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LINEAR WEINGARTEN SPACELIKE HYPERSURFACES IN LORENTZ SPACE FORMS WITH PRESCRIBED GAUSS MAP

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Abstract

This paper address the geometry of complete linear Weingarten spacelike hypersurfaces in the Lorentz space forms. First, a divergence lemma concerning linear Weingarten spacelike hypersurfaces is obtained. Then, with the aid of this lemma, by supposing suitable restrictions on the Gauss map, we show that such hypersurfaces must be totally umbilical, which are some extension of the recent results of Aquino, Bezerra and Lima [7] and Aquino, Lima and Velásquez [11].

1. Introduction

Let $\mathbf{L}_1^{n+1}$ be an $(n+1)$-dimensional Lorentz space, that is, a semi-Riemannian manifold of index 1. When $\mathbf{L}_1^{n+1}$ is simply connected and has constant sectional curvature, it is called a Lorentz space form. The Lorentz-Minkowski space $\mathbf{L}^{n+1}$, the de Sitter space $\mathbf{S}_1^{n+1}$ and the anti-de Sitter space $\mathbf{H}_1^{n+1}$ are the standard Lorentz space forms of constant sectional curvature 0, 1 and -1, respectively. In order to simplify our notation, we will denote by $\mathbf{L}_1^{n+1}(c) \subset \mathbf{R}_q^{n+1+|c|}$, $q = 1 + \frac{1}{2}(|c| - c)$ the Lorentz-Minkowski space $\mathbf{L}^{n+1}$, de Sitter space $\mathbf{S}_1^{n+1}$ and anti-de Sitter space $\mathbf{H}_1^{n+1}$ according $c = 0$, $c = 1$ and $c = -1$, respectively.

A smooth immersion $\varphi : M^n \hookrightarrow \mathbf{L}_1^{n+1}(c)$ of an n-dimensional connected manifold M is said to be a *spacelike* hypersurface if the induced metric via φ is a Riemannian metric on M. As is usual, the spacelike hypersurface is said to be *complete* if the Riemannian induced metric is a complete metric on M.

The interest in the study of rigidity of complete spacelike hypersurfaces in Lorentz manifolds have been widely approached for many authors in recent years, from both the physical and mathematical points of view. From a mathematical viewpoint, a basic question related to this topic is the Bernstein-type properties. It was proved by Calabi [12] (for $n \leq 4$) and by Cheng and Yau [21]

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(for all n) that a complete maximal spacelike hypersurface in $\mathbf{L}^{n+2}$ is totally geodesic. In [29], Nishikawa obtained similar results for other Lorentzian manifolds. In particular, he proved that a complete maximal spacelike hypersurface in $\mathbf{S}_1^{n+1}$ is totally geodesic.

On the other hand, it is well known that the geometry of the Gauss map of a spacelike hypersurface immersed into a Lorentz space form can impose several restrictions on its own geometry. For the study of spacelike hypersurfaces with constant mean curvature in Lorentz space forms $\mathbf{L}_1^{n+1}(c)$, Aiyama [1] and Xin [38], simultaneous and independently, used the generalized maximum principle of Omori-Yau [31, 41] in order to characterize the spacelike hyperplanes as the only complete constant mean curvature spacelike hypersurfaces in $\mathbf{L}^{n+1}$ having the image of its Gauss map contained in a geodesic ball of $\mathbf{H}^n$ (see also [33] for a weaker first version of this result given by Palmer). Afterwards, Xin and Ye [39] improved such previous results showing that if the image of the Gauss map of a complete constant mean curvature spacelike hypersurface of $\mathbf{L}^{n+1}$ lies in a horoball of $\mathbf{H}^n$, then it must be a hyperplane. Also working in this context, Aledo and Alías [2] showed that a complete constant mean curvature hypersurface in $\mathbf{S}_1^{n+1}$ is a spacelike geodesic round sphere if the image of its hyperbolic Gauss map contained in a geodesic ball of the hyperbolic space $\mathbf{H}^n$. Besides, working with a suitable warped product model of $\mathbf{H}_1^{n+1}$, Camargo et al. [13] showed that if M is a complete spacelike hypersurface with constant mean curvature and bounded scalar curvature in anti-de Sitter space $\mathbf{H}_1^{n+1}$, such that the gradient of its height function with respect to a timelike vector has integrable norm, then M must be totally umbilical. Later, Aquino and Lima [9] considered the umbilicity of complete constant mean curvature spacelike hypersurfaces in de Sitter space $\mathbf{S}_1^{n+1}$ and anti-de Sitter space $\mathbf{H}_1^{n+1}$ by supposing suitable restrictions on the Gauss map.

For the study of spacelike hypersurfaces with constant scalar curvature in Lorentz space forms $\mathbf{L}_1^{n+1}(c)$, Aquino, Bezerra and Lima [7] studied the geometry of complete spacelike hypersurfaces with constant scalar curvature immersed into the de Sitter space $\mathbf{S}_1^{n+1}$, and showed that such hypersurfaces must be totally umbilical, provided that its Gauss map has some suitable behavior. Meanwhile, by supposing suitable restrictions on the Gauss map, recently, Aquino, de Lima and Velásquez [11] characterized totally umbilical hypersurfaces as the only complete hypersurfaces immersed in $\mathbf{H}_1^{n+1}$ with constant scalar curvature.

As a natural generalization of hypersurfaces with constant scalar curvature or with constant mean curvature, linear Weingarten hypersurfaces have been studied by many authors in Riemannian space forms ([8, 10, 17, 19, 20, 25, 37]) and in Lorentz space forms ([15, 16, 23, 26, 27, 40]). Recall that a hypersurface in Lorentz space forms $\mathbf{L}_1^{n+1}(c)$ is said to be **linear Weingarten** if its normalized scalar curvature R and mean curvature H satisfy $R = aH + b$ for some constants $a, b \in \mathbf{R}$. In case of Riemannian space forms, Li, Suh and Wei [25] studied linear Weingarten hypersurfaces in unit sphere $\mathbf{S}^{n+1}$ and proved the first rigidity result under the assumption that hypersurfaces are compact. Later, this rigidity result is extended to the case of complete linear Weingarten hypersurfaces with two distinct principal curvatures in real space forms by Shu [37]. Aquino, Lima and Velásquez [10] established a new characterization theorem concerning complete linear Weingarten hypersurfaces immersed in real space forms under the assumption that the mean curvature attains its maximum along hypersurfaces and an appropriated restriction on the norm of the traceless part of the second fundamental form. The first author and Wang [19] also considered the rigidity of linear Weingarten hypersurfaces in Riemannian space forms. Recently, the authors [18] obtained a rigidity result for complete linear Weingarteen hypersurfaces in hyperbolic space by supposing suitable restrictions on the Gauss map of such hypersurfaces which extended that ones in [6] without the assumption of constant scalar curvature. In case of Lorentz space forms, Lima and Velásquez [27] showed that a compact spacelike linear Weingarten hypersurface with $R = aH + b$ immersed in de Sitter space $\mathbf{S}_1^{n+1}$ is a totally umbilical spacelike hypersurface if $(n-1)a^2 + 4n(1-b) \geq 0$ and the second fundamental form is bounded. The first author [15] also studied complete linear Weingarten submanifolds in semi-Riemannian space forms $\mathbf{L}_p^{n+p}(c)$ with parallel normalized mean curvature vector.

In this paper, motivated by the works above described and the approaches developed in [15, 16, 18], we deal with complete linear Weingarten spacelike hypersurfaces immersed into the Lorentz space forms $\mathbf{L}_1^{n+1}(c)$. Using Lemma 3.4 below jointly with Generalized Maximum Principle due to Yau [42], a rigidity result for complete linear Weingarten spacelike hypersurfaces immersed in $\mathbf{L}_1^{n+1}(c)$ is obtained by supposing suitable restrictions on its Gauss map, which is an extension of the result in [7, 11] without the assumption of constant normalized scalar curvature.

In what follow, we will state our main result. Let $\varphi : M^n \hookrightarrow \mathbf{L}_1^{n+1}(c)$ be a linear Weingarten spacelike hypersurface, given a vector $v \in \mathbf{R}_q^{n+1+|c|}$, $q = 1 + \frac{1}{2}(|c| - c)$, let $v^\top$ denote the orthogonal projection of v onto the tangent bundle TM, that is

$$v^\top = v + f_v N - c l_v \varphi,$$

where $l_v = \langle \varphi, v \rangle$, $f_v = \langle N, v \rangle$ and N is the unit normal vector field of M. Denote by $\mathfrak{L}^1(M)$ the L^1 space of integrable functions on M.

Theorem 1.1. *Let $\varphi : M^n \hookrightarrow \mathbf{L}_1^{n+1}(c)$ $(n \geq 3)$ be a complete linear Weingarten spacelike hypersurface with bounded mean curvature H and $R = aH + b$, where a, b are constants and $a \geq 0$, $b < c$. Suppose that for some nonzero vector $v \in \mathbf{R}_q^{n+1+|c|}$, $q = 1 + \frac{1}{2}(|c| - c)$, we have $|v^\top| \in \mathfrak{L}^1(M)$. Then we have the following conclusions*:

(1) *When $c = 1$, M is a totally umbilical spacelike hypersurface of $\mathbf{S}_1^{n+1}$ if one of the following conditions is satisfied*:

(i) *v is timelike*;

(ii) *v is null and the image of the hyperbolic Gauss map of M is contained in the closure of a domain enclosed by a horosphere of $\mathbf{H}^{n+1}$ determined by v*;

(iii) *v is spacelike and the image of the hyperbolic Gauss map of M is contained in the closure of a hemisphere of* $\mathbf{H}^{n+1}$ *determined by v.*

(2) *When $c = -1$, M is a totally umbilical spacelike hypersurface of* $\mathbf{H}_1^{n+1}$ *if v is timelike and the image of the Gauss map of M is contained in a region bounded by two totally umbilical spacelike hypersurfaces of* $\mathbf{H}_1^{n+1}$ *determined by v.*

Remark 1.2. Choosing $a = 0$ and $R = aH + b = b < c$ in Theorem 1.1, we obtain the Theorem 4.1 in [7] and Theorem 3.5 in [11] for $c = 1$ and $c = -1$ respectively.

2. Preliminaries

In this section we will introduce some basic facts and notations which will be used in this paper. Let $\varphi : M^n \hookrightarrow \mathbf{L}_1^{n+1}(c) \subset \mathbf{R}_q^{n+1+|c|}$, $q = 1 + \frac{1}{2}(|c| - c)$ be an immersed spacelike hypersurface in Lorentz space forms $\mathbf{L}_1^{n+1}(c)$ with N its Gauss map. Besides ∇^0, $\overline{\nabla}$ and ∇ denote the Levi-Civita connections in $\mathbf{R}_q^{n+1+|c|}$, $\mathbf{L}_1^{n+1}(c)$ and M respectively. Then the Gauss and Weingarten formulae for M in $\mathbf{L}_1^{n+1}(c) \subset \mathbf{R}_q^{n+1+|c|}$ are given, respectively, by

$$\nabla_X^0 Y = \overline{\nabla}_X Y - c\langle X, Y\rangle\varphi = \nabla_X Y - \langle AX, Y\rangle N - c\langle X, Y\rangle\varphi, \tag{2.1}$$

and

$$AX = -\overline{\nabla}_X N = -\nabla_X^0 N, \tag{2.2}$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M)$, where $A : \mathfrak{X}(M) \to \mathfrak{X}(M)$ stands for the shape operator of M with respect to a choice of timelike orientation N for M.

On the one hand, as in [32], the curvature tensor R of the spacelike hypersurface M is given by

$$R(X, Y)Z = \nabla_{[X, Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where [ ] denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(M)$. A fact well known is that the curvature tensor R of a spacelike hypersurface M immersed in $\mathbf{L}_1^{n+1}(c)$ can be described in terms of its shape operator A by the so-called Gauss equation given by

$$R(X, Y)Z = c(\langle Y, Z\rangle X - \langle X, Z\rangle Y) - \langle AX, Z\rangle AY + \langle AY, Z\rangle AX,$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(M)$. From Gauss equation, we can deduce that

$$|A|^2 = n^2H^2 + n(n-1)(R - c), \tag{2.3}$$

where R and $|A|^2$ are the normalized scalar curvature and the norm square of the second fundamental form of M^n respectively. On the other hand, the Codazzi equation of M^n is given by

$$(\nabla_Y A)X = (\nabla_X A)Y, \tag{2.4}$$

where $\nabla_X A$ denotes the covariant derivative of A.

Denote by Φ the totally umbilical tensor of M, which is given by $\Phi = A + HI$, where I is the identity operator on TM. Then, by an easy computation, we have $tr(\Phi) = 0$ and

$$|\Phi|^2 = |A|^2 - nH^2 \geq 0, \tag{2.5}$$

where the equality $|\Phi|^2 = 0$ holds if and only if M is totally umbilical. We know that A is a self-adjoint linear operator on TM, and its eigenvalues $\kappa_1, \kappa_2, \ldots, \kappa_n$ are the principal curvatures of the hypersurface. Associated to the shape operator A there are n algebraic invariants given by

$$S_r = \sigma_r(\kappa_1, \kappa_2, \ldots, \kappa_n), \quad 1 \leq r \leq n, \tag{2.6}$$

where $\sigma_r : \mathbf{R}^n \to \mathbf{R}$ is the elementary symmetric function in $\mathbf{R}^n$ gievn by

$$\sigma_r(\kappa_1, \kappa_2, \ldots, \kappa_n) = \sum_{i_1 < \cdots < i_r} \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_r}.$$

Observe that the characteristic polynomial of A can be written in terms of the S_r as

$$\det(tI - A) = \sum_{r=0}^{n} (-1)^r S_r t^{n-r},$$

where $S_0 = 1$ by construction. The r-th $(0 \leq r \leq n)$ mean curvature H_r of the hypersurface is defined by

$$\binom{n}{r} H_r = (-1)^r S_r = \sigma_r(-\kappa_1, -\kappa_2, \ldots, -\kappa_n).$$

We observe that $H_0 = 1$, while

$$H_1 = -\frac{1}{n} \sum_{i=1}^{n} \kappa_i = -\frac{1}{n}\, tr(A) = H$$

is the usual mean curvature of M^n. The choice of the sign $(-1)^r$ in the definition of H_r is motivated by the fact that the mean curvature vector is given by $\vec{H} = HN$. Therefore, $H(p) > 0$ at a point $p \in M$ if and only if $\vec{H}(p)$ is in the same time-orientation as $N(p)$ (in the sense that $\langle \vec{H}, N \rangle_p < 0$).

According to our definition of the r-mean curvatures, the *Newton transformation* P_r on M are given by setting $P_0 = I$ and, for $1 \leq r \leq n$,

$$P_r = (-1)^r S_r I + A P_{r-1} = \binom{n}{r} H_r I + A P_{r-1}.$$

Let us recall that each P_r is also a self-adjoint linear operator on each tangent plane TM which commutes with A. Indeed A and P_r can be simultaneously diagonalized: if $\{e_1, e_2, \ldots, e_n\}$ are the eigenvectors of A corresponding to the eigenvalues $\kappa_1, \kappa_2, \dots, \kappa_n$ respectively, then they are also the eigenvectors of P_r with corresponding eigenvalues given by

$$\mu_{i,r} = (-1)^r \sum_{\substack{1 \le i_1 < \cdots < i_r \le n \\ i_1, \dots, i_r \ne i}} \kappa_{i_1} \cdots \kappa_{i_r} = \sum_{\substack{1 \le i_1 < \cdots < i_r \le n \\ i_1, \dots, i_r \ne i}} (-\kappa_{i_1}) \cdots (-\kappa_{i_r})$$

for every $1 \le i \le n$.

On the other hand, given $f \in C^\infty(M)$, for each $0 \le r \le n$, the second-order differential operator L_r is defined as follows

$$L_r f = tr(P_r \nabla^2 f),$$

where ∇^2 stands for the hessian tensor and given by

$$\nabla^2 f(X, Y) = \langle \nabla_X(\nabla f), Y \rangle, \quad X, Y \in \mathfrak{X}(M).$$

Let $\{e_1, e_2, \dots, e_n\}$ be a local orthonormal frame on M and observe that

$$\begin{aligned} \text{(2.7)} \qquad \operatorname{div}(P_r(\nabla f)) &= \sum_{i=1}^n \langle (\nabla_{e_i} P_r)(\nabla f), e_i \rangle + \sum_{i=1}^n \langle P_r(\nabla_{e_i} \nabla f), e_i \rangle \\ &= \langle \operatorname{div} P_r, \nabla f \rangle + L_r(f), \end{aligned}$$

where div denotes the divergence on M and

$$\operatorname{div} P_r = \operatorname{tr}(\nabla P_r) = \sum_{i=1}^n (\nabla_{e_i} P_r)(e_i).$$

Since $\mathbf{L}_1^{n+1}(c)$ has constant sectional curvature, the Newton transformations P_r are divergence free, that is, div $P_r = 0$ ([3]). Then we can rewrite L_r as

$$\text{(2.8)} \qquad L_r f = \operatorname{div}(P_r \nabla f).$$

From equation (2.8), we conclude that the operator L_r is elliptic if, and only if, P_r is positive definite. We observe that $L_0 = \triangle$ is always elliptic. In particular, when $r = 1$ the operator L_1 agrees (up to the sign) with the operator $\square$, which was introduced by Cheng and Yau [22].

For linear Weingarten spacelike hypersurfaces in Lorentz space forms $\mathbf{L}_1^{n+1}(c)$ with $R = aH + b$, combining (2.3) with (2.5), we have

$$\text{(2.9)} \qquad |\Phi|^2 = n(n-1)(H^2 + aH) + n(n-1)(b - c).$$

Let $P: TM \to TM$ be the operator given by $P = \left(nH + \frac{n-1}{2}a\right)I + A$. It is not difficult to prove that P is self-adjoint. Now we define a operator L associated with P acting on any function $f \in \mathscr{C}^2(M)$ by

$$\text{(2.10)} \qquad L(f) = \operatorname{div}(P(\nabla f)) = \left(\square + \frac{n-1}{2} a\Delta\right) f,$$

where $\Box$ is the Cheng-Yau's operator in [22], i.e. $\Box f = \mathrm{div}(P_1(\nabla f))$. Thus, L is a second-order differential operator and L is elliptic or parabolic if and only if P is positive definite or non-negative definite.

3. Some Lemmas

Firstly, by fixing a vector $v \in \mathbf{R}_q^{n+1+|c|}$, $q = 1 + \frac{1}{2}(|c| - c)$, we consider the *height* and *angle* functions which are naturally attached to an immersion $\varphi : M^n \hookrightarrow \mathbf{L}_1^{n+1}(c)$ defined by

$$l_v = \langle \varphi, v \rangle \quad \textit{and} \quad f_v = \langle N, v \rangle.$$

By a straightforward computation, we conclude that $\nabla l_v = v^\top$ and $\nabla f_v = -A(v^\top)$, where $v^\top$ is the orthogonal projection of v onto the tangent bundle TM, that is

$$v^\top = v + f_v N - c l_v \varphi.$$

Using Gauss formula (2.1) and Weingarten formula (2.2), it is not difficult to verify that

$$\nabla_X \nabla l_v = \nabla_X v^\top = -f_v AX - c l_v X, \tag{3.1}$$

for all $x \in \mathfrak{X}(M)$. Now, we use (3.1) jointly with Codazzi equation (2.4) to deduce

$$\nabla_X \nabla f_v = f_v A^2 X + c l_v AX - (\nabla_{v^\top} A)X,$$

for all $x \in \mathfrak{X}(M)$. Moreover, based on the paper due to Reilly [35], it is possible to obtain the following identities related with the action of L_r on these functions [3, 5, 34]:

$$L_r l_v = (-1)^{r+1}[(r+1)S_{r+1} f_v + c(n-r)S_r l_v], \tag{3.2}$$

and

$$L_r f_v = (-1)^{r+1}\{[(r+2)S_{r+2} - S_1 S_{r+1}]f_v - c(r+1)S_{r+1} l_v + v^\top(S_{r+1})\}, \tag{3.3}$$

where $c_r = (n-r)\binom{n}{r} = (r+1)\binom{n}{r+1}$. In particular, letting $r = 0$ and $r = 1$ in above equations respectively, we get the following relations

$$\triangle l_v = nHf_v - cnl_v, \tag{3.4}$$

$$\triangle f_v = |A|^2 f_v - cnHl_v + nv^\top(H), \tag{3.5}$$

$$\Box l_v = 2S_2 f_v + c(n-1)S_1 l_v, \tag{3.6}$$

$$\Box f_v = (3S_3 - S_1 S_2)f_v - 2cS_2 l_v + v^\top(S_2). \tag{3.7}$$

Now, we present the main analytical tools which will be used to prove our results. First one is a classic algebraic lemma due to Okumura in [30] and the equality case was proved by Alencar and do Carmo in [4].

LEMMA 3.1. *Let $\mu_1, \mu_2, \dots, \mu_n$ be real numbers such that*

$$\sum_{i=1}^{n} \mu_i = 0 \quad and \quad \sum_{i=1}^{n} \mu_i^2 = \beta^2,$$

where $\beta \geq 0$. Then

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_{i=1}^{n} \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{3.8}$$

and equality holds if, and only if, either at least $(n-1)$ of the numbers μ_i are equal.

In what follows, we quote a suitable characterization of totally umbilical hypersurfaces in a semi-Riemannian space form due to Kim et al. [24], which corresponds to a converse for a theorem due to Sharma and Duggal in [36].

LEMMA 3.2. *Let M^n be a connected semi-Riemannian hypersurface of a semi-Riemannian space form $\mathbf{Q}^{n+1}(c)$. Suppose that $\mathbf{Q}^{n+1}(c)$ carries a conformal vector field V whose tangential component $V^{\top}$ on M^n becomes a conformal vector field. Then, one of the following holds*:
(i) *M^n is a totally umbilical hypersurface*;
(ii) *the restriction of V to M^n reduces to a tangent vector field on M^n.*

Next important lemma is based on Yau's result ([42]) which was obtained by Caminha in [14].

LEMMA 3.3 ([14]). *Let X be a smooth vector field on the n-dimensional complete noncompact oriented Riemannian manifold M, such that* div X *does not change sign on M. If $|X| \in \mathfrak{L}^1(M)$, then* div $X = 0$.

To close this section, we will present the following key lemma which plays an important role in the proof of Theorem 1.1.

LEMMA 3.4. *Let $\varphi : M^n \hookrightarrow \mathbf{L}_1^{n+1}(c)$ be a complete linear Weingarten spacelike hypersurface immersed in Lorentz space forms $\mathbf{L}_1^{n+1}(c)$ with $R = aH + b$, where a, b are constants. Then we have*

$$\begin{aligned} \operatorname{div}\left(P(\nabla f_v) + (n-1)(b-c)\nabla l_v + \frac{a}{2}P_1(\nabla l_v)\right) \\ = \left[\sum_{i=1}^{n}(\kappa_i + H)^3 + (n-2)\left(H + \frac{a}{2}\right)|\Phi|^2\right] f_v. \end{aligned} \tag{3.9}$$

Proof. Let $p \in M$ and $\{e_1, \dots, e_n\}$ be a local orthonormal frame on a neighborhood U of p, geodesic at p and diagonalizing A at p, with $Ae_i = \kappa_i e_i$, for $1 \leq i \leq n$. From Codazzi equation, we have that

$$\begin{aligned}\operatorname{div} A(\nabla f_v) &= \sum_{i=1}^{n} \langle \nabla_{e_i} A(\nabla f_v), e_i \rangle \\ &= \sum_{i=1}^{n} \langle (\nabla_{e_i} A)(\nabla f_v) + A(\nabla_{e_i} \nabla f_v), e_i \rangle \\ &= \sum_{i=1}^{n} \langle (\nabla_{\nabla f_v} A)(e_i), e_i \rangle + \sum_{i=1}^{n} \langle A(\nabla_{e_i} \nabla f_v), e_i \rangle \\ &= tr(\nabla_{\nabla f_v} A) + \sum_{i=1}^{n} \nabla^2 f_v(e_i, Ae_i),\end{aligned} \tag{3.10}$$

where ∇^2 stands for the hessian tensor. Since

$$\begin{aligned}-\langle \nabla(nH), \nabla f_v \rangle &= -\sum_{j} \langle e_j(nH)e_j, \nabla f_v \rangle = \sum_{i,j} \langle e_j \langle Ae_i, e_i \rangle e_j, \nabla f_v \rangle \\ &= \sum_{i,j} \langle (\nabla_{e_j} A)e_i, e_i \rangle \langle e_j, \nabla f_v \rangle = \sum_{i,j} \langle (\nabla_{e_i} A)e_j, e_i \rangle \langle e_j, \nabla f_v \rangle \\ &= \sum_{i,j} \langle (\nabla_{e_i} A)e_i, e_j \rangle \langle e_j, \nabla f_v \rangle = \sum_{i} \langle (\nabla_{e_i} A)e_i, \nabla f_v \rangle \\ &= \sum_{i} \langle e_i, (\nabla_{e_i} A)\nabla f_v \rangle = \sum_{i} \langle e_i, (\nabla_{\nabla f_v} A)e_i \rangle = tr(\nabla_{\nabla f_v} A),\end{aligned}$$

then, we have

$$\begin{aligned}&\operatorname{div}\left(\left(nH + \frac{n-1}{2}a\right)\nabla f_v\right) \\ &= \left\langle \nabla\left(nH + \frac{n-1}{2}a\right), \nabla f_v \right\rangle + \left(nH + \frac{n-1}{2}a\right)\triangle f_v \\ &= \langle \nabla(nH), \nabla f_v \rangle + \left(nH + \frac{n-1}{2}a\right)\triangle f_v \\ &= -tr(\nabla_{\nabla f_v} A) + \left(nH + \frac{n-1}{2}a\right)\triangle f_v.\end{aligned} \tag{3.11}$$

Combining equations (3.10), (3.11) with (3.5), we obtain

$$\begin{aligned}L(f_v) &= \operatorname{div}\left(\left(nH + \frac{n-1}{2}a\right)\nabla f_v + A(\nabla f_v)\right) \\ &= \left(nH + \frac{n-1}{2}a\right)\triangle f_v + \sum_{i=1}^{n} \nabla^2 f_v(e_i, Ae_i) \\ &= \left(nH + \frac{n-1}{2}a\right)(|A|^2 f_v - cnHl_v + nv^{\top}(H)) + \sum_{i=1}^{n} \nabla^2 f_v(e_i, Ae_i).\end{aligned} \tag{3.12}$$

From Gauss equation and $R = aH + b$, we know that

$$n^2H^2 - |A|^2 + n(n-1)aH = n(n-1)(c-b)$$

is a constant on M. Now observing that $|A|^2 = \sum_{i=1}^n \langle Ae_i, Ae_i\rangle$, $(\nabla_{e_i}e_j)(p) = 0$ for all $i, j \in 1, 2, \dots, n$ and equation (3.1), we obtain

$$\begin{aligned}
(3.13)\qquad \sum_{i=1}^n \nabla^2 f_v(e_i, Ae_i) &= \sum_i \langle \nabla_{e_i}\nabla f_v, Ae_i\rangle = -\sum_i \langle \nabla_{e_i}A(v^\top), Ae_i\rangle \\
&= -\sum_i \langle (\nabla_{e_i}A)v^\top, Ae_i\rangle - \sum_i \langle A(\nabla_{e_i}v^\top), Ae_i\rangle \\
&= -\sum_i \kappa_i \langle (\nabla_{e_i}A)v^\top, e_i\rangle - \sum_i \langle \nabla_{e_i}v^\top, A^2e_i\rangle \\
&= -\sum_i \kappa_i \langle v^\top, (\nabla_{e_i}A)e_i\rangle - \sum_i \kappa_i^2 \langle \nabla_{e_i}v^\top, e_i\rangle \\
&= -\sum_i \kappa_i \langle v^\top, \nabla_{e_i}(Ae_i)\rangle + \sum_i \kappa_i^2 \langle f_vAe_i + cl_ve_i, e_i\rangle \\
&= -\sum_i \kappa_i \langle v^\top, e_i(\kappa_i)e_i\rangle + f_v\sum_i \kappa_i^3 + cl_v|A|^2 \\
&= -\frac{1}{2}\sum_i \langle v^\top, e_i(\kappa_i^2)e_i\rangle + f_v\sum_i \kappa_i^3 + cl_v|A|^2 \\
&= -\frac{1}{2}v^\top(|A|^2) + f_v\sum_{i=1}^n \kappa_i^3 + c|A|^2l_v \\
&= -\frac{1}{2}v^\top(n^2H^2 + n(n-1)(aH + b - c)) \\
&\quad + f_v\sum_{i=1}^n \kappa_i^3 + c|A|^2l_v \\
&= -n^2Hv^\top(H) - \frac{n(n-1)}{2}av^\top(H) + f_v\sum_{i=1}^n \kappa_i^3 + c|A|^2l_v.
\end{aligned}$$

Then, from (3.12) and (3.13), we infer that

$$(3.14)\qquad L(f_v) = \left(nH + \frac{n-1}{2}a\right)(|A|^2f_v - cnHl_v) + f_v\sum_{i=1}^n \kappa_i^3 + c|A|^2l_v.$$

A simple computation provides us the following equality

$$(3.15)\qquad \sum_{i=1}^n \kappa_i^3 = -nH^3 - 3H|\Phi|^2 + \sum_{i=1}^n (\kappa_i + H)^3.$$

From (3.14), (3.15), we conclude that

$$
\begin{aligned}
(3.16)\qquad L(f_v) &= \left(nH + \frac{n-1}{2}a\right)(|A|^2 f_v - cnHl_v) \\
&\quad + \left(-nH^3 - 3H|\Phi|^2 + \sum_{i=1}^{n}(\kappa_i + H)^3\right) f_v + c|A|^2 l_v \\
&= \left(-nH^3 - 3H|\Phi|^2 + \sum_{i=1}^{n}(\kappa_i + H)^3 + \left(nH + \frac{n-1}{2}a\right)|A|^2\right) f_v \\
&\quad - c\left(n^2H^2 + \frac{n(n-1)}{2}aH - |A|^2\right) l_v \\
&= \left(-nH^3 - 3H|\Phi|^2 + \sum_{i=1}^{n}(\kappa_i + H)^3 + \left(nH + \frac{n-1}{2}a\right)|A|^2\right) f_v \\
&\quad - (n^2H^2 + n(n-1)aH - |A|^2)cl_v + \frac{n(n-1)}{2}caHl_v.
\end{aligned}
$$

Considering $n^2H^2 + n(n-1)aH - |A|^2 = n(n-1)(c-b)$ is a constant and (3.4), we have

$$
\begin{aligned}
(3.17)\qquad & (n^2H^2 + n(n-1)aH - |A|^2)cl_v \\
&= \frac{1}{n}(n^2H^2 + n(n-1)aH - |A|^2)(nHf_v - \triangle l_v).
\end{aligned}
$$

Inserting (3.17) into (3.16), the expression in (3.16) becomes

$$
\begin{aligned}
(3.18)\qquad L(f_v) &= \left(-nH^3 - 3H|\Phi|^2 + \sum_{i=1}^{n}(\kappa_i + H)^3 + \left(nH + \frac{n-1}{2}a\right)|A|^2\right) f_v \\
&\quad + \frac{1}{n}(n^2H^2 + n(n-1)aH - |A|^2)(\triangle l_v - nHf_v) + \frac{n(n-1)}{2}caHl_v \\
&= \frac{1}{n}(n^2H^2 + n(n-1)aH - |A|^2)\triangle l_v + \frac{n(n-1)}{2}caHl_v \\
&\quad + \left[\sum_{i=1}^{n}(\kappa_i + H)^3 + \left((n-2)H + \frac{n-1}{2}a\right)|\Phi|^2 - \frac{n(n-1)}{2}aH^2\right] f_v \\
&= (n-1)(c-b)\triangle l_v + \frac{n(n-1)}{2}caHl_v \\
&\quad + \left[\sum_{i=1}^{n}(\kappa_i + H)^3 + \left((n-2)H + \frac{n-1}{2}a\right)|\Phi|^2 - \frac{n(n-1)}{2}aH^2\right] f_v.
\end{aligned}
$$

On the other hand, (3.6) implies that

$$\frac{n(n-1)}{2}caHl_v = \frac{a}{2}(2S_2 f_v - \Box l_v). \tag{3.19}$$

Inserting (3.19) into (3.18) and considering

$$2S_2 = n^2H^2 - |A|^2 = n(n-1)H^2 - |\Phi|^2,$$

we have

$$\begin{aligned} L(f_v) - (n-1)(c-b)\triangle l_v + \frac{a}{2}\Box l_v &= \left[\frac{a}{2}(2S_2 - n(n-1)H^2) + \sum_{i=1}^{n}(\kappa_i + H)^3 \right. \\ &\quad \left. + \left((n-2)H + \frac{n-1}{2}a\right)|\Phi|^2\right] f_v \\ &= \left[\sum_{i=1}^{n}(\kappa_i + H)^3 + (n-2)\left(H + \frac{a}{2}\right)|\Phi|^2\right] f_v, \end{aligned} \tag{3.20}$$

which implies (3.9). □

4. Proof of the main theorem

In order to prove our main result, we will describe some particular regions of the hyperbolic space $\mathbf{H}^{n+1}$. We recall that $\mathbf{H}^{n+1}$ admits a foliation by means of totally umbilical hypersurfaces

$$L_\tau = \{p \in \mathbf{H}^{n+1} \mid \langle p, v\rangle = \tau\},$$

where $v \in \mathbf{L}^{n+2}$ is a fixed vector and $\tau^2 + \langle v, v\rangle > 0$ ([28]).

In particular, when v is a nonzero null vector, we have that such hypersurfaces L_τ are exactly the *horospheres* of $\mathbf{H}^{n+1}$. In this case, we will refer to the *interior domain* enclosed by L_τ the set

$$\{p \in \mathbf{H}^{n+1} \mid \langle p, v\rangle < \tau\},$$

and the *exterior domain* enclosed by L_τ the set

$$\{p \in \mathbf{H}^{n+1} \mid \langle p, v\rangle > \tau\}.$$

On the other hand, when v is a spacelike vector, the level set

$$L_0 = \{p \in \mathbf{H}^{n+1} \mid \langle p, v\rangle = 0\}$$

defines a totally geodesic hypersphere in $\mathbf{H}^{n+1}$. So, in analogy to the context of the Euclidean sphere $\mathbf{S}^{n+1}$, we will refer to such hypersphere as the *equator* of $\mathbf{H}^{n+1}$ determined by v. This equator divides $\mathbf{H}^{n+1}$ into two connected components, which (proceeding with our analogy between $\mathbf{S}^{n+1}$ and $\mathbf{H}^{n+1}$) will be called *hemispheres* of $\mathbf{H}^{n+1}$ determined by v.

For the simplicity of proof, we first show the following proposition.

Proposition 4.1. *Let $\varphi : M^n \hookrightarrow \mathbf{L}_1^{n+1}(c)$, $n \geq 3$, be a complete linear Weingarten spacelike hypersurface with bounded mean curvature H and $R = aH + b$, where a, b are constants and $a \geq 0$, $b < c$. Suppose the angle function f_v determined by a nonzero vector $v \in \mathbf{R}_q^{n+1+|c|}$, $q = 1 + \frac{1}{2}(|c| - c)$ with $|v^\top| \in \mathfrak{L}^1(M)$ does not change sign on M, then $|\Phi|^2 f_v = 0$ on M.*

Proof. Initially from Gauss equation (2.3) and $b < c$, we obtain that

$$n^2H^2 + n(n-1)aH = |A|^2 + n(n-1)(c-b) > 0, \tag{4.1}$$

which implies that the mean curvature H does not vanishes on M. Hence, we can choose an orientation for M in such a way that $H > 0$.

Since

$$\sum_{i=1}^{n}(\kappa_i + H) = 0 \quad \text{and} \quad \sum_{i=1}^{n}(\kappa_i + H)^2 = |\Phi|^2,$$

it follows from Lemma 3.1 that

$$\sum_{i=1}^{n}(\kappa_i + H)^3 \geq -\frac{n-2}{\sqrt{n(n-1)}}|\Phi|^3.$$

Thus we obtain

$$\sum_{i=1}^{n}(\kappa_i + H)^3 + (n-2)\left(H + \frac{a}{2}\right)|\Phi|^2 \geq (n-2)|\Phi|^2\left(H + \frac{a}{2} - \frac{|\Phi|}{\sqrt{n(n-1)}}\right). \tag{4.2}$$

From (2.9), a simple computation implies

$$\left(H + \frac{a}{2}\right)^2 - \left(\frac{|\Phi|}{\sqrt{n(n-1)}}\right)^2 = \frac{a^2}{4} + c - b > 0 \tag{4.3}$$

if $b < c$. Therefore, since we have assumed that $a \geq 0$, $b < c$, from equations (4.2) and (4.3), we obtain

$$\begin{aligned} &\sum_{i=1}^{n}(\kappa_i + H)^3 + (n-2)\left(H + \frac{a}{2}\right)|\Phi|^2 \\ &\qquad \geq (n-2)|\Phi|^2\left(H + \frac{a}{2} - \frac{|\Phi|}{\sqrt{n(n-1)}}\right) \geq 0. \end{aligned} \tag{4.4}$$

on M. Therefore, Lemma 3.4, equation (4.4) and the fact f_v does not change sign on M allow us to conclude that

$$\operatorname{div}\left(P(\nabla f_v) + (n-1)(b-c)\nabla l_v + \frac{a}{2}P_1(\nabla l_v)\right)$$

does not change sign on M.

Since H is bounded, it follows from Gauss equation (2.3) that $|A|^2$ is bounded on M. Consequently, from Cauchy-Schwarz inequality, we have that

$$\begin{aligned}\left|P(\nabla f_v) + (n-1)(b-c)\nabla l_v + \frac{a}{2}P_1(\nabla l_v)\right| \\ &= \left|-\left(nH + \frac{n-1}{2}a\right)A(v^\top) - A^2(v^\top) + \frac{a}{2}(nH+A)(v^\top) + (n-1)(b-c)v^\top\right| \\ &\le \left|nH|A| + \frac{n}{2}a|A| + |A|^2 + \frac{a}{2}nH + (n-1)(c-b)\right||v^\top| \in \mathfrak{L}^1(M).\end{aligned}$$

Hence, from Lemma 3.3 we obtain

$$\operatorname{div}\left(P(\nabla f_v) + (n-1)(b-c)\nabla l_v + \frac{a}{2}P_1(\nabla l_v)\right) = 0.$$

Equation (4.4) and the hypothesis on $n \ge 3$ imply that $|\Phi|^2 f_v = 0$ on M. □

Proof of Theorem 1.1. When $c = 1$, we note that, when v is a null vector, our hypothesis on the hyperbolic Gauss map of M means that the corresponding angle function $f_v = \langle N, v\rangle$ has strict sign on M, and when v is a timelike vector, the simple fact that M is a spacelike hypersurface implies that f_v always has strict sign on M. Applying Proposition 4.1 we conclude that $|\Phi|^2 = 0$ on M when v is either a timelike or null vector. Therefore, M must be totally umbilical. Furthermore, when v is a spacelike vector, we have f_v does not change sign on M from the assumption of Gauss map. A straightforward computation shows that

$$\left|\nabla^2 l_v - \frac{1}{n}(\triangle l_v)g\right|^2 = |\nabla^2 l_v|^2 - \frac{1}{n}(\triangle l_v)^2 = |\Phi|^2 f_v^2,$$

where g stands for the Riemannian metric of M. Thus $|\Phi|^2 f_v = 0$ implies that

$$\nabla^2 l_v = \frac{1}{n}(\triangle l_v)g.$$

Consequently, $\nabla l_v = v^\top$ is a conformal vector field on M. On the other hand, taking into account once more that $|v^\top| \in \mathfrak{L}^1(M)$, v can not be a tangent vector to M. Therefore, from Lemma 3.2 we conclude that M is a totally umbilical hypersurface.

When $c = -1$, our hypothesis on the Gauss map of M implies that there exists a timelike vector $v \in \mathbf{R}_2^{n+2}$ such that the corresponding angle function $f_v = \langle N, v \rangle$ has strict sign on M, Applying Proposition 4.1 again, we conclude that $|\Phi|^2 = 0$ on M. Consequently, M must be a totally umbilical hypersurface. $\square$

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