} \end{pmatrix} \\
&\triangleq \begin{pmatrix} a_1(u) \\ a_2(u) \end{pmatrix},
\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
a_1(u) = &- \frac{u^2(1 - r_{11}(s_u + u^{-2/\alpha_1} t, s_u)) - u^2(r(|\tau_u - u^{-2/\alpha_1} t|) - r(|\tau_u|))}{1 + r(|\tau_u|)} \\
&+ \frac{(x - yr(|\tau_u|))(1 - r_{11}(s_u + u^{-2/\alpha_1} t, s_u))}{1 - r^2(|\tau_u|)} \\
&+ \frac{(y - xr(|\tau_u|))(r(|\tau_u|) - r(|\tau_u - u^{-2/\alpha_1} t|))}{1 - r^2(|\tau_u|)}
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
a_2(u) = &- \frac{u^2(1 - r_{22}(t_u + u^{-2/\alpha_2} t, t_u)) - u^2(r(|\tau_u + u^{-2/\alpha_2} t|) - r(|\tau_u|))}{1 + r(|\tau_u|)} \\
&+ \frac{(y - xr(|\tau_u|))(1 - r_{22}(t_u + u^{-2/\alpha_1} t, t_u))}{1 - r^2(|\tau_u|)} \\
&+ \frac{(x - yr(|\tau_u|))(r(|\tau_u|) - r(|\tau_u + u^{-2/\alpha_2} t|))}{1 - r^2(|\tau_u|)}.
\end{aligned} \tag{4.8}$$

Applying the mean value theorem twice, we see that for u large enough,

$$\begin{aligned}\left|r\left(\left|\tau_u+u^{-2/\alpha}t\right|\right)-r\left(|\tau_u|\right)\right| &\le \left|u^{-2/\alpha}t\right|\cdot \max_{\substack{s\text{ is between}\\ |\tau_u|\text{ and }|\tau_u+u^{-2/\alpha}t|}}\left|r'(s)\right| \\ &\le \left|u^{-2/\alpha}t\right|\cdot \max_{|s|\le 2C_0\sqrt{\log u}/u}\left|r'(s)\right| \\ &\le \left|u^{-2/\alpha}t\right|\cdot \max_{|s|\le 2C_0\sqrt{\log u}/u}\left(|s|\cdot\max_{|t|\le|s|}\left|r''(t)\right|\right) \\ &\le 2C_0|t|\sqrt{\log u}\cdot u^{-1-2/\alpha}\cdot \max_{|t|\le 2C_0\sqrt{\log u}/u}\left|r''(t)\right| \\ &\le 4C_0\left|r''(0)\right||t|\sqrt{\log u}\cdot u^{-1-2/\alpha},\end{aligned} \tag{4.9}$$

where the second inequality holds because of $u^{-2/\alpha}=o(\sqrt{\log u}/u)$, as $u\to\infty$ and the last inequality holds since $r''(\cdot)$ is continuous in a neighborhood of zero. Thus, (4.9) implies that, as $u\to\infty$,

$$u^2\left|r\left(\left|\tau_u+u^{-2/\alpha}t\right|\right)-r\left(|\tau_u|\right)\right|\le 4C_0\left|r''(0)\right||t|\sqrt{\log u}\cdot u^{1-2/\alpha}\to 0, \tag{4.10}$$

where the convergence is uniform for all s_u and t_u that satisfy $|\tau_u|\le C_0\sqrt{\log u}/u$. We also notice that for $i=1,2$ and all $s\in\mathbb{R}^N$,

$$1-r_{ii}\left(s+u^{-2/\alpha}t,s\right)=c_iu^{-2}|t|^{\alpha_i}+o\left(u^{-2}\right)\qquad\text{as } u\to\infty. \tag{4.11}$$

By (4.6), (4.10), and (4.11), we conclude that, as $u\to\infty$,

$$\mathbb{E}\left(\begin{pmatrix}\xi_u(t)\\ \eta_u(t)\end{pmatrix}\middle|\begin{matrix}X_1(s_u)=u-\dfrac{x}{u}\\ X_2(t_u)=u-\dfrac{y}{u}\end{matrix}\right)\to\begin{pmatrix}-\dfrac{c_1|t|^{\alpha_1}}{1+\rho}\\ -\dfrac{c_2|t|^{\alpha_2}}{1+\rho}\end{pmatrix}, \tag{4.12}$$

where the convergence is uniform w.r.t. s_u and t_u satisfying $|\tau_u|\le C_0\sqrt{\log u}/u$.

Next, we consider the conditional covariance matrix of $(\xi_u(t)-\xi_u(s),\eta_u(t)-\eta_u(s))^T$.

$$\begin{aligned}&\operatorname{Var}\left(\begin{pmatrix}\xi_u(t)-\xi_u(s)\\ \eta_u(t)-\eta_u(s)\end{pmatrix}\middle|\begin{matrix}X_1(s_u)\\ X_2(t_u)\end{matrix}\right)\\ &\quad=\operatorname{Var}\begin{pmatrix}\xi_u(t)-\xi_u(s)\\ \eta_u(t)-\eta_u(s)\end{pmatrix}\\ &\qquad-\operatorname{Cov}\left(\begin{pmatrix}\xi_u(t)-\xi_u(s)\\ \eta_u(t)-\eta_u(s)\end{pmatrix},\begin{pmatrix}X_1(s_u)\\ X_2(t_u)\end{pmatrix}\right)\\ &\qquad\times\operatorname{Var}\begin{pmatrix}X_1(s_u)\\ X_2(t_u)\end{pmatrix}^{-1}\operatorname{Cov}\left(\begin{pmatrix}\xi_u(t)-\xi_u(s)\\ \eta_u(t)-\eta_u(s)\end{pmatrix},\begin{pmatrix}X_1(s_u)\\ X_2(t_u)\end{pmatrix}\right)^T.\end{aligned} \tag{4.13}$$

Let $h_u(t,s) := r(|\tau_u + u^{-2/\alpha_2}t - u^{-2/\alpha_1}s|)$. Applying (4.10) and (4.11), we obtain

$$
\begin{aligned}
&\operatorname{Var}\begin{pmatrix}\xi_u(t)-\xi_u(s)\\ \eta_u(t)-\eta_u(s)\end{pmatrix}\\
&=\begin{pmatrix}
\begin{array}{l}2u^2\bigl(1-r_{11}\bigl(s_u\\ \qquad +u^{-2/\alpha_1}s, s_u+u^{-2/\alpha_1}t\bigr)\bigr)\end{array} & \begin{array}{l}u^2\bigl(h_u(t,t)-h_u(s,t)\\ \qquad -h_u(t,s)+h_u(s,s)\bigr)\end{array}\\
\begin{array}{l}u^2\bigl(h_u(t,t)-h_u(s,t)\\ \qquad -h_u(t,s)+h_u(s,s)\bigr)\end{array} & \begin{array}{l}2u^2\bigl(1-r_{22}\bigl(t_u\\ \qquad +u^{-2/\alpha_2}s, t_u+u^{-2/\alpha_2}t\bigr)\bigr)\end{array}
\end{pmatrix} \qquad (4.14)\\
&=\begin{pmatrix} 2c_1|t-s|^{\alpha_1}\bigl(1+o(1)\bigr) & o(1)\\ o(1) & 2c_2|t-s|^{\alpha_2}\bigl(1+o(1)\bigr)\end{pmatrix},
\end{aligned}
$$

where $o(1)$ converges to zero uniformly w.r.t. τ_u satisfying $|\tau_u| \le C_0\sqrt{\log u}/u$, as $u \to \infty$. Also, we have

$$
\begin{aligned}
&\operatorname{Cov}\left[\begin{pmatrix}\xi_u(t)-\xi_u(s)\\ \eta_u(t)-\eta_u(s)\end{pmatrix}, \begin{pmatrix}X_1(s_u)\\ X_2(t_u)\end{pmatrix}\right]\\
&=\begin{pmatrix}
\begin{array}{l}u\bigl(r_{11}\bigl(s_u+u^{-2/\alpha_1}t, s_u\bigr)\\ \qquad -r_{11}\bigl(s_u+u^{-2/\alpha_1}s, s_u\bigr)\bigr)\end{array} & \begin{array}{l}u\bigl(r\bigl(\bigl|\tau_u-u^{-2/\alpha_1}t\bigr|\bigr)\\ \qquad -r\bigl(\bigl|\tau_u-u^{-2/\alpha_1}s\bigr|\bigr)\bigr)\end{array}\\
\begin{array}{l}u\bigl(r\bigl(\bigl|\tau_u+u^{-2/\alpha_2}t\bigr|\bigr)\\ \qquad -r\bigl(\bigl|\tau_u+u^{-2/\alpha_2}s\bigr|\bigr)\bigr)\end{array} & \begin{array}{l}u\bigl(r_{22}\bigl(t_u+u^{-2/\alpha_2}t, t_u\bigr)\\ \qquad -r_{22}\bigl(t_u+u^{-2/\alpha_2}s, t_u\bigr)\bigr)\end{array}
\end{pmatrix} \qquad (4.15)\\
&=\begin{pmatrix} o(1) & o(1)\\ o(1) & o(1)\end{pmatrix},
\end{aligned}
$$

as $u \to \infty$, and

$$
\operatorname{Var}\begin{pmatrix}X_1(s_u)\\ X_2(t_u)\end{pmatrix}^{-1} = \frac{1}{1-r^2(|\tau_u|)}\begin{pmatrix} 1 & -r\bigl(|\tau_u|\bigr)\\ -r\bigl(|\tau_u|\bigr) & 1\end{pmatrix}. \tag{4.16}
$$

By (4.13)–(4.16), we conclude that as $u \to \infty$,

$$
\operatorname{Var}\left(\begin{pmatrix}\xi_u(t)-\xi_u(s)\\ \eta_u(t)-\eta_u(s)\end{pmatrix}\middle| \begin{matrix}X_1(s_u)\\ X_2(t_u)\end{matrix}\right) \to \begin{pmatrix} 2c_1|t-s|^{\alpha_1} & 0\\ 0 & 2c_2|t-s|^{\alpha_2}\end{pmatrix}, \tag{4.17}
$$

where the convergence is uniform w.r.t. τ_u satisfying $|\tau_u| \le C_0\sqrt{\log u}/u$. Hence, the uniform convergence of f.d.d. in Lemma 4.1 follows from (4.12), (4.17) and Lemma 4.2.

Now we prove the second part of Lemma 4.1. The continuous mapping theorem (see, e.g., Billingsley [8], page 30) can be used to prove (4.3) holds when s_u and t_u are fixed. Since we need to prove uniform convergence w.r.t. s_u and t_u, we use a discretization method instead. Let

$$
\begin{aligned}
f(u,x,y) := \mathbb{P}\Bigl(&\max_{s\in\mathbb{S}} \xi_u(s) > x, \max_{t\in\mathbb{T}} \eta_u(t) > y \Bigm| \\
&X_1(s_u) = u - \frac{x}{u}, X_2(t_u) = u - \frac{y}{u}\Bigr)
\end{aligned} \tag{4.18}
$$

and

$$f(x,y) := \mathbb{P}\Big(\max_{s\in\mathbb{S}} \xi(s) > x, \max_{t\in\mathbb{T}} \eta(t) > y\Big). \tag{4.19}$$

Without loss of generality, suppose that $\mathbb{S} = [a,b]^N$ and $\mathbb{T} = [c,d]^N$, where $a < b, c < d$. For any $\delta \in (0,1)$, let $m = \lfloor \frac{b-a}{\delta} \rfloor$, $n = \lfloor \frac{d-c}{\delta} \rfloor$ and let

$$\begin{aligned} \mathcal{S}_m &:= \big\{ s_{\mathbf{k}} | s_{\mathbf{k}} = (x_{k_1}, \dots, x_{k_N}), \mathbf{k} = (k_1, \dots, k_N) \in \{0, 1, \dots, m+1\}^N \big\}, \\ \mathcal{T}_n &:= \big\{ t_{\mathbf{l}} | t_{\mathbf{l}} = (y_{l_1}, \dots, y_{l_N}), \mathbf{l} = (l_1, \dots, l_N) \in \{0, 1, \dots, n+1\}^N \big\}, \end{aligned}$$

where x_i, y_i are defined as

$$\begin{aligned} &a = x_0 < x_1 < \cdots < x_m \le x_{m+1} = b, \qquad x_i = a + i\delta, i = 0, 1, \dots, m, \\ &c = y_0 < y_1 < \cdots < y_n \le y_{n+1} = d, \qquad y_i = c + i\delta, i = 0, 1, \dots, n. \end{aligned} \tag{4.20}$$

Then $[a,b]^N \times [c,d]^N$ can be divided into δ-cubes with vertices in $\mathcal{S}_m \times \mathcal{T}_n$.

The function $f(u,x,y)$ in (4.18) is bounded below by

$$\begin{aligned} f_{m,n}(u,x,y) := \mathbb{P}\bigg(&\max_{s\in\mathcal{S}_m} \xi_u(s) > x, \max_{t\in\mathcal{T}_n} \eta_u(t) > y \bigg| \\ &X_1(s_u) = u - \frac{x}{u}, X_2(t_u) = u - \frac{y}{u} \bigg) \end{aligned} \tag{4.21}$$

and is bounded above by $g_{m,n}(u,x,y)$ which is defined as

$$\begin{aligned} &\mathbb{P}\bigg(\max_{s\in\mathcal{S}_m} \xi_u(s) > x - \varepsilon, \max_{t\in\mathcal{T}_n} \eta_u(t) > y - \varepsilon \bigg| X_1(s_u) = u - \frac{x}{u}, X_2(t_u) = u - \frac{y}{u} \bigg) \\ &\quad + \mathbb{P}\bigg(\max_{s\in\mathbb{S}} \xi_u(s) > x, \max_{s\in\mathcal{S}_m} \xi_u(s) \le x - \varepsilon \bigg| X_1(s_u) = u - \frac{x}{u}, X_2(t_u) = u - \frac{y}{u} \bigg) \\ &\quad + \mathbb{P}\bigg(\max_{t\in\mathbb{T}} \eta_u(t) > y, \max_{t\in\mathcal{T}_n} \eta_u(t) \le y - \varepsilon \bigg| X_1(s_u) = u - \frac{x}{u}, X_2(t_u) = u - \frac{y}{u} \bigg) \\ &\quad \triangleq f_{m,n}(u, x-\varepsilon, y-\varepsilon) + s_{m,n}(u,x,y) + t_{m,n}(u,x,y), \end{aligned} \tag{4.22}$$

where $\varepsilon > 0$ is any small constant. Let

$$f_{m,n}(x,y) := \mathbb{P}\Big(\max_{s\in\mathcal{S}_m} \xi(s) > x, \max_{t\in\mathcal{T}_n} \eta(t) > y\Big). \tag{4.23}$$

Since the finite dimensional distributions of $(\xi_u(\cdot), \eta_u(\cdot))$ converge uniformly to those of $(\xi(\cdot), \eta(\cdot))$, we have

$$\lim_{u\to\infty} \max_{|\tau_u| \le C_0\sqrt{\log u}/u} \Big| f_{m,n}(u,x,y) - f_{m,n}(x,y) \Big| = 0. \tag{4.24}$$

The continuity of the trajectory of $(\xi(\cdot), \eta(\cdot))$ yields

$$\lim_{\substack{m\to\infty \\ n\to\infty}} f_{m,n}(x, y) = f(x, y). \tag{4.25}$$

By (4.24) and (4.25), we conclude

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \lim_{u\to\infty} \max_{|\tau_u|\le C_0\sqrt{\log u}/u} \left| f_{m,n}(u, x, y) - f(x, y) \right| = 0. \tag{4.26}$$

Let us consider the conditional probability $s_{m,n}(u, x, y)$ in (4.22).

$$\begin{aligned}
& s_{m,n}(u, x, y) \\
& \quad \le \mathbb{P}\left(\max_{|s-t|\le\delta} \left|\xi_u(s) - \xi_u(t)\right| > \varepsilon \,\middle|\, X_1(s_u) = u - \frac{x}{u}, X_2(t_u) = u - \frac{y}{u} \right) \\
& \quad \le \frac{1}{\varepsilon}\mathbb{E}\left(\max_{|s-t|\le\delta} \left|\xi_u(s) - \xi_u(t)\right| \,\middle|\, X_1(s_u) = u - \frac{x}{u}, X_2(t_u) = u - \frac{y}{u} \right) \\
& \quad = \frac{1}{\varepsilon}\mathbb{E}_{\mathbb{P}_u}\left(\max_{|s-t|\le\delta} \left|x(s) - x(t)\right| \right),
\end{aligned} \tag{4.27}$$

where $\mathbb{P}_u$ is the probability measure on $(C(\mathbb{S}), \mathcal{B}(C(\mathbb{S}))$ defined as

$$\mathbb{P}_u(A) := \mathbb{P}\left(\xi_u(\cdot) \in A \,\middle|\, X_1(s_u) = u - \frac{x}{u}, X_2(t_u) = u - \frac{y}{u} \right),$$

for all $A \in \mathcal{B}(C(\mathbb{S}))$ and $x(\cdot)$ is the coordinate random element on $(C(\mathbb{S}), \mathcal{B}(C(\mathbb{S})), \mathbb{P}_u)$, i.e., $x(t, \omega) = \omega(t), \forall \omega \in C(\mathbb{S})$ and $t \in \mathbb{S}$. Consider the canonical metric

$$\begin{aligned}
d_u(s, t) &:= \left[\mathbb{E}_{\mathbb{P}_u}\left(\left|x(s) - x(t)\right|^2\right)\right]^{1/2} \\
&= \left[\mathbb{E}\left(\left|\xi_u(s) - \xi_u(t)\right|^2 \,\middle|\, X_1(s_u) = u - \frac{x}{u}, X_2(t_u) = u - \frac{y}{u} \right)\right]^{1/2}.
\end{aligned}$$

By (4.17), for u large enough and all s_u, t_u such that $|\tau_u| \le C_0\sqrt{\log u}/u$, we have

$$d_u(s, t) \le 2\sqrt{c_1}|s - t|^{\alpha_1/2}, \tag{4.28}$$

which implies $\forall s \in \mathbb{S}$,

$$\left\{ t \in \mathbb{S} \,\middle|\, |t - s| \le (\varepsilon/2\sqrt{c_1})^{2/\alpha_1} \right\} \subseteq \left\{ t \in \mathbb{S} \,\middle|\, d_u(s, t) \le \varepsilon \right\}.$$

Hence

$$N_{d_u}(\mathbb{S}, \varepsilon) \le C_0 \varepsilon^{-2N/\alpha_1}, \tag{4.29}$$

where $N_{d_u}(\mathbb{S}, \varepsilon)$ denotes the minimum number of d_u-balls with radius ε that are needed to cover $\mathbb{S}$. By Dudley's theorem (see, e.g., Theorem 1.3.3 in Adler and Taylor [4]) and (4.28), we have

$$\mathbb{E}_{\mathbb{P}_u}\Big(\max_{|s-t|\le\delta}\big|x(s)-x(t)\big|\Big) \le K\int_0^{2\sqrt{c_1}\delta^{\alpha_1/2}} \sqrt{\log N_{d_u}(\mathbb{S}, \varepsilon)}\, d\varepsilon, \tag{4.30}$$

where $K < \infty$ is a constant (which does not depend on δ) and, thanks to (4.29), the last integral goes to 0 as $\delta \to 0$ (or, equivalently, as $m \to \infty, n \to \infty$). By (4.27) and (4.30), we conclude that

$$\lim_{\substack{m\to\infty\\ n\to\infty}} \limsup_{u\to\infty} \max_{|\tau_u|\le C_0\sqrt{\log u}|/u} \big|s_{m,n}(u,x,y)\big| = 0. \tag{4.31}$$

A similar argument shows that

$$\lim_{\substack{m\to\infty\\ n\to\infty}} \limsup_{u\to\infty} \max_{|\tau_u|\le C_0\sqrt{\log u}|/u} \big|t_{m,n}(u,x,y)\big| = 0. \tag{4.32}$$

Since

$$\begin{aligned}
&\big|f(u,x,y) - f(x,y)\big| \\
&\quad\le \big|f_{m,n}(u,x,y) - f(x,y)\big| + \big|g_{m,n}(u,x,y) - f(x,y)\big| \\
&\quad\le \big|f_{m,n}(u,x,y) - f(x,y)\big| + \big|f_{m,n}(u,x-\varepsilon,y-\varepsilon) - f(x-\varepsilon,y-\varepsilon)\big| \\
&\qquad + \big|f(x-\varepsilon,y-\varepsilon) - f(x,y)\big| + \big|s_{m,n}(u,x,y)\big| + \big|t_{m,n}(u,x,y)\big|,
\end{aligned} \tag{4.33}$$

we combine (4.26), (4.31) and (4.32) to obtain

$$\begin{aligned}
&\limsup_{u\to\infty} \max_{|\tau_u|\le C_0\sqrt{\log u}|/u} \big|f(u,x,y) - f(x,y)\big| \\
&\quad\le \big|f(x-\varepsilon,y-\varepsilon) - f(x,y)\big| \\
&\qquad + \lim_{\substack{m\to\infty\\ n\to\infty}} \limsup_{u\to\infty} \max_{|\tau_u|\le C_0\sqrt{\log u}/u} \Big(\big|f_{m,n}(u,x,y) - f(x,y)\big| + \big|s_{m,n}(u,x,y)\big| \\
&\qquad + \big|t_{m,n}(u,x,y)\big| + \big|f_{m,n}(u,x-\varepsilon,y-\varepsilon) - f(x-\varepsilon,y-\varepsilon)\big|\Big) \\
&\quad= \big|f(x-\varepsilon,y-\varepsilon) - f(x,y)\big|.
\end{aligned}$$

Since the last term $\to 0$ as $\varepsilon \downarrow 0$, we have completed the proof of the second part of the lemma. □

Now we are ready to prove the main lemmas in Section 3.

Proof of Lemma 3.1. Let $\phi(a,b)$ be the density of $(X_1(s_u), X_2(t_u))^T$, i.e.,

$$\phi(a,b) = \frac{1}{2\pi\sqrt{1-r^2(|\tau_u|)}} \exp\left\{-\frac{1}{2}\frac{a^2 - 2r(|\tau_u|)ab + b^2}{1-r^2(|\tau_u|)}\right\}. \tag{4.34}$$

By conditioning and a change of variables, the LHS of (3.1) becomes

$$\begin{aligned}
&\mathbb{P}\Big(\max_{s\in s_u+u^{-2/\alpha_1}\mathbb{S}} X_1(s)>u, \max_{t\in t_u+u^{-2/\alpha_2}\mathbb{T}} X_2(t)>u\Big)\\
&\quad=\int_{\mathbb{R}^2}\mathbb{P}\Big(\max_{s\in s_u+u^{-2/\alpha_1}\mathbb{S}} X_1(s)>u, \max_{t\in t_u+u^{-2/\alpha_2}\mathbb{T}} X_2(t)>u\Big|X_1(s_u)=u-\frac{x}{u},\\
&\qquad X_2(t_u)=u-\frac{y}{u}\Big)\phi\Big(u-\frac{x}{u},u-\frac{y}{u}\Big)u^{-2}\,dx\,dy\\
&\quad=\frac{1}{2\pi\sqrt{1-r^2(|\tau_u|)}u^2}\exp\Big(-\frac{u^2}{1+r(|\tau_u|)}\Big)\int_{\mathbb{R}^2} f(u,x,y)\tilde{\phi}(u,x,y)\,dx\,dy,
\end{aligned}\tag{4.35}$$

where $f(u,x,y)$ is defined in (4.18) with $\xi_u(\cdot)$, $\eta_u(\cdot)$ in (4.1), and where

$$\begin{aligned}
&\tilde{\phi}(u,x,y)\\
&\quad:=\exp\Big\{-\frac{1}{2(1-r^2(|\tau_u|))}\Big(\frac{x^2+y^2}{u^2}-2\big(1-r\big(|\tau_u|\big)\big)(x+y)-2r\big(|\tau_u|\big)\frac{xy}{u^2}\Big)\Big\}.
\end{aligned}$$

Since $\max_{|\tau_u|\le C_0\sqrt{\log u}/u}|r(|\tau_u|)-\rho|\to 0$ as $u\to\infty$, it is easy to check that

$$\max_{|\tau_u|\le C_0\sqrt{\log u}/u}\big|\tilde{\phi}(u,x,y)-e^{(x+y)/(1+\rho)}\big|\to 0 \qquad \text{as } u\to\infty. \tag{4.36}$$

Recall $H_\alpha(\cdot)$ in (2.1) and $f(x,y)$ in (4.19). Since $\xi(\cdot)$, $\eta(\cdot)$ are independent, and

$$\begin{aligned}
\big\{\xi(t),t\in\mathbb{R}^N\big\}&\stackrel{d}{=}\Big\{(1+\rho)\Big[\chi_1\Big(\Big(\frac{\sqrt{c_1}}{1+\rho}\Big)^{2/(\alpha_1)}t\Big)-\Big|\Big(\frac{\sqrt{c_1}}{1+\rho}\Big)^{2/\alpha_1}t\Big|^{\alpha_1}\Big],t\in\mathbb{R}^N\Big\},\\
\big\{\eta(t),t\in\mathbb{R}^N\big\}&\stackrel{d}{=}\Big\{(1+\rho)\Big[\chi_2\Big(\Big(\frac{\sqrt{c_2}}{1+\rho}\Big)^{2/\alpha_2}t\Big)-\Big|\Big(\frac{\sqrt{c_2}}{1+\rho}\Big)^{2/\alpha_2}t\Big|^{\alpha_2}\Big],t\in\mathbb{R}^N\Big\},
\end{aligned}$$

where $\stackrel{d}{=}$ means equality of all finite dimensional distributions, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} f(x,y)e^{(x+y)/(1+\rho)}\,dx\,dy\\
&\quad=\int_{\mathbb{R}} e^{x/(1+\rho)}\mathbb{P}\Big(\max_{s\in\mathbb{S}}\xi(s)>x\Big)\,dx\int_{\mathbb{R}} e^{y/(1+\rho)}\mathbb{P}\Big(\max_{t\in\mathbb{T}}\eta(t)>y\Big)\,dy\\
&\quad=(1+\rho)^2 H_{\alpha_1}\Big(\frac{c_1^{1/\alpha_1}\mathbb{S}}{(1+\rho)^{2/\alpha_1}}\Big)H_{\alpha_2}\Big(\frac{c_2^{1/\alpha_2}\mathbb{T}}{(1+\rho)^{2/\alpha_2}}\Big).
\end{aligned}\tag{4.37}$$

By (4.35) and (4.37), to conclude the lemma, it suffices to prove

$$\lim_{u\to\infty}\int_{\mathbb{R}^2}\max_{|\tau_u|\le C_0\sqrt{\log u}/u}\big|f(u,x,y)\tilde{\phi}(u,x,y)-f(x,y)e^{(x+y)/(1+\rho)}\big|\,dx\,dy=0. \tag{4.38}$$

First, applying Lemma 4.1 together with (4.36), we have

$$\max_{|\tau_u|\le C_0\sqrt{\log u}/u}\left|f(u,x,y)\tilde{\phi}(u,x,y)-f(x,y)e^{(x+y)/(1+\rho)}\right|\to 0 \qquad \text{as } u\to\infty. \tag{4.39}$$

Second, as in Ladneva and Piterbarg [18], we can find an integrable dominating function $g\in L(\mathbb{R}^2)$ such that for u large enough,

$$\max_{|\tau_u|\le C_0\sqrt{\log u}/u}\left|f(u,x,y)\tilde{\phi}(u,x,y)-f(x,y)e^{(x+y)/(1+\rho)}\right|\le g(x,y). \tag{4.40}$$

Therefore, (4.38) follows from the dominated convergence theorem. This finishes the proof. □

Proof of Lemma 3.2. We first claim that for any compact sets $\mathbb{S}$ and $\mathbb{T}$, the identity

$$H_\alpha(\mathbb{S})+H_\alpha(\mathbb{T})-H_\alpha(\mathbb{S}\cup\mathbb{T})=H_\alpha(\mathbb{S},\mathbb{T}) \tag{4.41}$$

holds. Indeed, if we let $X=\sup_{t\in\mathbb{S}}(\chi(t)-|t|^\alpha)$ and $Y=\sup_{t\in\mathbb{T}}(\chi(t)-|t|^\alpha)$, then

$$\begin{aligned}H_\alpha(\mathbb{S})+H_\alpha(\mathbb{T})-H_\alpha(\mathbb{S}\cup\mathbb{T})&=\mathbb{E}\big(e^X\big)+\mathbb{E}\big(e^Y\big)-\mathbb{E}\big(e^{\max(X,Y)}\big)\\&=\mathbb{E}\big(e^X1_{\{X<Y\}}\big)+\mathbb{E}\big(e^Y1_{\{X\ge Y\}}\big)=\mathbb{E}\big(e^{\min(X,Y)}\big)=H_\alpha(\mathbb{S},\mathbb{T}).\end{aligned}$$

Now let $\mathbb{T}_1=[0,T]^N$, $\mathbb{T}_2=[\mathbf{m}T,(\mathbf{m}+1)T]$ and $\mathbb{T}_3=[\mathbf{n}T,(\mathbf{n}+1)T]$. Consider the events

$$A=\Big\{\max_{s\in s_u+u^{-2/\alpha_1}\mathbb{T}_1}X_1(s)>u\Big\},\qquad B=\Big\{\max_{s\in s_u+u^{-2/\alpha_1}\mathbb{T}_2}X_1(s)>u\Big\},$$
$$C=\Big\{\max_{t\in t_u+u^{-2/\alpha_2}\mathbb{T}_1}X_2(t)>u\Big\},\qquad D=\Big\{\max_{t\in t_u+u^{-2/\alpha_2}\mathbb{T}_3}X_2(t)>u\Big\}.$$

It is easy to check that the LHS of (3.2) is equal to

$$\begin{aligned}&\mathbb{P}(A\cap B\cap C\cap D)\\&\quad=\big[\mathbb{P}(A\cap C)+\mathbb{P}(B\cap C)-\mathbb{P}\big((A\cup B)\cap C\big)\big]\\&\qquad+\big[\mathbb{P}(A\cap D)+\mathbb{P}(B\cap D)-\mathbb{P}\big((A\cup B)\cap D\big)\big]\\&\qquad-\big[\mathbb{P}\big(A\cap(C\cup D)\big)+\mathbb{P}\big(B\cap(C\cup D)\big)-\mathbb{P}\big((A\cup B)\cap(C\cup D)\big)\big].\end{aligned} \tag{4.42}$$

Let $R(u)=\frac{(1+\rho)^2}{2\pi\sqrt{1-\rho^2}}u^{-2}\exp(-\frac{u^2}{1+r(|\tau_u|)})$ and $q_{\alpha,c}=\frac{(1+\rho)^{2/\alpha}}{c^{1/\alpha}}$. By Lemma 3.1, we have

$$\begin{aligned}\mathbb{P}(A\cap C)&=R(u)H_{\alpha_1}\Big(\frac{\mathbb{T}_1}{q_{\alpha_1,c_1}}\Big)H_{\alpha_2}\Big(\frac{\mathbb{T}_1}{q_{\alpha_2,c_2}}\Big)\big(1+\gamma_1(u)\big),\\ \mathbb{P}(B\cap C)&=R(u)H_{\alpha_1}\Big(\frac{\mathbb{T}_2}{q_{\alpha_1,c_1}}\Big)H_{\alpha_2}\Big(\frac{\mathbb{T}_1}{q_{\alpha_2,c_2}}\Big)\big(1+\gamma_2(u)\big),\\ \mathbb{P}\big((A\cup B)\cap C\big)&=R(u)H_{\alpha_1}\Big(\frac{\mathbb{T}_1\cup\mathbb{T}_2}{q_{\alpha_1,c_1}}\Big)H_{\alpha_2}\Big(\frac{\mathbb{T}_1}{q_{\alpha_2,c_2}}\Big)\big(1+\gamma_3(u)\big),\end{aligned}$$

where, for $i = 1, 2, 3$, $\gamma_i(u) \to 0$ uniformly w.r.t. τ_u satisfying $|\tau_u| \le C_0\sqrt{\log u}/u$, as $u \to \infty$. These, together with (4.41), imply

$$\begin{aligned}&\mathbb{P}(A \cap C) + \mathbb{P}(B \cap C) - \mathbb{P}\big((A \cup B) \cap C\big)\\ &\quad = R(u) H_{\alpha_2}\left(\frac{\mathbb{T}_1}{q_{\alpha_2,c_2}}\right) H_{\alpha_1}\left(\frac{\mathbb{T}_1}{q_{\alpha_1,c_1}}, \frac{\mathbb{T}_2}{q_{\alpha_1,c_1}}\right)\big(1 + o(1)\big).\end{aligned} \tag{4.43}$$

Similarly, we have

$$\begin{aligned}&\mathbb{P}(A \cap D) + \mathbb{P}(B \cap D) - \mathbb{P}\big((A \cup B) \cap D\big)\\ &\quad = R(u) H_{\alpha_2}\left(\frac{\mathbb{T}_3}{q_{\alpha_2,c_2}}\right) H_{\alpha_1}\left(\frac{\mathbb{T}_1}{q_{\alpha_1,c_1}}, \frac{\mathbb{T}_2}{q_{\alpha_1,c_1}}\right)\big(1 + o(1)\big)\end{aligned} \tag{4.44}$$

and

$$\begin{aligned}&\mathbb{P}\big(A \cap (C \cup D)\big) + \mathbb{P}\big(B \cap (C \cup D)\big) - \mathbb{P}\big((A \cup B) \cap (C \cup D)\big)\\ &\quad = R(u) H_{\alpha_2}\left(\frac{\mathbb{T}_1 \cup \mathbb{T}_3}{q_{\alpha_2,c_2}}\right) H_{\alpha_1}\left(\frac{\mathbb{T}_1}{q_{\alpha_1,c_1}}, \frac{\mathbb{T}_2}{q_{\alpha_1,c_1}}\right)\big(1 + o(1)\big).\end{aligned} \tag{4.45}$$

By (4.42)–(4.45), we have

$$\begin{aligned}&\mathbb{P}(A \cap B \cap C \cap D)\\ &\quad = R(u) H_{\alpha_1}\left(\frac{\mathbb{T}_1}{q_{\alpha_1,c_1}}, \frac{\mathbb{T}_2}{q_{\alpha_1,c_1}}\right) H_{\alpha_2,c_2}\left(\frac{\mathbb{T}_1}{q_{\alpha_2,c_2}}, \frac{\mathbb{T}_3}{q_{\alpha_2,c_2}}\right)\big(1 + o(1)\big),\end{aligned}$$

which concludes the lemma. □

Proof of Lemma 3.3. Let $f(|t|) = \frac{1}{1+r(|t|)}$. Recall $\tau_{\mathbf{kl}}$ defined in (3.9) and $|\tau_{\mathbf{kl}}| \le 2\delta(u)$, when u is large. By Taylor's expansion,

$$f\big(|\tau_{\mathbf{kl}}|\big) = f(0) + \tfrac{1}{2} f''(0)|\tau_{\mathbf{kl}}|^2\big(1 + \gamma_{\mathbf{kl}}(u)\big),$$

where $f(0) = \frac{1}{1+\rho}$, $f''(0) = \frac{-r''(0)}{(1+\rho)^2}$ and, as $u \to \infty$, $\gamma_{\mathbf{kl}}(u)$ converges to zero uniformly w.r.t. all $(\mathbf{k}, \mathbf{l}) \in \mathcal{C}$. Therefore, for any $\varepsilon > 0$, we have

$$\sum_{(\mathbf{k},\mathbf{l})\in\mathcal{C}} e^{-(1/2)f''(0)(1+\varepsilon)u^2|\tau_{\mathbf{kl}}|^2} \le h(u) \le \sum_{(\mathbf{k},\mathbf{l})\in\mathcal{C}} e^{-(1/2)f''(0)(1-\varepsilon)u^2|\tau_{\mathbf{kl}}|^2} \tag{4.46}$$

when u is large enough. For $a > 0$, let

$$h(u, a) := \sum_{(\mathbf{k},\mathbf{l})\in\mathcal{C}} e^{-au^2|\tau_{\mathbf{kl}}|^2}. \tag{4.47}$$

In order to prove (3.12), it suffices to prove that

$$\lim_{u\to\infty} u^N d_1^N(u) d_2^N(u) h(u,a) = \left(\frac{\pi}{a}\right)^{N/2} mes_N(A_1 \cap A_2). \tag{4.48}$$

To this end, we write

$$\begin{aligned} &u^N d_1^N(u) d_2^N(u) h(u,a) \\ &= \frac{1}{u^N} \sum_{(\mathbf{k},\mathbf{l})\in\mathcal{C}} e^{-a\sum_{j=1}^N (l_j u d_2(u) - k_j u d_1(u))^2} \cdot \big(ud_1(u)\big)^N \big(ud_2(u)\big)^N. \end{aligned} \tag{4.49}$$

Let

$$p(u) := \frac{1}{u^N} \sum_{(\mathbf{k},\mathbf{l})\in\mathcal{C}} \min_{(s,t)\in u\Delta_{\mathbf{k}}^{(1)} \times u\Delta_{\mathbf{l}}^{(2)}} e^{-a|t-s|^2} \cdot \big(ud_1(u)\big)^N \big(ud_2(u)\big)^N,$$

$$q(u) := \frac{1}{u^N} \sum_{(\mathbf{k},\mathbf{l})\in\mathcal{C}} \max_{(s,t)\in u\Delta_{\mathbf{k}}^{(1)} \times u\Delta_{\mathbf{l}}^{(2)}} e^{-a|t-s|^2} \cdot \big(ud_1(u)\big)^N \big(ud_2(u)\big)^N.$$

It follows from (4.49) that

$$p(u) \le u^N d_1^N(u) d_2^N(u) h(u,a) \le q(u) \tag{4.50}$$

and

$$p(u) \le \frac{1}{u^N} \int_{\substack{s\in uA_1, t\in uA_2 \\ |t-s|\le C\sqrt{\log u}}} e^{-a|t-s|^2}\, dt\, ds \le q(u). \tag{4.51}$$

Observe that

$$\begin{aligned} &\frac{1}{u^N} \iint_{\substack{s\in uA_1, t\in uA_2 \\ |t-s|\le C\sqrt{\log u}}} e^{-a|t-s|^2}\, dt\, ds \\ &= \frac{1}{u^N} \iint_{\substack{y\in uA_1, x+y\in uA_2 \\ |x|\le C\sqrt{\log u}}} e^{-a|x|^2}\, dx\, dy \\ &= \frac{1}{u^N} \int_{|x|\le C\sqrt{\log u}} e^{-a|x|^2}\, dx \int_{\mathbb{R}^N} 1_{\{y\in uA_1\cap(uA_2-x)\}}\, dy \\ &= \int_{|x|\le C\sqrt{\log u}} e^{-a|x|^2}\, dx \int_{\mathbb{R}^N} 1_{\{z\in A_1\cap(A_2-x/u)\}}\, dz \\ &\to mes_N(A_1\cap A_2) \int_{\mathbb{R}^N} e^{-a|x|^2}\, dx = \left(\frac{\pi}{a}\right)^{N/2} mes_N(A_1\cap A_2), \end{aligned} \tag{4.52}$$

as $u \to \infty$, where the convergence holds by the dominated convergence theorem. Indeed, $\int_{\mathbb{R}^N} 1_{\{z \in A_1 \cap (A_2 - x/u)\}} dz$ is bounded by $\max_{|\varepsilon|<1} mes_N(A_1 \cap (A_2 - \varepsilon))$ uniformly for $|x| \le C\sqrt{\log u}$ when u is large enough.

It follows from (4.50)–(4.50) that, for concluding (4.48), it remains to verify

$$D(u) := q(u) - p(u) \to 0 \qquad \text{as } u \to \infty. \tag{4.53}$$

Define

$$\hat{\mathcal{D}} := \big\{(s,t) \in A_1 \times A_2 : |t-s| \le \delta(u) + \sqrt{N} d_1(u) + \sqrt{N} d_2(u)\big\}. \tag{4.54}$$

By the definition of $\mathcal{C}$ in (3.7), we see that $\mathcal{D} \subseteq \bigcup_{(\mathbf{k},\mathbf{l}) \in \mathcal{C}} \Delta_{\mathbf{k}}^{(1)} \times \Delta_{\mathbf{l}}^{(2)} \subseteq \hat{\mathcal{D}}$. Since $d_1(u) = o(\delta(u))$ and $d_2(u) = o(\delta(u))$ as $u \to \infty$, the set $\hat{\mathcal{D}}$ is a subset of $\tilde{\mathcal{D}} := \{(s,t) \in A_1 \times A_2 : |t-s| \le 2\delta(u)\}$ when u is large.

Write $D(u)$ in (4.53) as a sum over $(\mathbf{k},\mathbf{l}) \in \mathcal{C}$. To estimate the cardinality of $\mathcal{C}$, we notice that

$$mes_{2N}(\tilde{\mathcal{D}}) = \iint_{s \in A_1, t \in A_2} 1_{\{|t-s| \le 2\delta(u)\}}\, ds\, dt \tag{4.55}$$

$$= \int_{|x| \le 2\delta(u)} \int_{y \in A_1 \cap (A_2 - x)} dy\, dx \le K\delta(u)^N, \tag{4.56}$$

for all u large enough, where $K = 2^{N+1}\pi^{N/2}\Gamma^{-1}(N/2) \max_{|\varepsilon| \le 1} mes_N(A_1 \cap (A_2 - \varepsilon))$. Hence, for large u, the number of summands in (4.49) is bounded by

$$\#\big\{(\mathbf{k},\mathbf{l}) | (\mathbf{k},\mathbf{l}) \in \mathcal{C}\big\} \le \frac{mes_{2N}(\tilde{\mathcal{D}})}{mes_{2N}(\Delta_{\mathbf{k}}^{(1)} \times \Delta_{\mathbf{l}}^{(2)})} \le \frac{K\delta(u)^N}{d_1^N(u) d_2^N(u)}. \tag{4.57}$$

Next, by applying the inequality $e^{-x} - e^{-y} \le y - x$ for $y \ge x > 0$ to each summand in $D(u)$, we obtain

$$\begin{aligned} &\max_{(s,t) \in u\Delta_{\mathbf{k}}^{(1)} \times u\Delta_{\mathbf{l}}^{(2)}} e^{-a|t-s|^2} - \min_{(s,t) \in u\Delta_{\mathbf{k}}^{(1)} \times u\Delta_{\mathbf{l}}^{(2)}} e^{-a|t-s|^2} \\ &\le a\Big(\max_{(s,t) \in u\Delta_{\mathbf{k}}^{(1)} \times u\Delta_{\mathbf{l}}^{(2)}} |t-s|^2 - \min_{(s,t) \in u\Delta_{\mathbf{k}}^{(1)} \times u\Delta_{\mathbf{l}}^{(2)}} |t-s|^2\Big) \\ &= a \max\big(|t-s| + |t_1 - s_1|\big)\big(|t-s| - |t_1 - s_1|\big), \end{aligned} \tag{4.58}$$

where the last maximum is taken over $(s,t,s_1,t_1) \in u\Delta_{\mathbf{k}}^{(1)} \times u\Delta_{\mathbf{l}}^{(2)} \times u\Delta_{\mathbf{k}}^{(1)} \times u\Delta_{\mathbf{l}}^{(2)}$.

Since $|t-s| \le 2\delta(u)$ for all $(t,s) \in u\Delta_{\mathbf{k}}^{(1)} \times u\Delta_{\mathbf{l}}^{(2)}$ when u is large, the inequality $||t-s| - |t_1 - s_1|| \le |t - t_1| + |s - s_1|$ implies that (4.58) is at most

$$4a\sqrt{N}u^2\delta(u)\big(d_1(u) + d_2(u)\big) \tag{4.59}$$

when u is large enough. By (4.59) and (4.57), we can verify that

$$
\begin{aligned}
D(u) &\le \frac{1}{u^N}\frac{K(\delta(u))^N}{d_1^N(u)d_2^N(u)} 4a\sqrt{N}u^2\delta(u)\big(d_1(u)+d_2(u)\big)\big(ud_1(u)\big)^N\big(ud_2(u)\big)^N \\
&\le C_0(\log u)^{(N+1)/2}\big(u^{1-2/\alpha_1}+u^{1-2/\alpha_2}\big) \to 0 \qquad \text{as } u\to\infty.
\end{aligned}
$$

Therefore (4.48) holds. Similarly, we can check that the same statement holds while changing the set $\mathcal{C}$ to $\mathcal{C}^\circ$. $\square$

Proof of Lemma 3.4. Inequality (3.22) holds immediately by Lemma 6.2 in Piterbarg [23]. Hence, we only consider the case when $\mathbf{m}\neq\mathbf{0}$. Suppose that $\{X(t), t\in\mathbb{R}^N\}$ is a real valued continuous Gaussian process with $\mathbb{E}[X(t)]=0$ and covariance function $r(t)$ satisfying $r(t)=1-|t|^\alpha+o(|t|^\alpha)$ for a constant $\alpha\in(0,2)$. Applying Lemma 6.1 in Piterbarg [23], we see that for any $S>0$,

$$
\begin{aligned}
&\mathbb{P}\Big(\max_{t\in u^{-2/\alpha}[0,S]^N} X(t)>u,\ \max_{t\in u^{-2/\alpha}[\mathbf{m}S,(\mathbf{m}+1)S]} X(t)>u\Big) \\
&\quad= \mathbb{P}\Big(\max_{t\in u^{-2/\alpha}[0,S]^N} X(t)>u\Big)+\mathbb{P}\Big(\max_{t\in u^{-2/\alpha}[\mathbf{m}S,(\mathbf{m}+1)S]} X(t)>u\Big) \\
&\qquad -\mathbb{P}\Big(\max_{t\in u^{-2/\alpha}([0,S]^N\cup[\mathbf{m}S,(\mathbf{m}+1)S])} X(t)>u\Big) \\
&\quad= \big(H_\alpha([0,S]^N)+H_\alpha\big([\mathbf{m}S,(\mathbf{m}+1)S]\big)-H_\alpha\big([0,S]^N\cup\big[\mathbf{m}S,(\mathbf{m}+1)S\big]\big)\big) \\
&\qquad\times \frac{1}{\sqrt{2\pi}u}e^{-(1/2)u^2}\big(1+o(1)\big) \\
&\quad= H_\alpha\big([0,S]^N,\big[\mathbf{m}S,(\mathbf{m}+1)S\big]\big)\frac{1}{\sqrt{2\pi}u}e^{-(1/2)u^2}\big(1+o(1)\big) \qquad \text{as } u\to\infty,
\end{aligned}
\tag{4.60}
$$

where the last equality holds thanks to (4.41).

On the other hand, by applying Lemma 6.3 in Piterbarg [23] and the inequality $\inf_{s\in[0,1]^N,t\in[\mathbf{m},\mathbf{m}+1]}|s-t|\ge|m_{i_0}|-1$ (recall that i_0 is defined in Lemma 3.4), we have

$$
\begin{aligned}
&\mathbb{P}\Big(\max_{t\in u^{-2/\alpha}[0,S]^N} X(t)>u,\ \max_{t\in u^{-2/\alpha}[\mathbf{m}S,(\mathbf{m}+1)S]} X(t)>u\Big) \\
&\quad\le C_0S^{2N}\frac{1}{\sqrt{2\pi}u}e^{-(1/2)u^2}\exp\Big(-\frac{1}{8}\big(|m_{i_0}|-1\big)^\alpha S^\alpha\Big)
\end{aligned}
\tag{4.61}
$$

for all u large enough. It follows from (4.60) and (4.61) that

$$
H_\alpha\big([0,S]^N,\big[\mathbf{m}S,(\mathbf{m}+1)S\big]\big)\le C_0S^{2N}\exp\big(-\tfrac{1}{8}\big(|m_{i_0}|-1\big)^\alpha S^\alpha\big), \tag{4.62}
$$

which implies (3.24) by letting $S=\frac{c^{1/\alpha}T}{(1+\rho)^{2/\alpha}}$.

When $|m_{i_0}| = 1$, the above upper bound is not sharp. Instead, we derive (3.23) in Lemma 3.4 as follows. For concreteness, suppose that $i_0 = N$ and $m_N = 1$. By applying Lemmas 6.1–6.3 in Piterbarg [23], we have

$$\begin{aligned}
&\mathbb{P}\Big(\max_{t\in u^{-2/\alpha}[0,S]^N} X(t) > u, \max_{t\in u^{-2/\alpha}[\mathbf{m}S,(\mathbf{m}+1)S]} X(t) > u\Big) \\
&\quad\le \mathbb{P}\Big(\max_{t\in u^{-2/\alpha}(\prod_{j=1}^{N-1}[m_jS,(m_j+1)S]\times[S,S+\sqrt{S}])} X(t) > u\Big) \qquad (4.63)\\
&\qquad + \mathbb{P}\Big(\max_{t\in u^{-2/\alpha}[0,S]^N} X(t) > u, \max_{t\in u^{-2/\alpha}(\prod_{j=1}^{N-1}[m_jS,(m_j+1)S]\times[S+\sqrt{S},2S+\sqrt{S}])} X(t) > u\Big)\\
&\quad\le C_0 S^{N-1/2}\frac{1}{\sqrt{2\pi}u}e^{-(1/2)u^2} + C_0 S^{2N}\frac{1}{\sqrt{2\pi}u}e^{-(1/2)u^2}e^{-(1/8)S^{\alpha/2}}\\
&\quad\le C_0 S^{N-1/2}\frac{1}{\sqrt{2\pi}u}e^{-(1/2)u^2}
\end{aligned}$$

for u and S large. Hence, when $|m_{i_0}| = 1$, we have

$$H_\alpha\big([0,S]^N, \big[\mathbf{m}S,(\mathbf{m}+1)S\big]\big) \le C_0 S^{N-1/2} \qquad (4.64)$$

for large S. This implies (3.23) by letting $S = \frac{c^{1/\alpha}T}{(1+\rho)^{2/\alpha}}$.

Notice that

$$\#\Big\{m\in\mathbb{Z}^N \,\big|\, \max_{1\le i\le N}|m_i| = k\Big\} = (2k+1)^N - (2k-1)^N, \qquad k = 1,2,\ldots. \qquad (4.65)$$

By (3.23), (3.24) and the fact that $\int_T^\infty x^{N-1}e^{-ax^\alpha}\,dx \sim \frac{1}{a\alpha}T^{N-\alpha}e^{-aT^\alpha}$ as $T\to\infty$, we have

$$\begin{aligned}
\sum_{\mathbf{m}\neq\mathbf{0}} \mathcal{H}_{\alpha,c}(\mathbf{m}) &= \sum_{k=1}^\infty \sum_{|m_{i_0}|=k} \mathcal{H}_{\alpha,c}(\mathbf{m})\\
&\le C_0\big(3^N-1\big)T^{N-1/2}\\
&\quad + C_0\sum_{k=2}^\infty\big[(2k+1)^N-(2k-1)^N\big]T^{2N}e^{-(c/(8(1+\rho)^2))(k-1)^\alpha T^\alpha}\\
&\le C_0\big(3^N-1\big)T^{N-1/2} + C_0T^{2N}\int_1^\infty x^{N-1}e^{-(c/(8(1+\rho)^2))x^\alpha T^\alpha}\,dx \le C_0T^{N-1/2}
\end{aligned}$$

for T large enough. This completes the proof of Lemma 3.4. □

Proof of Lemma 3.5. The proof is similar to that of Lemma 3.3. Indeed, we only need to modify (4.49) and (4.55) in the proof of Lemma 3.3. For any $y = (y_1,\ldots,y_N)\in\mathbb{R}^N$ and $1\le i\le j\le N$, let $y_{i:j} = (y_i, \dots, y_j)$. On one hand, with a different scaling, $h(u,a)$ in (4.49) has the following asymptotics:

$$
\begin{aligned}
& u^{2N-M} d_1^N(u) d_2^N(u) h(u,a) \\
& \approx \frac{1}{u^M} \iint_{\substack{y \in uA_1, x+y \in uA_2 \\ |x| \le C\sqrt{\log u}}} e^{-a|x|^2}\, dx\, dy \\
& = \frac{1}{u^M} \int_{|x| \le C\sqrt{\log u}} e^{-a|x|^2} \Bigg(\int_{\mathbb{R}^M} 1_{\{y_{1:M} \in uA_{1,M} \cap (uA_{2,M} - x_{1:M})\}}\, dy_{1:M} \\
& \quad \times \prod_{j=M+1}^{N} \int_{\mathbb{R}} 1_{\{y_j \in [uS_j, uT_j] \cap [uT_j - x_j, uR_j - x_j]\}}\, dy_j \Bigg) dx \qquad (4.66) \\
& = \int_{|x| \le C\sqrt{\log u}} e^{-a|x|^2} \prod_{j=M+1}^{N} x_j 1_{\{x_j > 0\}} \left(\int_{\mathbb{R}^M} 1_{\{z_{1:M} \in A_{1,M} \cap (A_{2,M} - x_{1:M}/u)\}}\, dz_{1:M} \right) dx \\
& \to mes_M(A_{1,M} \cap A_{2,M}) \int_{\mathbb{R}^M} e^{-a|x_{1:M}|^2}\, dx_{1:M} \prod_{j=M+1}^{N} \int_0^\infty x_j e^{-ax_j^2}\, dx_j \\
& = 2^{M-N} \pi^{M/2} a^{M/2-N} mes_M(A_{1,M} \cap A_{2,M}),
\end{aligned}
$$

as $u \to \infty$. On the other hand, when u is large enough, $mes_{2N}(\tilde{\mathcal{D}})$ defined in (4.55) can be bounded above by

$$
\begin{aligned}
mes_{2N}(\tilde{\mathcal{D}}) &= \iint_{s \in A_1, t \in A_2} 1_{\{|t-s| \le 2\delta(u)\}}\, ds\, dt \\
&= \int_{|x| \le 2\delta(u)} \left(\int_{y_{1:M} \in A_{1,M} \cap (A_{2,M} - x_{1:M})} dy_{1:M} \right) \prod_{j=M+1}^{N} x_j 1_{\{x_j > 0\}}\, dx \\
&= \delta(u)^{2N-M} \int_{|z| \le 2} \left(\int_{y_{1:M} \in A_{1,M} \cap (A_{2,M} - z_{1:M}\delta(u))} dy_{1:M} \right) \prod_{j=M+1}^{N} z_j 1_{\{z_j > 0\}}\, dz \qquad (4.67) \\
&\le K\delta(u)^{2N-M},
\end{aligned}
$$

where $K = \max_{|\varepsilon| \le 1} mes_M(A_{1,M} \cap (A_{2,M} - \varepsilon)) \int_{|z| \le 2} \prod_{j=M+1}^{N} z_j 1_{\{z_j > 0\}}\, dz$.

By (4.66) and (4.67), (3.36) can be obtained through the same argument in the proof of Lemma 3.3. We omit the details. □

We end this section with the proof of Lemma 4.2.

Proof of Lemma 4.2. Let $f_{u,\tau_u}(\cdot)$ and $f(\cdot)$ be the density function of $X(u,\tau_u)$ and X, respectively. It suffices to prove that for all $x \in \mathbb{R}^N$,

$$\int_{\{y\le x\}} f(y) \max_{\tau_u} \left| \frac{f_{u,\tau_u(y)}}{f(y)} - 1 \right| dy \to 0 \qquad \text{as } u \to \infty, \tag{4.68}$$

where $\{y \le x\} = \prod_{i=1}^N (-\infty, x_i]$.

First, we will find an upper bound for $\max_{\tau_u} |f_{u,\tau_u}(y)/f(y) - 1|$. For any $\varepsilon > 0$, define

$$\Gamma(u,\tau_u) = \big(\gamma_{ij}(u,\tau_u)\big)_{i,j=1,\dots,n} := \frac{1}{\varepsilon}\big(\Sigma(u,\tau_u) - \Sigma\big),$$
$$e(u,\tau_u) = \big(e_i(u,\tau_u)\big)_{i=1,\dots,n} := \frac{1}{\varepsilon}\big(\mu(u,\tau_u) - \mu\big).$$

By assumption (4.4), there exists a constant $U > 0$ such that for all $u > U$,

$$\max_{\tau_u}\big|\mu_j(u,\tau_u) - \mu_j\big| < \varepsilon, \qquad \max_{\tau_u}\big|\sigma_{ij}(u,\tau_u) - \sigma_{ij}\big| < \varepsilon, \qquad i,j = 1,\dots,n,$$

which implies $|\gamma_{ij}(u,\tau_u)| \le 1$ and $|e_i(u,\tau_u)| \le 1$ for $u > U$.

Let $\Sigma^{-1} = (v_{ij})_{i,j=1,\dots,n}$ be the inverse of Σ. When ε is small, the determinant of $\Sigma(u,\tau_u)$ satisfies

$$\big|\Sigma(u,\tau_u)\big| = \big|\Sigma + \varepsilon\Gamma(u,\tau_u)\big| = |\Sigma|\big(1 + \varepsilon\,\mathrm{tr}\big(\Sigma^{-1}\Gamma(u,\tau_u)\big) + O\big(\varepsilon^2\big)\big),$$

where $O(\varepsilon^2)/\varepsilon^2$ is uniformly bounded w.r.t. τ_u for large u (see, e.g., Magnus and Neudecker [19], page 169). Hence, when ε is small enough, we have

$$\left| \frac{|\Sigma(u,\tau_u)|}{|\Sigma|} - 1 \right| \le 2\varepsilon\big|\mathrm{tr}\big(\Sigma^{-1}\Gamma(u,\tau_u)\big)\big| \le 2\varepsilon \sum_{i,j} |v_{ij}|. \tag{4.69}$$

Since $|\gamma_{ij}(u,\tau_u)| \le 1, \forall i,j = 1,\dots,n, \forall \tau_u$ for large u, as $\varepsilon \to 0$, the inverse of $\Sigma(u,\tau_u)$ can be written as

$$\Sigma(u,\tau_u)^{-1} = \Sigma^{-1} - \varepsilon\Sigma^{-1}\Gamma(u,\tau_u)\Sigma^{-1} + O\big(\varepsilon^2\big),$$

where $O(\varepsilon^2)/\varepsilon^2$ is a matrix whose entries are uniformly bounded and independent of τ_u for large u (see, e.g., Meyer [20], page 618). Hence,

$$\begin{aligned}
d_{u,\tau_u}(y) &:= -\tfrac{1}{2}\big[\big(y-\mu(u,\tau_u)\big)^T\Sigma^{-1}(u,\tau_u)\big(y-\mu(u,\tau_u)\big) - (y-\mu)^T\Sigma^{-1}(y-\mu)\big] \\
&= -\tfrac{1}{2}(y-\mu)^T\big(-\varepsilon\Sigma^{-1}\Gamma(u,\tau_u)\Sigma^{-1} + O\big(\varepsilon^2\big)\big)(y-\mu) \\
&\quad + \varepsilon e^T(u,\tau_u)\big(\Sigma^{-1} - \varepsilon\Sigma^{-1}\Gamma(u,\tau_u)\Sigma^{-1} + O\big(\varepsilon^2\big)\big)(y-\mu) \\
&\quad - \tfrac{1}{2}\varepsilon^2 e^T(u,\tau_u)\big(\Sigma^{-1} - \varepsilon\Sigma^{-1}\Gamma(u,\tau_u)\Sigma^{-1} + O\big(\varepsilon^2\big)\big)e(u,\tau_u).
\end{aligned}$$

Since $|\gamma_{ij}(u, \tau_u)|$ and $|e_i(u, \tau_u)|$ are uniformly bounded by 1 w.r.t. τ_u for all $u > U$, we derive that for any $y \in \mathbb{R}^N$,

$$\max_{\tau_u}\left|d_{u,\tau_u}(y)\right| \to 0 \qquad \text{as } u \to \infty. \tag{4.70}$$

By (4.69) and (4.70), for $y \in \mathbb{R}^N$,

$$\max_{\tau_u}\left|\frac{f_{u,\tau_u(y)}}{f(y)} - 1\right| = \max_{\tau_u}\left|e^{d_{u,\tau_u}(y)}\frac{|\Sigma(u,\tau_u)|^{-1/2}}{|\Sigma|^{-1/2}} - 1\right| \to 0 \qquad \text{as } u \to \infty. \tag{4.71}$$

If we could further find an integrable function $g(y)$ on $\mathbb{R}^N$,

$$f(y)\max_{\tau_u}\left|\frac{f_{u,\tau_u}(y)}{f(y)} - 1\right| \le g(y), \tag{4.72}$$

then (4.68) holds by the dominated convergence theorem.

Given a constant C_0, let $A_I := \{(a_{ij})_{i,j=1}^n \in \mathbb{R}^{N\times N} \,|\, \max_{i,j}|a_{i,j}| \le C_0\}, b_I := \{(b_i)_{i=1}^n \in \mathbb{R}^N \,|\, \max_i |b_i| \le C_0\}$. Then there exist constants C_2, C_3, such that

$$\left|x^T A x\right| \le C_2 x^T x, \qquad \left|b^T x\right| \le C_3 + x^T x \qquad \forall x \in \mathbb{R}^N, \forall A \in A_I, \forall b \in b_I.$$

Hence, there exists a constant $C_4 > 0$ such that

$$\left|d_{u,\tau_u}(y)\right| \le C_4\varepsilon(y-\mu)^T(y-\mu) + C_4\varepsilon. \tag{4.73}$$

By (4.69) and (4.73), for small ε and large u, there exists a constant K such that

$$\max_{\tau_u}\left|\frac{f_{u,\tau_u(y)}}{f(y)} - 1\right| \le K e^{C_4\varepsilon(y-\mu)^T(y-\mu)} + 1.$$

On the other hand, for all $y \in \mathbb{R}^N$,

$$f(y) \le (2\pi)^{-n/2}|\Sigma|^{-1/2}e^{-(\lambda/2)(y-\mu)^T(y-\mu)},$$

where λ is the minimum eigenvalue of Σ^{-1}. If we choose $\varepsilon < \frac{\lambda}{2C_4}$ and define

$$g(y) := (2\pi)^{-n/2}|\Sigma|^{-1/2}e^{-(\lambda/2)(y-\mu)^T(y-\mu)}\left(K e^{C_4\varepsilon(y-\mu)^T(y-\mu)} + 1\right),$$

then (4.72) holds and hence we have completed the proof. □

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