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High-resolution product quantization for Gaussian processes under sup-norm distortion

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We derive high-resolution upper bounds for optimal product quantization of pathwise continuous Gaussian processes with respect to the supremum norm on $[0, T]^d$. Moreover, we describe a product quantization design which attains this bound. This is achieved under very general assumptions on random series expansions of the process. It turns out that product quantization is asymptotically only slightly worse than optimal functional quantization. The results are applied to fractional Brownian sheets and the Ornstein–Uhlenbeck process.

Keywords: Gaussian process; high-resolution quantization; product quantization; series expansion

1. Introduction

In this paper, we investigate the functional quantization problem for pathwise continuous Gaussian processes $X = (X_t)_{t\in I}$, $I = [0, T]^d$, where the path space $E = \mathcal{C}(I)$ is endowed with the supremum norm. For any real separable space $(E, \|\cdot\|)$ and $r \in (0, \infty)$, optimal quantization means the best approximation in $L^r_E(\mathbb{P})$ of a random vector $X : (\Omega, \mathcal{A}, \mathbb{P}) \to E$ by random vectors $\widehat{X} : (\Omega, \mathcal{A}, \mathbb{P}) \to E$ taking finitely many values in E. If $N \in \mathbb{N}$, $\operatorname{card}(\widehat{X}(\Omega)) \le N$, then $\widehat{X}$ is called N-quantization. This leads to the minimal level N-*quantization* error defined by

$$e_{N,r}(X, E) := \inf\{(\mathbb{E}\|X - \widehat{X}\|^r)^{1/r} : \widehat{X} N\text{-quantization of } X\}, \tag{1.1}$$

provided $X \in L^r_E(\mathbb{P})$. When $E = \mathbb{R}^d$, this problem is known as *optimal vector quantization* and has been extensively investigated since the early 1950s, with some applications to signal processing and transmission (see Gersho and Gray [11]) and to model-based clustering in statistics (see e.g., Tarpey [25]). Beyond these classical applications, optimal quantization has been used as a space discretization device to solve nonlinear problems, such as those arising in optimal stopping theory (American-style option pricing, reflected BSDE, Bally and Pagès [2]), nonlinear filtering (Pagès and Pham [22]), forward-backward SDE (see Delarue and Menozzi [5]) and SPDE (see Gobet *et al.* [12]). The mathematical foundations are treated in Graf and Luschgy [13]. Much attention has been paid to the infinite-dimensional case. This is the so-called *functional quantization* of stochastic processes: the aim is to quantize some processes viewed as random vectors taking values in their path spaces. Recently, a first application of functional quantization to statistical clustering of functional data has been investigated (see Tarpey and Kinateder [26] and

Tarpey *et al.* [27]). The simplest application of functional quantization as a numerical method is to use it as an alternative to Monte Carlo simulation, using the quadrature formula

$$\mathbb{E}(F(X)) \approx \mathbb{E}(F(\widehat{X})) = \sum_{a\in\alpha} F(a)\mathbb{P}(\widehat{X}=a), \qquad \text{where } \alpha = \widehat{X}(\Omega),$$

for sufficiently regular functionals $F: E \to \mathbb{R}$. If $\widehat{X}$ is an L^r-optimal N-quantization and F is Lipschitz continuous, then the induced error is bounded by $[F]_{\mathrm{Lip}} e_{N,r}(X, E)$, $r \geq 1$. Some numerical applications are being developed for the pricing of path-dependent options (such as regular Asian options) in various models using $E = L^2([0, T], dt)$ (Black and Scholes, Heston, see Pagès and Printems [23], Wilbertz [29]). However, many important functionals of processes, like those related to barrier options or to options on maximum, are only continuous with respect to the sup-norm on $E = \mathcal{C}([0, T])$.

Let us now describe what we will call the *product quantization* scheme. Let X be a centered E-valued Gaussian random vector. Let $\xi_1, \xi_2, \ldots$ be i.i.d. $\mathcal{N}(0, 1)$-distributed random variables and let $(f_j)_{j\geq1}$ be a sequence in E such that $\sum_{j=1}^{\infty} \xi_j f_j$ converges a.s. in E and

$$X \stackrel{d}{=} \sum_{j=1}^{\infty} \xi_j f_j. \tag{1.2}$$

Let us call such a sequence *admissible* for X. For background on expansions for Gaussian random vectors, the reader is referred to Bogachev [4] and Ledoux and Talagrand [17]. One checks that $(f_j)_{j\geq1}$ is admissible for X if and only if $(f_j)_{j\geq1}$ is a normalized tight frame in the reproducing kernel Hilbert space (Cameron–Martin space) $H = H_X$, that is, $\{f_j, j \geq 1\} \subset H$ and $\sum_{j\geq1}(f_j, h)_H^2 = \|h\|_H^2$ for all $h \in H$ (see Luschgy and Pagès [21]). Then a sufficient (but not necessary) condition is that $(f_j)_{j\geq1}$ is an orthonormal basis of H_X.

For $m, N_1, \ldots, N_m \in \mathbb{N}$ with $\prod_{j=1}^{m} N_j \leq N$, let $\widehat{\xi}_j$ be an L^r-optimal N_j-quantization for ξ_j, that is, $(\mathbb{E}|\xi_j - \widehat{\xi}_j|^r)^{1/r} = e_{N_j,r}(\xi_j, \mathbb{R})$. An L^r-product N-quantization of X with respect to $(f_j)_{j\geq1}$ is then defined by

$$\widehat{X} := \widehat{X}^{(N_1,\ldots,N_m)} := \sum_{j=1}^{m} \hat{\xi}_j f_j \tag{1.3}$$

and the quantization error induced by $\widehat{X}$ is

$$(\mathbb{E}\|X - \widehat{X}\|^r)^{1/r}.$$

Note that if $\alpha_j = \widehat{\xi}_j(\Omega)$, then the codebook $\alpha = \widehat{X}(\Omega)$ of $\widehat{X}$ satisfies $\alpha = \{\sum_{j=1}^{m} a_j f_j : a \in \prod_{j=1}^{m} \alpha_j\}$ and

$$\left(\mathbb{E} \min_{a\in\alpha} \|X - a\|^r\right)^{1/r} \leq (\mathbb{E}\|X - \widehat{X}\|^r)^{1/r}.$$

The minimal Nth product quantization error is then defined by

$$e_{N,r}^{(\mathrm{prod})}(X, E) := \inf\{(\mathbb{E}\|X - \widehat{X}\|^r)^{1/r} : (f_j)_{j\geq1} \in E^{\mathbb{N}} \text{ admissible for } X,$$

$$\widehat{X} L^r\text{-product } N\text{-quantization w.r.t. } (f_j)\}. \tag{1.4}$$

Clearly, we have

$$e_{N,r}(X, E) \leq e_{N,r}^{(\text{prod})}(X, E). \tag{1.5}$$

We address the issue of high-resolution product quantization in $E = \mathcal{C}(I)$ under the sup-norm, which concerns the performance of $\widehat{X} = \widehat{X}^{(N_1,\dots,N_m)}$ under a suitable choice of the marginal quantization levels N_j and the behaviour of $e_{N,r}^{(\text{prod})}(X, \mathcal{C}(I))$ as $N \to \infty$. For a broad class of Gaussian processes, we derive high-resolution upper estimates for $e_{N,r}^{(\text{prod})}(X, \mathcal{C}(I))$. Furthermore, we describe a product quantization design $\widehat{X}$ which attains this bound. Combining these estimates with precise high-resolution formulas for $e_{N,r}(X, \mathcal{C}(I))$ (see Dereich *et al.* [6], Dereich and Scheutzow [7], Graf *et al.* [14]), one may typically conclude that

$$e_{N,r}^{(\text{prod})}(X, \mathcal{C}(I)) = O\big((\log\log N)^c e_{N,r}(X, \mathcal{C}(I))\big),$$

for some suitable constant $c > 0$. This suggests that the asymptotic quality of product quantization, which is based on easy computations, is only slightly worse than optimal quantization. The optimality of this rate for product quantization rate remains open, although one may reasonably guess that it is optimal.

The paper is organized as follows. In Section 2, we derive high-resolution upper estimates for $e_{N,r}^{(\text{prod})}(X, \mathcal{C}(I))$ under very general assumptions on expansions. Section 3 contains a collection of examples, including fractional Brownian sheets, Riemann–Liouville processes and the Ornstein–Uhlenbeck process.

It is convenient to use the symbols $\sim$ and $\approx$, where $a_n \sim b_n$ means $a_n/b_n \to 1$ and $a_n \approx b_n$ means $a_n = O(b_n)$ and $a_n = \Omega(b_n)$. Throughout, all logarithms are natural logarithms and $[x]$ denotes the integer part of the real number x.

2. High-resolution product quantization

We investigate high-resolution product functional quantization of centered continuous Gaussian processes $X = (X_t)_{t\in I}$ on $I = [0, T]^d$ in the space $E = \mathcal{C}(I)$ equipped with the sup-norm $\|x\| = \sup_{t\in I} |x(t)|$. Let

$$e_{N,r}^{(\text{prod})}(X) := e_{N,r}^{(\text{prod})}(X, \mathcal{C}(I)).$$

The subsequent setting comprises a broad class of processes.

Let $(f_j)_{j\geq 1} \in \mathcal{C}(I)^{\mathbb{N}}$ satisfy the following assumptions:

(A1) $\|f_j\| \leq C_1 j^{-\vartheta} \log(1 + j)^\gamma$ for every $j \geq 1$ with $\vartheta > 1/2, \gamma \geq 0$ and $C_1 < \infty$;

(A2) f_j is a-Hölder-continuous and $[f_j]_a \leq C_2 j^b$ for every $j \geq 1$ with $a \in (0, 1], b \in \mathbb{R}$ and $C_2 < \infty$, where

$$[f]_a = \sup_{s\neq t} \frac{|f(s) - f(t)|}{|s - t|^a}$$

(and $|t|$ denotes the l_2-norm of $t \in \mathbb{R}^d$).

In the sequel, finite constants depending only on the parameters $T, \vartheta, \gamma, a, b, C_1, C_2, d$ and r are denoted by C and may differ from one formula to another one. Other dependencies are explicitly indicated.

First, observe that by (A1), $\sum_{j=1}^{\infty} f_j(t)^2 \leq \sum_{j=1}^{\infty} \|f_j\|^2 < \infty$ for every $t \in I$, so we can define a centered Gaussian process Y by $Y_t := \sum_{j=1}^{\infty} \xi_j f_j(t)$. Using (A1) and (A2), we have, for $\rho \in (0, 1]$,

$$\begin{aligned} |f_j(s) - f_j(t)| &= |f_j(s) - f_j(t)|^{\rho} |f_j(s) - f_j(t)|^{1-\rho} \\ &\leq ([f_j]_a |s-t|^a)^{\rho} (2\|f_j\|)^{1-\rho} \\ &\leq C_{\rho} j^{\rho(b+\vartheta)-\vartheta} \log(1+j)^{\gamma(1-\rho)} |s-t|^{a\rho} \end{aligned}$$

and hence

$$\sum_{j=1}^{\infty} [f_j]_{a\rho}^2 < \infty \qquad \text{for every } \rho < \frac{\vartheta - 1/2}{(b+\vartheta)_+}.$$

This yields

$$\mathbb{E}|Y_s - Y_t|^2 = \sum_{j=1}^{\infty} |f_j(s) - f_j(t)|^2 \leq \left(\sum_{j=1}^{\infty} [f_j]_{a\rho}^2 \right) |s-t|^{2a\rho} \tag{2.1}$$

and using the Gaussian feature of Y, we obtain from the Kolmogorov criterion that Y has a continuous modification X. Consequently, (f_j) is admissible for X and

$$X = \sum_{j=1}^{\infty} \xi_j f_j \qquad \text{a.s.} \tag{2.2}$$

For $r \in [1, \infty)$, the quantization error induced by the L^r-product N-quantization $\widehat{X} := \widehat{X}^{(N_1,\ldots,N_m)}$ (see (1.3)) satisfies

$$\begin{aligned} (\mathbb{E}\|X - \widehat{X}\|^r)^{1/r} &= \|X - \widehat{X}\|_{L_E^r(\mathbb{P})} \\ &\leq \left\| \sum_{j=1}^{m} (\xi_j - \hat{\xi}_j) f_j \right\|_{L_E^r(\mathbb{P})} + \left\| \sum_{j \geq m+1} \xi_j f_j \right\|_{L_E^r(\mathbb{P})} \\ &\leq \sum_{j=1}^{m} \|\xi_j - \hat{\xi}_j\|_{L^r(\mathbb{P})} \|f_j\| + \left\| \sum_{j \geq m+1} \xi_j f_j \right\|_{L_E^r(\mathbb{P})} \end{aligned}$$

so that

$$(\mathbb{E}\|X-\widehat{X}\|^r)^{1/r} \le \sum_{j=1}^{m} \|f_j\| e_{N_j,r}(\mathcal{N}(0,1)) + \left(\mathbb{E}\left\|\sum_{j\ge m+1} \xi_j f_j\right\|^r\right)^{1/r}. \tag{2.3}$$

For $r \in (0,1)$, we have

$$(\mathbb{E}\|X-\widehat{X}\|^r)^{1/r} \le \mathbb{E}\|X-\widehat{X}\| \le \sum_{j=1}^{m} \|f_j\| \mathbb{E}|\xi_j - \widehat{\xi}_j| + \mathbb{E}\left\|\sum_{j\ge m+1} \xi_j f_j\right\|. \tag{2.4}$$

Let us now consider the truncation error.

Theorem 1. *Assume that* $(f_j)_{j\ge 1} \in \mathcal{C}(I)^{\mathbb{N}}$ *satisfies* (A1)–(A2). *Then*, *for every* $n \ge 2$ *and* $r \in (0,\infty)$,

$$\left(\mathbb{E}\left\|\sum_{j\ge n} \xi_j f_j\right\|^r\right)^{1/r} \le \frac{C(\log n)^{\gamma+1/2}}{n^{\vartheta-1/2}}$$

and

$$\left(\mathbb{E}\left\|\sum_{j\ge n} \xi_j f_j\right\|^r\right)^{1/r} \le \frac{C(\log n)^{\gamma}}{n^{\vartheta-1/2}}, \qquad \textit{if } b+\vartheta \le 0.$$

Proof. By equivalence of Gaussian moments,

$$\left(\mathbb{E}\left\|\sum_{j\ge n} \xi_j f_j\right\|^r\right)^{1/r} \le D\mathbb{E}\left\|\sum_{j\ge n} \xi_j f_j\right\| \tag{2.5}$$

for some constant D depending on r (cf. Ledoux and Talagrand [17], Corollary 3.2). The upper estimate for $\mathbb{E}\|\sum_{j\ge n} \xi_j f_j\|$ is based on corresponding estimates for finite blocks of exponentially increasing length. For $m \ge 1$, set

$$Z = Z^{(m)} := \sum_{j=2^{m-1}+1}^{2^m} \xi_j f_j.$$

For a given $N \ge 1$, consider the grid $G_N = \{\frac{(2i-1)T}{2N} : i = 1,\dots,N\}^d$. Then

$$\|Z\| \le \sup_{t\in G_N} |Z_t| + \sup_{|s-t|\le CN^{-1}} |Z_s - Z_t|.$$

It follows from the Gaussian maximal inequality that

$$\mathbb{E}\sup_{t\in G_N} |Z_t| \le C\sqrt{\log(1+N^d)} \sup_{t\in G_N} \sqrt{\mathbb{E}Z_t^2}.$$

Using (A1), we have, for every $t \in I$,

$$\mathbb{E}Z_t^2 \leq \sum_{j=2^{m-1}+1}^{2^m} \|f_j\|^2 \leq C \sum_{j=2^{m-1}+1}^{2^m} j^{-2\vartheta} \log(1+j)^{2\gamma} \leq C 2^{m(1-2\vartheta)} m^{2\gamma}$$

so that

$$\mathbb{E} \sup_{t \in G_N} |Z_t| \leq C\sqrt{\log(1+N)} 2^{-m(\vartheta-1/2)} m^{\gamma}.$$

Moreover, using (A2), we have, for $|s-t| \leq CN^{-1}$,

$$\begin{aligned} |Z_s - Z_t| &\leq \sum_{j=2^{m-1}+1}^{2^m} |\xi_j||f_j(s) - f_j(t)| \\ &\leq C|s-t|^a \sum_{j=2^{m-1}+1}^{2^m} |\xi_j|[f_j]_a \\ &\leq CN^{-a} \sum_{j=2^{m-1}+1}^{2^m} |\xi_j| j^b \end{aligned}$$

and hence

$$\mathbb{E} \sup_{|s-t| \leq CN^{-1}} |Z_s - Z_t| \leq CN^{-a} \sum_{j=2^{m-1}+1}^{2^m} j^b \leq CN^{-a} 2^{m(1+b)}.$$

Thus we have established the estimate

$$\mathbb{E}\|Z^{(m)}\| \leq C\left(\sqrt{\log(1+N)} 2^{-m(\vartheta-1/2)} m^{\gamma} + N^{-a} 2^{m(1+b)}\right). \tag{2.6}$$

As concerns the choice of N, set $N := [2^{um}] + 1$, with $u \in (0, \infty)$ satisfying $1 + b - au \leq \frac{1}{2} - \vartheta$. Equation (2.6) then becomes

$$\mathbb{E}\|Z^{(m)}\| \leq C 2^{-m(\vartheta-1/2)} m^{\gamma+1/2}. \tag{2.7}$$

We note that in the case $b + \vartheta \leq -1/2$, we may choose $N = 1$ and thereby obtain a power reduction from $m^{\gamma+1/2}$ to m^{γ}. This can be improved. In fact, we have

$$\begin{aligned} \mathbb{E}|Z_s - Z_t|^2 &= \sum_{j=2^{m-1}+1}^{2^m} |f_j(s) - f_j(t)|^2 \\ &\leq C|s-t|^{2a} \sum_{j-2^{m-1}+1}^{2^m} j^{2b} \leq C|s-t|^{2a} 2^{m(1+2b)} \end{aligned}$$

so that

$$d_Z(s,t) := (\mathbb{E}|Z_s - Z_t|^2)^{1/2} \le C|s-t|^a 2^{m(b+1/2)}.$$

If $N(\varepsilon, d_Z)$ denotes the covering numbers of I with respect to the intrinsic semi-metric d_Z, then, by chaining,

$$\mathbb{E} \sup_{|s-t|\le CN^{-1}} |Z_s - Z_t| \le \mathbb{E} \sup_{d_Z(s,t)\le\delta} |Z_s - Z_t| \le C\int_0^\delta \sqrt{\log N(\varepsilon, d_Z)}\,\mathrm{d}\varepsilon,$$

where $\delta := CN^{-a}2^{m(b+1/2)}$ (cf. Van der Waart and Wellner [28], page 101). Since

$$N(\varepsilon, d_Z) \le C\biggl(\frac{2^{m(b+1/2)}}{\varepsilon}\biggr)^{d/a}, \qquad 0<\varepsilon\le\varepsilon_0,$$

and $\int_0^1 \sqrt{\log(1/x)}\,\mathrm{d}x < +\infty$, we obtain, for sufficiently large N,

$$\int_0^\delta \sqrt{\log N(\varepsilon, d_Z)}\,\mathrm{d}\varepsilon \le C2^{m(b+1/2)}\int_0^1 \sqrt{\log(1/x)}\,\mathrm{d}x \le C2^{m(b+1/2)}.$$

Consequently,

$$\begin{aligned}\mathbb{E}\|Z^{(m)}\| &\le C\bigl(\sqrt{\log(1+N)}2^{-m(\vartheta-1/2)}m^\gamma + 2^{m(b+1/2)}\bigr)\\ &\le C2^{-m(\vartheta-1/2)}m^\gamma \qquad \text{if } b+\vartheta\le 0.\end{aligned}\tag{2.8}$$

We now complete the proof. For $n\ge 2$, choose $m=m(n)\ge 1$ such that $2^{m-1}<n\le 2^m$. Then

$$\biggl\|\sum_{j\ge n}\xi_j f_j\biggr\| \le \sum_{j\ge m+1}\|Z^{(j)}\| + \biggl\|\sum_{j=n}^{2^m}\xi_j f_j\biggr\|.$$

Since $\mathbb{E}\|\sum_{n\le j\le 2^m}\xi_j f_j\| \le \mathbb{E}\|Z^{(m)}\|$ by the Anderson inequality (cf. Bogachev [4], Corollary 3.3.7), we deduce from equation (2.7) that

$$\mathbb{E}\biggl\|\sum_{j\ge n}\xi_j f_j\biggr\| \le C\sum_{j\ge m}\frac{j^{\gamma+1/2}}{2^{j(\vartheta-1/2)}} \le \frac{Cm^{\gamma+1/2}}{2^{m(\vartheta-1/2)}} \le \frac{C(\log n)^{\gamma+1/2}}{n^{\vartheta-1/2}}.$$

If $b+\vartheta\le 0$, then it follows from (2.8) that

$$\mathbb{E}\biggl\|\sum_{j\ge n}\xi_j f_j\biggr\| \le \frac{C(\log n)^\gamma}{n^{\vartheta-1/2}}.$$

Combining these estimates with (2.5) yields the assertion. $\square$

Remarks.

• The rate for the truncation error depends only on ϑ and γ, that is, on the decay of the size of functions f_j (provided $b+\vartheta>0$). The occurrence of expansions with $b+\vartheta\le 0$ seems to be a rare event and otherwise b plays no role (see the subsequent example). The case $\gamma=0$ typically corresponds to one-parameter processes with $I=[0,T]$.

• The $e^{(\mathrm{prod})}_{N,r}$-problem comprises the optimization of admissible sequences and, in view of (2.3) and (2.4), is thus related to the l-numbers of X defined by

$$l_{n,r}(X)=l_{n,r}(X,\mathcal{C}(I)):=\inf\left\{\left(\mathbb{E}\left\|\sum_{j\ge n}\xi_j g_j\right\|^r\right)^{1/r}:(g_j)\text{ admissible for }X\text{ in }\mathcal{C}(I)\right\}. \tag{2.9}$$

Rate-optimal solutions of the $l_{n,r}$-problem, in the sense of $l_{n,r}(X)\approx(\mathbb{E}\|\sum_{j\ge n}\xi_j g_j\|^r)^{1/r}$ as $n\to\infty$, have recently been investigated (see Kühn and Linde [16], Dzhaparidze and van Zanten [8–10], Ayache and Taqqu [1]). Admissible sequences of type (A1) and (A2) seem to be promising candidates.

Example 1 (*Weierstrass processes*). Let

$$f_j(t)=j^{-\vartheta}\sin(j^{b+\vartheta}t),\qquad j\ge 1,\vartheta>1/2,b\in\mathbb{R},t\in[0,T].$$

Then $\|f_j\|\le j^{-\vartheta}$ and $[f_j]_1=j^b$. Since $f_j(0)=0$, we also have $\|f_j\|\le Tj^b$, so (A1) and (A2) are satisfied, with $\tilde{\vartheta}=\max\{\vartheta,-b\}$ and $a=1$. The covariance function of $X=\sum_{j=1}^{\infty}\xi_j f_j$ is given by

$$\mathbb{E}X_sX_t=\sum_{j\ge 1}j^{-2\vartheta}\sin(j^{b+\vartheta}s)\sin(j^{b+\vartheta}t).$$

Now, in the "Weierstrass case" $b+\vartheta>0$, we obtain, from Theorem 1,

$$\left(\mathbb{E}\left\|\sum_{j\ge n}\xi_j f_j\right\|^r\right)^{1/r}\le\frac{C\sqrt{\log n}}{n^{\vartheta-1/2}},$$

while in the "non-Weierstrass case," $b+\vartheta\le 0$ appears the better rate:

$$\left(\mathbb{E}\left\|\sum_{j\ge n}\xi_j f_j\right\|^r\right)^{1/r}\le\frac{C}{n^{-b-1/2}}.$$

We pass to the minimal product quantization error $e^{(\mathrm{prod})}_{N,r}(X)$.

Theorem 2. *Assume that X admits an admissible set $(f_j)_{j\ge1}$ in $\mathcal{C}(I)$ satisfying* (A1) *and* (A2). *We then have, for every $N\ge 3$ and $r\in(0,\infty)$,*

$$e^{(\mathrm{prod})}_{N,r}(X)\le\frac{C(\log\log N)^{\vartheta+\gamma}}{(\log N)^{\vartheta-1/2}} \tag{2.10}$$

and

$$e_{N,r}^{(\text{prod})}(X) \le \frac{C(\log\log N)^{\vartheta+\gamma-1/2}}{(\log N)^{\vartheta-1/2}} \qquad \text{if } b+\vartheta \le 0.$$

Furthermore, the L^r-product N-quantization $\widehat{X}$ with respect to (f_j), with tuning parameters defined in (2.11) *and* (2.15) *below, achieves these rates.*

Proof. Let $r \in [1,\infty)$ and set $\nu_j := j_0^{-\vartheta}\log(1+j_0)^\gamma$ if $j < j_0 := [e^{\gamma/\vartheta}]$ and $\nu_j := j^{-\vartheta}\log(1+j)^\gamma$ if $j \ge j_0$. The sequence $(\nu_j)_j$ is then decreasing. Since

$$\lim_{k\to\infty} ke_{k,r}(\mathcal{N}(0,1),\mathbb{R}) \text{ exists in } (0,\infty)$$

(cf. Graf and Luschgy [13]), we deduce from (2.3), (A1) and Theorem 1 the estimate

$$(\mathbb{E}\|X-\widehat{X}\|^r)^{1/r} \le C\left(\sum_{j=1}^m \nu_j N_j^{-1} + \frac{\log(1+m)^{\gamma+1/2}}{m^{\vartheta-1/2}}\right),$$

for every $m, N_1,\dots,N_m \in \mathbb{N}$ with $\prod_{j=1}^m N_j \le N$. (The case $b+\vartheta \le 0$ is treated analogously.) Consequently,

$$e_{N,r}^{(\text{prod})}(X) \le C\inf\left\{\sum_{j=1}^m \nu_j N_j^{-1} + \frac{\log(1+m)^{\gamma+1/2}}{m^{\vartheta-1/2}} : m, N_1,\dots,N_m \in \mathbb{N}, \prod_{j=1}^m N_j \le N\right\}. \tag{2.11}$$

For a given $N \in \mathbb{N}$, we may first optimize the integer bit allocation given by the N_j's for fixed m and then optimize m. To this end, note that the continuous allocation problem reads

$$\inf\left\{\sum_{j=1}^m \nu_j y_j^{-1} : y_j > 0, \prod_{j=1}^m y_j \le N\right\} = \sum_{j=1}^m \nu_j z_j^{-1} = N^{-1/m} m\left(\prod_{j=1}^m \nu_j\right)^{1/m},$$

where

$$z_j = N^{1/m}\nu_j\left(\prod_{k=1}^m \nu_k\right)^{-1/m}$$

and $z_1 \ge \cdots \ge z_m$. One can produce an (approximate) integer solution by setting

$$N_j = [z_j] = \left[N^{1/m}\nu_j\left(\prod_{k=1}^m \nu_k\right)^{-1/m}\right], \qquad j \in \{1,\dots,m\}, \tag{2.12}$$

provided $z_m \geq 1$. Then

$$\sum_{j=1}^{m} \nu_j N_j^{-1} \leq 2mN^{-1/m} \left(\prod_{j=1}^{m} \nu_j \right)^{1/m} \leq CmN^{-1/m} \nu_m.$$

Since the constraint on m reads $m \in I(N)$ with

$$I(N) := \left\{ m \in \mathbb{N} : N^{1/m} \nu_m \left(\prod_{j=1}^{m} \nu_j \right)^{-1/m} \geq 1 \right\}, \tag{2.13}$$

we arrive at

$$e_{N,r}^{(\mathrm{prod})}(X) \leq C \inf_{m \in I(N)} \left(\frac{N^{-1/m} \log(1+m)^{\gamma}}{m^{\vartheta-1}} + \frac{\log(1+m)^{\gamma+1/2}}{m^{\vartheta-1/2}} \right), \tag{2.14}$$

for every $N \in \mathbb{N}$. We check that $I(N)$ is finite, $I(N) = \{1, \ldots, m^*(N)\}$, $m^*(N)$ increases to infinity and

$$m^*(N) \sim \frac{\log N}{\vartheta} \qquad \text{as } N \to \infty. \tag{2.15}$$

Finally, let

$$m = m(N) \in I(N), \qquad \text{with } m(N) \leq \frac{2 \log N}{\log\log N} \qquad \text{for } N \geq 3 \tag{2.16}$$

such that

$$m(N) \sim \frac{2 \log N}{\log\log N} \qquad \text{as } N \to \infty.$$

This is possible in view of (2.14). Using (2.4), the case $r \in (0,1)$ follows from $r = 1$ since the L^r-optimal N_j-quantizations $\widehat{\xi}_j$ satisfy $\mathbb{E}|\xi_j - \widehat{\xi}_j| \leq CN_j^{-1}$, $j \geq 1$; see Graf *et al.* [15]. □

We may reasonably conjecture that for many specific processes, the above rate is the true one. This would imply that product quantization achieves the optimal rate for quantization, namely the rate of convergence to zero of $e_{N,r}(X) := e_{N,r}(X, \mathcal{C}(I))$, only up to a $\log\log N$ term in formula (2.16). This is in contrast to the Hilbert space setting, where the optimal rate is attained by product quantization (cf. Luschgy and Pagès [20]). To be precise, we summarize the results on $e_{N,r}(X)$ in the present setting.

Proposition 1. (a) *Assume that X admits an admissible sequence in $\mathcal{C}(I)$ satisfying* (A1) *and* (A2). *Then*

$$e_{N,r}(X) = O\left(\frac{(\log\log N)^{\gamma+1/2}}{(\log N)^{\vartheta-1/2}} \right) \tag{2.17}$$

and

$$e_{N,r}(X) = O\left(\frac{(\log\log N)^{\gamma}}{(\log N)^{\vartheta-1/2}}\right), \qquad \text{if } b+\vartheta \leq 0. \tag{2.18}$$

(b) *Assume that* X *admits an admissible sequence satisfying* (A1). *Let* μ *be a finite Borel measure on* I *and let* $V : \mathcal{C}(I) \to L^2(I,\mu)$ *denote the natural embedding. Then*

$$e_{N,r}\big(V(X), L^2(\mu)\big) = O\left(\frac{(\log\log N)^{\gamma}}{(\log N)^{\vartheta-1/2}}\right)$$

and

$$e^{(\mathrm{prod})}_{N,2}\big(V(X), L^2(\mu)\big) = O\left(\frac{(\log\log N)^{\gamma}}{(\log N)^{\vartheta-1/2}}\right).$$

Proof. (a) The proof is not constructive. We use Proposition 4.1 in Li and Linde [18], which relates l-numbers (see (2.9)) and small ball probabilities (but this relation is not always sharp). By combining this relation and Theorem 1, we obtain

$$-\log\big(\mathbb{P}(\|X\| \leq \varepsilon)\big) = O\left(\varepsilon^{-1/(\vartheta-1/2)}\left(\log\left(\frac{1}{\varepsilon}\right)\right)^{(\gamma+1/2)/(\vartheta-1/2)}\right),$$

$$-\log\big(\mathbb{P}(\|X\| \leq \varepsilon)\big) = O\left(\varepsilon^{-1/(\vartheta-1/2)}\left(\log\left(\frac{1}{\varepsilon}\right)\right)^{\gamma/(\vartheta-1/2)}\right), \qquad \text{if } b+\vartheta \leq 0$$

as $\varepsilon \to 0$. We may then apply a known, precise relationship between these probabilities and $e_{N,r}(X)$ (cf. Dereich *et al.* [6], Graf *et al.* [14]) and this leads to the desired estimate.

(b) Let $(f_j)_{j\geq 1}$ be an admissible sequence in $\mathcal{C}(I)$ for X satisfying (A1) and consider an L^2-product N-quantization of $V(X)$ based on $(Vf_j)_{j\geq 1}$,

$$\widehat{V(X)}^N = \sum_{j=1}^{m} \hat{\xi}_j V(f_j),$$

where $\widehat{\xi_j}$ are L^2-optimal Voronoi N_j-quantizers; see Luschgy and Pagès [19]. Then, using the independence of $\xi_j - \hat{\xi}_j$, $j \geq 1$, and the stationarity property $\hat{\xi_j} = \mathbb{E}(\xi_j|\hat{\xi}_j)$ of the quantization $\hat{\xi}_j$, we have

$$\begin{aligned}
&\mathbb{E}\left\|\sum_{j=1}^{\infty} \xi_j V(f_j) - \widehat{V(X)}^N\right\|^2_{L^2(\mu)} \\
&\quad = \sum_{j=1}^{m} \mathbb{E}|\xi_j - \hat{\xi}_j|^2 \|Vf_j\|^2_{L^2(\mu)} + \sum_{j\geq m+1} \|Vf_j\|^2_{L^2(\mu)} \\
&\quad \leq C\left(\sum_{j=1}^{m} N_j^{-2} j^{-2\vartheta} \log(1+j)^{2\gamma} + \sum_{j\geq m+1} j^{-2\vartheta} \log(1+j)^{2\gamma}\right).
\end{aligned}$$

We then argue along the lines of Luschgy and Pagès [19] to conclude that

$$e^{(\mathrm{prod})}_{N,2}\big(V(X), L^2(\mu)\big) = O\left(\frac{(\log\log N)^{\gamma}}{(\log N)^{\vartheta-1/2}}\right). \qquad \square$$

Sometimes, (2.17) provides the true rate for $e_{N,r}(X)$ (as for the two-parameter Brownian sheet), sometimes it yields the best known upper bound (as for the d-parameter Brownian sheet with $d \geq 3$) and sometimes (2.18) provides the true rate (as for Brownian motion). The latter typically occurs when the rate of $e_{N,r}(X)$ and the "Hilbert rate" of $e_{N,r}(V(X), L^2(\mathrm{d}t))$ coincide (see Section 3). It remains an open question to find conditions for this to happen.

3. Examples

3.1. Fractional Brownian motions and fractional Brownian sheets

We consider the Dzaparidze–van Zanten expansion of the *fractional Brownian motion* $X = (X_t)_{t\in[0,T]}$ with Hurst index $\rho \in (0,1)$ and covariance function

$$\mathbb{E}X_s X_t = \tfrac{1}{2}(s^{2\rho} + t^{2\rho} - |s-t|^{2\rho}).$$

These authors discovered, in Dzhaparidze and van Zanten [9], that the sequence

$$\begin{aligned} f^1_j(t) &= \frac{T^{\rho}c_{\rho}\sqrt{2}}{|J_{1-\rho}(x_j)|x_j^{\rho+1}}\sin\left(\frac{x_j t}{T}\right), && j \geq 1, \\ f^2_j(t) &= \frac{T^{\rho}c_{\rho}\sqrt{2}}{|J_{-\rho}(y_j)|y_j^{\rho+1}}\left(1-\cos\left(\frac{y_j t}{T}\right)\right), && j \geq 1, \end{aligned} \tag{3.1}$$

in $\mathcal{C}([0,T])$ is admissible for X, where J_ν denotes the Bessel function of the first kind of order ν, $0 < x_1 < x_2 < \cdots$ are the positive zeros of $J_{-\rho}$, $0 < y_1 < y_2 < \cdots$ the positive zeros of $J_{1-\rho}$ and $c^2_\rho = \Gamma(1+2\rho)\sin(\pi\rho)/\pi$.

Using the asymptotic properties

$$x_j \sim y_j \sim \pi j, \qquad J_{1-\rho}(x_j) \sim J_{-\rho}(y_j) \sim \frac{\sqrt{2}}{\pi} j^{-1/2} \qquad \text{as } j \to \infty$$

(cf. Dzhaparidze and van Zanten [9]), one observes that a suitable arrangement of the functions (3.1) (like $f_{2j} = f^1_j$, $f_{2j-1} = f^2_j$) satisfies (A1) and (A2) with parameters $\vartheta = \rho + 1/2$, $\gamma = 0$, $a = 1$ and $b = 1/2 - \rho$. Consequently,

$$e^{(\mathrm{prod})}_{N,r}(FBM) = O\left(\frac{(\log\log N)^{\rho+1/2}}{(\log N)^{\rho}}\right), \tag{3.2}$$

while (see Dereich and Scheutzow [7], Graf *et al.* [14])

$$e_{N,r}(FBM) \approx (\log N)^{-\rho}. \tag{3.3}$$

The tensor products of functions (3.1) are admissible for the *fractional Brownian sheet* X over $[0, T]^d$ with covariance function

$$\mathbb{E}X_s X_t = \left(\tfrac{1}{2}\right)^d \prod_{i=1}^{d} (s_i^{2\rho_i} + t_i^{2\rho_i} - |s_i - t_i|^{2\rho_i}),$$

$\rho_i \in (0, 1)$, and satisfy conditions (A1) and (A2) with $\vartheta = \rho + 1/2$, $\rho = \min_{1\le i\le d} \rho_i$, $\gamma = \vartheta(m-1)$, where $m = \operatorname{card}\{i \in \{1, \dots, d\} : \rho_i = \rho\}$, $a = 1$ and $b = \max_{1\le i\le d}(1/2 - \rho_i))_+$. This is a consequence of the following lemma which ensures stability of conditions (A1) and (A2) under tensor products.

Lemma 1. *For $i \in \{1, \dots, d\}$, let $(f_j^i)_{j\ge 1} \in \mathcal{C}([0, T])^{\mathbb{N}}$ satisfy* (A1) *and* (A2) *with parameters $\vartheta_i, \gamma_i, a_i, b_i$ such that $\gamma_i = 0$. Then a decreasing arrangement of $(\bigotimes_{i=1}^d f_{j_i}^i)_{\underline{j}\in\mathbb{N}^d}$ satisfies* (A1) *and* (A2) *with parameters $\vartheta = \min_{1\le i\le d} \vartheta_i$, $\gamma = \vartheta(m-1)$, where $m = \operatorname{card}\{i \in \{1, \dots, d\} : \vartheta_i = \vartheta\}$, $a = \min_{1\le i\le d} a_i$ and $b = (\max_{1\le i\le d} b_i)_+$.*

Proof. For $\underline{j} = (j_1, \dots, j_d) \in \mathbb{N}^d$, set $f_{\underline{j}} = \bigotimes_{i=1}^d f_{j_i}^i$ so that $f_{\underline{j}}(t) = \prod_{j=1}^d f_{j_i}^i(t_i)$, $t \in [0, T]^d$. We have

$$\|f_{\underline{j}}\| \le \prod_{i=1}^{d} \|f_{j_i}^i\| \le C \prod_{i=1}^{d} j_i^{-\vartheta_i} \quad \text{and} \quad \left|f_{\underline{j}}(s) - f_{\underline{j}}(t)\right| \le C \max_{1\le i\le d} j_i^b |s-t|^a.$$

Let $u_{\underline{j}} := \prod_{i=1}^d j_i^{-\vartheta_i}$. Choose a bijective map $\psi : \mathbb{N} \to \mathbb{N}^d$ such that $u_k := u_{\psi(k)}$ is decreasing in $k \ge 1$. Set $f_k := f_{\psi(k)}$. Then

$$u_k \approx C k^{-\vartheta} (\log k)^{\vartheta(m-1)} \qquad \text{as } k \to \infty$$

(cf. Papageorgiou and Wasilkowski [24], Theorem 2.1). Consequently,

$$\|f_k\| \le C k^{-\vartheta} (\log k)^{\vartheta(m-1)}$$

and, for $\underline{j} = \psi(k)$,

$$j_i \le \prod_{i=1}^{d} j_i \le \prod_{i=1}^{d} j_i^{\vartheta_i/\vartheta} \le C k (\log k)^{-(m-1)} \le C k,$$

hence

$$|f_k(s) - f_k(t)| \le C k^b |s-t|^a. \qquad \square$$

Therefore, by Theorem 2 and Proposition 1,

$$e_{N,r}^{(\mathrm{prod})}(FBS) = O\left(\frac{(\log\log N)^{m(\rho+1/2)}}{(\log N)^{\rho}}\right) \tag{3.4}$$

and

$$e_{N,r}(FBS) = O\left(\frac{(\log\log N)^{m(\rho+1/2)-\rho}}{(\log N)^{\rho}}\right). \tag{3.5}$$

The Hilbert space setting $E = L^2([0,T]^d, \mathrm{d}t)$ provides the lower estimate

$$e_{N,r}(FBS) = \Omega\left(\frac{(\log\log N)^{(m-1)(\rho+1/2)}}{(\log N)^{\rho}}\right) \tag{3.6}$$

(see Luschgy and Pagès [19,20]). The true rate of $e_{N,r}(FBS)$ is known only for the case $m=1$, where the true rate is the "Hilbert rate" (2.6) (see Dereich *et al.* [6]), and for the case $m=2$, where (3.5) is the true rate (see Belinsky and Linde [3], Graf *et al.* [14]). A reasonable conjecture is that (3.5) is also the true rate for $m \geq 3$.

3.2. Riemann–Liouville and other moving average processes

For $\psi \in L^2([0,T], \mathrm{d}t)$ and a standard Brownian motion W, let

$$X_t = \int_0^t \psi(t-s)\,\mathrm{d}W_s, \qquad t \in [0,T],$$

and assume that X has a pathwise continuous modification. Since

$$\begin{aligned}
\mathbb{E}X_s X_t &= \int_0^{s\wedge t} \psi(s-u)\psi(t-u)\,\mathrm{d}u, \\
f_j(t) &= \sqrt{\frac{2}{T}} \int_0^t \psi(t-s)\cos\left(\frac{\pi(j-1/2)s}{T}\right)\mathrm{d}s \\
&= \sqrt{\frac{2}{T}} \int_0^t \psi(s)\cos\left(\frac{\pi(j-1/2)(t-s)}{T}\right)\mathrm{d}s, \qquad j \geq 1,
\end{aligned} \tag{3.7}$$

is an admissible sequence for X. Observe that (3.7) provides well-defined continuous functions, even for $\psi \in L^1([0,T], \mathrm{d}t)$.

Lemma 2. *Let $\psi \in L^1([0,1], \mathrm{d}t)$.*

(a) *If $\varphi(t) = \int_0^t |\psi(s)|\,\mathrm{d}s$ is β-Hölder continuous with $\beta \in (0,1]$, then the sequence (f_j) from* (3.7) *satisfies* (A2) *with $a=\beta$ and $b=1$. In particular, if $\psi \in L^2([0,T], \mathrm{d}t)$, then* (A2) *is satisfied with $a=1/2$ and $b=1$.*

(b) *If ψ has finite variation over $[0,T]$, then* (A1) *is satisfied with $\vartheta = 1$ and $\gamma = 0$.*

Proof. Let $\lambda_j = (\pi(j-1/2)/T)^{-2}$. (*a*) For $s < t$, we have

$$f_j(s) - f_j(t) = \sqrt{\frac{2}{T}}\left\{\int_0^s \psi(u)\big(\cos\big((s-u)/\sqrt{\lambda_j}\big) - \cos\big((t-u)/\sqrt{\lambda j}\big)\big)\,\mathrm{d}u\right.$$

$$- \int_s^t \psi(u) \cos\big((t-u)/\sqrt{\lambda_j}\big)\, \mathrm{d}u \bigg\}$$

so that

$$|f_j(s) - f_j(t)| \le \sqrt{\frac{2}{T}} \left(\frac{|s-t|}{\sqrt{\lambda_j}} \|\psi\|_{L^1(\mathrm{d}t)} + \int_s^t |\psi(u)|\, \mathrm{d}u \right).$$

(b) We have

$$\begin{aligned} f_j(t) &= -\sqrt{2\lambda_j/T} \int_0^t \psi(s)\, \mathrm{d}\big(\sin\big((t-s)/\sqrt{\lambda_j}\big)\big) \\ &= \sqrt{2\lambda_j/T} \left(\psi(0) \sin\big(t/\sqrt{\lambda_j}\big) + \int_0^t \sin\big((t-s)/\sqrt{\lambda_j}\big)\, \mathrm{d}\psi(s) \right) \end{aligned}$$

so that

$$\|f_j\| \le \sqrt{2\lambda_j/T}\big(|\psi(0)| + \mathrm{Var}(\psi, [0,T])\big).$$ □

This lemma yields a universal upper bound,

$$e_{N,r}^{(\mathrm{prod})}(X) = O\left(\frac{\log\log N}{(\log N)^{1/2}} \right),$$

for functions ψ having finite variation.

In the sequel, we do not concern ourselves with improvements of the parameter b in (A2) since the condition $b + \vartheta \le 0$ cannot be achieved in this setting.

Lemma 3. *Let* $\psi \in L^1([0,T], \mathrm{d}t)$.

(a) *If* ψ *is positive and decreasing on* $(0,T]$ *and* $\varphi(t) = \int_0^t \psi(s)\, \mathrm{d}s$ *is* β*-Hölder continuous with* $\beta \in (0,1]$, *then the sequence* (f_j) *from* (3.7) *satisfies* $\|f_j\| \le C j^{-\beta}$. *If* $\beta > 1/2$, *then* (A1) *is satisfied with* $\vartheta = \beta$ *and* $\gamma = 0$.

(b) *If* $\psi(0) = 0$, ψ *is* β*-Hölder continuous with* $\beta \in (0,1]$ *and* ψ *is differentiable on* $(0,T]$ *such that* ψ' *is positive and decreasing on* $(0,T]$, *then* (A1) *is satisfied with* $\vartheta = 1+\beta$ *and* $\gamma = 0$.

Proof. Let $\lambda_j = (\pi(j-1/2)/T)^{-2}$. (*a*) For $t \le \sqrt{\lambda_j}$, we have

$$|f_j(t)| \le \sqrt{2/T}\varphi\big(\sqrt{\lambda_j}\big).$$

Using the second integral mean value formula, we obtain, for $t \in [\sqrt{\lambda_j}, T]$ and some $\delta_j \in [\sqrt{\lambda_j}, t]$,

$$|f_j(t)| \le \sqrt{2/T} \left(\left| \int_0^{\sqrt{\lambda_j}} \psi(s) \cos\big((t-s)/\sqrt{\lambda_j}\big)\, \mathrm{d}s \right| + \left| \int_{\sqrt{\lambda_j}}^t \psi(s) \cos\big((t-s)/\sqrt{\lambda_j}\big)\, \mathrm{d}s \right| \right)$$

$$
\begin{aligned}
&= \sqrt{2/T}\Big(\Big|\int_0^{\sqrt{\lambda_j}} \psi(s)\cos\big((t-s)/\sqrt{\lambda_j}\big)\,\mathrm{d}s\Big| + \psi\big(\sqrt{\lambda_j}\big)\Big|\int_{\sqrt{\lambda_j}}^{\delta_j} \cos\big((t-s)/\sqrt{\lambda_j}\big)\,\mathrm{d}s\Big|\Big)\\
&\le \sqrt{2/T}\big(\varphi\big(\sqrt{\lambda_j}\big) + 2\sqrt{\lambda_j}\,\psi\big(\sqrt{\lambda_j}\big)\big)\\
&\le 3\sqrt{2/T}\,\varphi\big(\sqrt{\lambda_j}\big).
\end{aligned}
$$

Consequently,

$$\|f_j\| \le 3\sqrt{2/T}\,\varphi\big(\sqrt{\lambda_j}\big) \le C\lambda_j^{\beta/2}.$$

(b) The function ψ is absolutely continuous on $[0, T]$, so an integration by parts yields

$$f_j(t) = \sqrt{2\lambda_j/T}\int_0^t \psi'(s)\sin\big((t-s)/\sqrt{\lambda_j}\big)\,\mathrm{d}s.$$

Arguing as in (a) (with ψ replaced by ψ'), we deduce that

$$\|f_j\| \le 3\sqrt{2\lambda_j/T}\,\psi\big(\sqrt{\lambda_j}\big) \le C\lambda_j^{(1+\beta)/2}. \qquad \square$$

Now, let $\psi(t) = t^{\rho-1/2}$ with $\rho \in (0, \infty)$. Then

$$X_t = X_t^\rho = \int_0^t (t-s)^{\rho-1/2}\,\mathrm{d}W_s, \qquad t \in [0, T] \tag{3.8}$$

so that X^ρ is a *Riemann–Liouville* process of order ρ. Using the $(\rho \wedge \frac{1}{2})$-Hölder continuity of the application $t \mapsto X_t^\rho$ from $[0, T]$ into $L^2(\mathbb{P})$ and the Kolmogorov criterion, we can check that X^ρ has a pathwise continuous modification.

Lemma 4. *Let* $\psi(t) = t^{\rho-1/2}$, $\rho \in (0, \infty)$. *Then the sequence* (f_j) *from* (3.7) *satisfies* (A2) *with* $a = \min\{1, \rho + 1/2\}$, $b = 1$ *and* (A1) *for* $\rho \in (0, 3/2]$ *with* $\vartheta = \rho + 1/2$ *and* $\gamma = 0$.

Proof. This is an immediate consequence of Lemmas 2 and 3. $\square$

We deduce, for Riemann–Liouville processes of order $\rho \in (0, 3/2]$, that

$$e_{N,r}^{(\mathrm{prod})}(RL) = O\left(\frac{(\log\log N)^{\rho+1/2}}{(\log N)^\rho}\right), \tag{3.9}$$

while for every $\rho \in (0, \infty)$ (see [18], Graf *et al.* [14]),

$$e_{N,r}(RL) \approx (\log N)^{-\rho}. \tag{3.10}$$

To go beyond $\rho = 3/2$, we must slightly change the way we quantize. Let $\psi(t) = t^{\rho-1/2}$, with $\rho > 3/2$, and choose $k \in \mathbb{N}$ such that $k + 1/2 < \rho \le k + 3/2$. Set $\lambda_j = (\pi(j - 1/2)/T)^{-2}$. For

$k \in \{2n-1, 2n\}$ $n \in \mathbb{N}$, integration by parts yields the expansion

$$f_j(t) = \sum_{m=1}^{n} (-1)^{m-1} \lambda_j^m \sqrt{2/T} \psi^{(2m-1)}(t) + (-1)^n \lambda_j^n \sqrt{2/T} \int_0^t \psi^{(2n)}(s) \cos\big((t-s)/\sqrt{\lambda_j}\big)\, \mathrm{d}s$$
$$=: g_j(t) + h_j(t), \qquad t \in [0,T].$$

Since $\psi^{(2n)}(t) = Ct^{\beta-1}$ if $k = 2n-1$ and $\psi^{(2n)}(t) = Ct^{\beta}$ if $k = 2n$ with $\beta = \rho - k - 1/2 \in (0,1]$, we deduce from Lemma 2 and Lemma 3 that the sequence (h_j) in $C([0,T])$ satisfies (A1) with $\vartheta = \rho + 1/2$, $\gamma = 0$ and (A2) with $a = \rho - k - 1/2$, $b = -k$ if $k = 2n-1$ and $a = 1$, $b = -k+1$ if $k = 2n$. Clearly, the sequence (g_j) also satisfies the conditions (A1) and (A2) (with $\vartheta = 2$, $\gamma = 0$, $b = -2$ and $a = \rho - k - 1/2$ if $k = 2n-1$ and $a = 1$ if $k = 2n$). Consequently, there exist centered continuous Gaussian processes $U = (U_t)_{t \in [0,T]}$ and Z such that $U = \sum_{j=1}^{\infty} \xi_j g_j$ a.s., $Z = \sum_{j=1}^{\infty} \xi_j h_j$ a.s.,

$$X = X^{\rho} \stackrel{d}{=} U + Z \tag{3.11}$$

and $U \in \operatorname{span}\{\psi^{(2m-1)} : m = 1, \ldots, n\}$ a.s. Observe that

$$U = \sum_{m=1}^{n} (-1)^{m-1} \sqrt{2/T} \psi^{(2m-1)} \eta_m,$$

where $\eta_m = \sum_{j=1}^{\infty} \lambda_j^m \xi_j$ is $\mathcal{N}(0, \sum_{j=1}^{\infty} \lambda_j^{2m})$-distributed.

Now use, for example, $[N^{1/2n}]$-quantizations of η_m and a $[\sqrt{N}]$-product quantization of Z for the quantization of X (which is clearly not optimal in practise, but remains rate optimal). Let $\widehat{\eta}_m$ be an L^r-optimal $[N^{1/2n}]$-quantization for η_m,

$$\hat{U}^{\sqrt{N}} := \sum_{m=1}^{n} (-1)^{m-1} \sqrt{2/T} \psi^{(2m-1)} \hat{\eta}_m,$$

and let $\hat{Z}^{\sqrt{N}}$ be the L^r-product $[\sqrt{N}]$-quantization of Z from Theorem 2. A (modified) L^r-product N-quantization of X with respect to (f_j) is then defined by

$$\widehat{X} := \hat{U}^{\sqrt{N}} + \hat{Z}^{\sqrt{N}}. \tag{3.12}$$

Using Theorem 2, we can show for the quantization error, that

$$\begin{aligned}
\|U + Z - \widehat{X}\|_{L^r_E} &\le C\big(\big\|U - \hat{U}^{\sqrt{N}}\big\|_{L^r_E} + \big\|Z - \hat{Z}^{\sqrt{N}}\big\|_{L^r_E}\big) \\
&\le C\left(\sum_{m=1}^{n} \sqrt{2/T} \big\|\psi^{(2m-1)}\big\| \|\eta_m - \hat{\eta}_m\|_{L^r} + \big\|Z - \hat{Z}^{\sqrt{N}}\big\|_{L^r_E}\right) \\
&\le \frac{C}{N^{1/2n}} + \frac{C(\log\log\sqrt{N})^{\rho+1/2}}{(\log\sqrt{N})^{\rho}}
\end{aligned}$$

$$\le \frac{C(\log\log N)^{\rho+1/2}}{(\log N)^{\rho}}$$

so that, with the above modification, (3.9) remains true for $\rho > 3/2$.

Now, consider the *stationary Ornstein–Uhlenbeck process* as the solution of the Langevin equation

$$\mathrm{d}X_t = -\beta X_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \qquad t \in [0,T],$$

with X_0 independent of W and $\mathcal{N}(0, \frac{\sigma^2}{2\beta})$-distributed, $\sigma > 0$, $\beta > 0$. It admits the explicit representation

$$X_t = e^{-\beta t} X_0 + \sigma \mathrm{e}^{-\beta t} \int_0^t \mathrm{e}^{\beta s}\,\mathrm{d}W_s \tag{3.13}$$

and

$$\mathbb{E} X_s X_t = \frac{\sigma^2}{2\beta} \mathrm{e}^{-\beta|s-t|}.$$

By Lemma 2, the admissible sequence

$$f_0(t) = \frac{\sigma}{\sqrt{2a}} \mathrm{e}^{-\beta t}, \qquad f_j(t) = \sigma\sqrt{\frac{2}{T}} \int_0^t \mathrm{e}^{-\beta(t-s)} \cos\left(\frac{\pi(j-1/2)s}{T}\right) \mathrm{d}s, \qquad j \ge 1,$$

satisfies conditions (A1) and (A2) with $\vartheta = 1$, $\gamma = 0$, $a = 1$ and $b = 1$. Consequently,

$$e_{N,r}^{(\mathrm{prod})}(OU) = O\left(\frac{\log\log N}{(\log N)^{1/2}}\right), \tag{3.14}$$

while (see Graf *et al.* [14])

$$e_{N,r}(OU) \approx (\log N)^{-1/2}. \tag{3.15}$$

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