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A NOTE ON HOLOMORPHIC QUADRATIC DIFFERENTIALS ON THE UNIT DISK

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Abstract

Let $Q(\Delta)$ be the set of all integrable holomorphic quadratic differentials on the unit disk Δ. The subset $Q_0(\Delta)$ of $Q(\Delta)$ is the set associated with T_0 classes in the universal Teichmüller space $T(\Delta)$. In this paper, it is shown that $Q_0(\Delta)$ is dense in $Q(\Delta)$. The infinitesimal version is also obtained.

1. Introduction

Let Δ be the unit disk in the complex plane $\mathbf{C}$. Denote by $Bel(\Delta)$ the Banach space of Beltrami differentials $\mu = \mu(z)\, d\bar{z}/dz$ on Δ with finite L^{∞}-norm and by $M(\Delta)$ the open unit ball in $Bel(\Delta)$. Denoted by $Q(\Delta)$ the Banach space of the integrable holomorphic quadratic differentials on Δ with L^1-norm

$$\|\varphi\| = \iint_{\Delta} |\varphi(z)|\, dxdy < \infty. \tag{1.1}$$

In what follows, let $SQ(\Delta)$ denote the unit sphere and $Q_1(\Delta)$ the closed unit ball of $Q(\Delta)$.

For each element $\mu \in M(\Delta)$ there exists a uniquely determined quasiconformal mapping f^{μ} of Δ onto itself such that f^{μ} keeps $1, -1, i$ fixed and has the complex dilatation μ. Two elements μ_1 and μ_2 are said to be Teichmüller equivalent (denoted by $\mu_1 \sim \mu_2$) if and only if $f^{\mu_1}|_{S^1} = f^{\mu_2}|_{S^1}$. We denote the Teichmüller equivalence class of μ by $[\mu]$ or by $[f^{\mu}]$. The universal Teichmüller space $T(\Delta)$ is defined as the quotient space of $M(\Delta)$ under the equivalence relation. The point $[0]$, the Teichmüller equivalence class of the trivial Beltrami differential $\mu = 0$, is called the basepoint of $T(\Delta)$.

Define

$$k_0([\mu]) := \inf\{\|\nu\|_{\infty} : \nu \in [\mu]\}.$$

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We say that μ is extremal in $[\mu]$ if $\|\mu\|_\infty = k_0([\mu])$ (the corresponding quasiconformal map f^μ is said to be extremal for its boundary values as well), uniquely extremal if $\|\nu\|_\infty > k_0(\mu)$ for any other $\nu \in [\mu]$.

As is well known, a Beltrami differential μ is extremal if and only if it has a so-called Hamilton sequence, namely, a sequence $\{\varphi_n \in SQ(\Delta)\}$, such that

$$\lim_{n\to\infty} \iint \mu\varphi_n(z)\, dxdy = \|\mu\|_\infty. \tag{1.2}$$

Define $h^*(\mu)$ to be the infimum over all compact subsets E contained in Δ of the essential supremum norm of the Beltrami differential $\mu(z)$ as z varies over $\Delta\backslash E$ and $h([\mu])$ to be the infimum of $h^*(\nu)$ taken over all representatives ν of the class $[\mu]$. It is obvious that $h([\mu]) \le k_0([\mu])$. Following Earle and Li Zhong [4], $[\mu]$ is called a Strebel point if $h([\mu]) < k_0([\mu])$; otherwise, τ is called a non-Strebel point. The result in [10] shows that the set of Strebel points is open and dense in $T(\Delta)$.

By the result in [4] $[\mu]$ is a non-Strebel point if and only if the extremal in $[\mu]$ has a degenerating Hamilton sequence. A sequence in $Q(\Delta)$ is called degenerating if it converges to 0 uniformly on compact subsets of Δ.

By Strebel's frame mapping criterion (see Chapter 4 in [6]), every Strebel point $[\mu]$ is represented by the uniquely-extremal Beltrami differential of the form $k\dfrac{\bar{\varphi}}{|\varphi|}$, where $k = k_0([\mu]) \in (0,1)$ and φ is a unit vector in $Q(\Delta)$. Define the set of Strebel differentials $Q_S(\Delta)$ by

$$Q_S(\Delta) = \left\{\varphi \in Q(\Delta)\backslash\{0\} : \exists k \in (0,1)\ \ s.t.\ \left[k\frac{\bar{\varphi}}{|\varphi|}\right] \text{ is a Strebel point}\right\}.$$

The complement $Q_N(\Delta)$ of $Q_S(\Delta)$ in $Q(\Delta)$ is called the set of non-Strebel differentials.

Define

$$T_0(\Delta) = \{[\mu] \in T(\Delta) : h([\mu]) = 0\}.$$

$T_0(\Delta)$ is a closed subspace of $T(\Delta)$ [7] and every point in $T_0(\Delta)\backslash\{[0]\}$ is a Strebel point. The set $Q_0(\Delta)$ of T_0-class differentials is defined by

$$Q_0(\Delta) = \left\{\varphi \in Q(\Delta)\backslash\{0\} : \exists k \in (0,1),\ s.t.\ \left[k\frac{\bar{\varphi}}{|\varphi|}\right] \in T_0(\Delta)\right\}.$$

In [10], Lakic proved that both $Q_S(\Delta)$ and $Q_N(\Delta)$ are dense in $Q(\Delta)$. Naturally, we ask the following problem.

PROBLEM 1. Is it true that $Q_0(\Delta)$ is dense in $Q(\Delta)$?

In this paper, we answer the problem affirmatively.

THEOREM 1. *$Q_0(\Delta)$ is dense in $Q(\Delta)$.*

The theorem is proved in the next section. An infinitesimal version of Theorem 1 is obtained in Section 3.

2. Proof of Theorem 1

To prove Theorem 1, we need several lemmas.

The following lemma derives from the result in Lecture 4 of L. Bers' Courant Institute book [1].

LEMMA 2.1. *Let $\mu(z)$ be a $C^{n+\alpha}$ $(0 < \alpha < 1,\ n = 0, 1, 2, \dots)$ function on a disk around zero, $\|\mu\|_\infty < 1$. Let a, b be two complex constants. Then any local solution $w = f(z)$ of the Beltrami equation*

$$w_{\bar z} = \mu(z) w_z, \quad w(0) = a, \quad w_z(0) = b, \tag{2.1}$$

is of class $C^{n+1+\alpha}$.

Proof. By the theorem in Lecture 4 of L. Bers' Courant Institute book [1], the system (2.1) possesses, in the neighborhood of the origin, a solution $w(z)$ of class $C^{n+1+\alpha}$. Furthermore, by the similarity principle, any local solution $w = f(z)$ of the system (2.1) is of class $C^{n+1+\alpha}$. □

Let C be a Jordan curve in $\mathbf{C}$. We say that C is smooth if there is a parametrization $C : w(t),\ t \in [0, 2\pi]$ such that $w'(t)$ is continuous and nowhere vanishing. The following result due to Kellogg can be found in [8] or [9] (Chapter X. Theorem 6, page 426).

LEMMA 2.2. *Let γ map Δ conformally onto the inner domain of the Jordan curve C of class $C^{1+\alpha}$, $0 < \alpha < 1$. Then γ' and $\log \gamma'$ are both C^α on $\bar\Delta$. In particular, $\gamma'(z)$ is bounded from above and below on $\bar\Delta$.*

A quasiconformal map $f : \Delta \to \Delta$ is called asymptotically conformal if for every $\varepsilon > 0$ there is a compact set E in Δ such that f is $(1 + \varepsilon)$-quasiconformal on $\Delta \backslash E$.

If $[\mu] \in T_0(\Delta)$, by Theorem 2 in Chapter 15 of [6] there is a so-called asymptotically extremal representative in $[\mu]$, say μ, such that f^μ is asymptotically conformal. Conversely, it is clear that if f^μ is asymptotically conformal, then $h([\mu]) = h^*(\mu) = 0$ and $[\mu] \in T_0(\Delta)$.

Let $q : S^1 \to S^1$ be a quasisymmetric map of the unit circle. The map q is said to be symmetric (see [7], [13]) if

$$\lim_{t \to 0} \frac{q(e^{i(t+t_0)}) - q(e^{it_0})}{q(e^{it_0}) - q(e^{i(t_0 - t)})} = 1, \tag{2.2}$$

uniformly in t_0 for all points $z_0 = e^{it_0} \in S^1$.

Strebel [13] and Fehlmann [5] showed respectively that a quasisymmetric map $q : S^1 \to S^1$ is symmetric if and only if q admits an asymptotically conformal extension to Δ.

To sum up, we have the following conclusion.

Lemma 2.3. *$[\mu] \in T_0(\Delta)$ if and only if the boundary map $q = f^{\mu}|_{S^1}$ is symmetric.*

Lemma 2.4. *Suppose φ belongs to $Q(\Delta)$ and is holomorphic in $\{z : |z| < 1 + \rho\}$ for some $\rho > 0$. If φ has no zero points on S^1, then $[\mu] \in T_0(\Delta)$ and $\varphi \in Q_0(\Delta)$ where*

$$\mu(z) = k\frac{\overline{\varphi(z)}}{|\varphi(z)|}, \quad k \in (0, 1).$$

Proof. By the discreteness of zero points of non-zero holomorphic functions, there exists some small $\delta > 0$ such that φ has no zero points in $R = \{z : 1 - \delta < |z| < 1 + \delta\}$. Therefore, $\mu(z)$ is C^∞ on R. Let $\tilde{f}$ be a quasiconformal mapping from $\{z : |z| < 1 + \delta\}$ onto the unit disk Δ with the complex dilatation μ. Applying Lemma 2.1 on R piecewise, we see that $\tilde{f}$ is C^∞ on R. It yields that $C = \tilde{f}(S^1)$ is a smooth Jordan curve of class C^∞.

Denote by J the inner Jordan domain of C and by γ a conformal mapping from Δ onto J. By Carathéodory's theorem ([2] or see Theorem 2 on page 41 in [9]), γ has a continuous extension to $\bar{\Delta}$. By Lemma 2.2, $\gamma'(z)$ is nowhere vanishing on $\bar{\Delta}$. It derives that the inverse map γ^{-1} is a C^1 diffeomorphism from $\bar{J}$ onto $\bar{\Delta}$.

Let $g = \gamma^{-1} \circ \tilde{f}$ be the quasiconformal mapping from Δ onto Δ. It is clear that g is at least C^1 on $\bar{\Delta}$ and hence $g|_{S^1}$ is a symmetric map of S^1 by the definition. Notice that the Beltrami differential of g is μ. Therefore, by Lemma 2.3 we have $[\mu] \in T_0(\Delta)$. $\square$

Proof of Theorem 1. Given a holomorphic quadratic differential φ in $Q(\Delta)\backslash\{0\}$, let $\varphi_r(z) = \varphi(rz)$ for $r \in (0, 1)$. Obviously, φ_r converges to φ uniformly on compact subset of Δ. Moreover, we have

$$\lim_{r \to 1} \|\varphi_r - \varphi\| = 0.$$

Since φ is holomorphic in Δ, the set of zero points of φ is discrete in Δ. We can choose a sequence $\{r_n\}$ in $(0, 1)$ such that $r_n \to 1$, $n \to \infty$ and φ_n $(= \varphi_{r_n})$ has no zero points on S^1.

Fix $k \in (0, 1)$ and consider the Beltrami differential

$$\mu_n(z) = k\frac{\overline{\varphi_n(z)}}{|\varphi_n(z)|}. \tag{2.3}$$

Since φ_n is holomorphic in $\left\{z : |z| < \frac{1}{r_n}\right\}$, it follows from Lemma 2.4 that $[\mu_n] \in T_0(\Delta)$ and hence $\varphi_n \in Q_0(\Delta)$. It is clear that

$$\lim_{n\to\infty} \|\varphi_n - \varphi\| = 0.$$

The proof of Theorem 1 is complete.

3. Infinitesimal version of Theorem 1

Two Beltrami differentials μ and ν in $Bel(\Delta)$ are said to be infinitesimally equivalent if

$$\iint_\Delta \mu\varphi \, dxdy = \iint_\Delta \nu\varphi \, dxdy, \quad \text{for any } \varphi \in Q(\Delta).$$

The tangent space $B(\Delta)$ of $T(\Delta)$ at the basepoint is defined as the set of the quotient space of $Bel(\Delta)$ under the equivalence relation. Denote by $[\mu]_B$ the equivalence class of μ in $B(\Delta)$.

$B(\Delta)$ is a Banach space with its standard sup-norm

$$\|[\mu]_B\| = \|\mu\| := \sup_{\varphi \in SQ(\Delta)} \left| \iint_\Delta \mu\varphi \, dxdy \right|$$

and infinitesimal metric

$$\begin{aligned} d([\mu]_B, [\nu]_B) &:= \|\mu - \nu\| \\ &= \sup_{\varphi \in SQ(\Delta)} \left| \iint_\Delta (\mu - \nu)\varphi \, dxdy \right|, \quad [\mu]_B, [\nu]_B \in B(\Delta). \end{aligned}$$

Define the (infinitesimal) boundary dilatation $b([\mu]_B)$ of $[\mu]_B$ to be the infimum over all elements in the equivalence class $[\mu]_B$ of the quantity $b^*(\nu)$. Here $b^*(\nu)$ is the infimum over all compact subsets E contained in Δ of the essential supremum of the Beltrami differential $\nu(z)$ as z varies over $\Delta \backslash E$. We call $[\mu]_B$ in $B(\Delta)$ an infinitesimal Strebel point if $b([\mu]_B) < \|\mu\|$. Clearly, the basepoint is not a Strebel point since $b([0]_B) = \|[0]_B\| = 0$. It follows from Reich's infinitesimal frame mapping theorem (see Theorem 2.4 in [11]) that if $[\mu]_B$ is an infinitesimal Strebel point, then there exists a unique vector φ in $Q(\Delta)$ such that μ and $\|\mu\| \frac{\bar{\varphi}}{|\varphi|}$ are infinitesimally equivalent.

In [3], the semi-norm β on $B(\Delta)$ is defined by

$$\beta(\mu) = \sup_{\varphi_n} \limsup_{n\to\infty} \left| \iint_\Delta \mu\varphi_n \, dxdy \right|,$$

where the supremum is taken over all degenerating sequences in $Q_1(\Delta)$. It is also proved in [3] that $\beta(\mu) = b([\mu]_B)$.

Define

$$B_0(\Delta) = \{[\mu]_B \in B(\Delta) : b([\mu]_B) = 0\}.$$

Then $B_0(\Delta)$ is a linear subspace of $B(\Delta)$ and every point in $B_0(\Delta)\backslash\{[0]_B\}$ is an infinitesimal Strebel point.

In a parallel way, one can define the sets of infinitesimal Strebel and non-Strebel differentials respectively. Whereas there is no difference between these two kinds of Strebel (or non-Strebel) differentials due to the property that a(n) (infinitesimal) Strebel point has no degenerating Hamilton sequence and vice versa. So, we still use $Q_N(\Delta)$ and $Q_S(\Delta)$ to denote them.

By use of a trick in [10] for the infinitesimal setting, one easily proves (also see [12]) that the set of infinitesimal Strebel points is open and dense in $B(\Delta)$.

We now define a new set of B_0-class (parallel to T_0-class) differentials independently. The set $Q_Z(\Delta)$ of B_0-class differentials is defined by

$$Q_Z(\Delta) = \left\{\varphi \in Q(\Delta)\backslash\{0\} : \exists k \in \mathbf{R},\ s.t.\ \left[k\frac{\bar{\varphi}}{|\varphi|}\right]_B \in B_0(\Delta)\right\}.$$

The infinitesimal counterpart of Theorem 1 is as follows.

THEOREM 2. *$Q_Z(\Delta)$ is dense in $Q(\Delta)$.*

Proof. Let φ be a holomorphic quadratic differential in $Q(\Delta)\backslash\{0\}$. Use the same denotation as in the proof of Theorem 1. Since μ_n is continuous on some ring domain $U_n = \{z : \rho_n \leq |z| \leq 1\}$, $\rho_n \in (0,1)$, where μ_n is defined by (2.3). It follows from the following Lemma 3.1 that $[\mu_n]_B$ belongs to $B_0(\Delta)$ and hence $\varphi_n \in Q_Z(\Delta)$. It yields directly that $Q_Z(\Delta)$ is dense in $Q(\Delta)$. □

LEMMA 3.1. *Suppose $\mu \in Bel(\Delta)$ is continuous in the ring domain $\{z : \rho \leq |z| \leq 1\}$, $\rho \in (0,1)$. Then $[\mu]_B \in B_0(\Delta)$.*

Proof. Since $b([\mu]_B) = \beta(\mu)$, it suffices to prove that $\beta(\mu) = 0$. For any given degenerating sequence $\{\psi_n\}$ in $SQ(\Delta)$, we need to prove that

$$\limsup_{n\to\infty}\left|\iint_\Delta \mu\psi_n\, dxdy\right| = 0. \tag{3.1}$$

Let $\epsilon > 0$ and

$$P(z,\bar{z}) = \sum_{k,l=0}^{N} c_{kl}z^k\bar{z}^l = \sum_{k,l=0}^{N} c_{kl}r^{k+l}e^{i(k-l)\theta}$$

be a polynomial in z and $\bar{z}$ for which

$$|\mu(z) - P(z,\bar{z})| \leq \epsilon, \quad \rho \leq |z| \leq 1.$$

Notice that

$$\int_0^{2\pi} e^{in\theta}\, d\theta = 0, \quad n = 1, 2, \ldots.$$

We have

$$\begin{aligned}
\iint_{\rho\le|z|\le 1} P(z,\bar z)\psi_n(z)\, dxdy &= \iint_{\rho\le|z|\le 1}\left[\sum_{k,l=0}^{N} c_{kl} z^k \bar z^l \sum_{m=0}^{\infty}\frac{\psi_n^{(m)}(0)}{m!} z^m\right] dxdy \\
&= \int_0^{2\pi}\int_\rho^1\left[\sum_{k,l=0}^{N}\sum_{m=0}^{\infty} c_{kl}\frac{\psi_n^{(m)}(0)}{m!} r^{m+k+l+1} e^{i(m+k-l)\theta}\right] drd\theta \\
&= \int_0^{2\pi}\int_\rho^1\left[\sum_{k,l=0}^{N}\sum_{m=0}^{N} c_{kl}\frac{\psi_n^{(m)}(0)}{m!} r^{m+k+l+1} e^{i(m+k-l)\theta}\right] drd\theta \\
&= \sum_{m=0}^{N}\frac{\psi_n^{(m)}(0)}{m!}\sum_{k,l=0}^{N} c_{kl}\int_\rho^1 r^{m+k+l+1}\, dr\int_0^{2\pi} e^{i(m+k-l)\theta}\, d\theta.
\end{aligned}$$

Since ψ_n converges to 0 uniformly on compact subsets of Δ, it holds that

$$\lim_{n\to\infty} \psi_n^{(m)}(0) = 0$$

for any fixed m. Therefore,

$$\lim_{n\to\infty}\iint_{\rho\le|z|\le 1} P(z,\bar z)\psi_n(z)\, dxdy = 0.$$

It follows readily that

$$\lim_{n\to\infty}\left|\iint_\Delta \mu(z)\psi_n(z)\, dxdy\right| = \left|\lim_{n\to\infty}\iint_{\rho\le|z|\le 1}(\mu(z) - P(z,\bar z))\psi_n(z)\, dxdy\right| \le \epsilon.$$

Thus, we obtain (3.1). □

Lemma 3.1 indicates that a Beltrami differential belongs to $B_0(\Delta)$ if it is continuous near the boundary S^1. It is unclear whether there is a similar conclusion for the T_0 classes. Precisely, we pose the following problem.

Problem 2. Suppose μ is continuous in the ring domain $\{z : \rho \le |z| \le 1\}$, $\rho \in (0,1)$. Can we say that $[\mu] \in T_0(\Delta)$?

The author believes that the boundary map $f^\mu|_{S^1}$ corresponding to μ is Lipschitz or even bi-Lipschitz if μ is continuous near S^1.

At last, we note that it is very interesting to determine the relation between the two sets $Q_0(\Delta)$ and $Q_Z(\Delta)$. The following problem is open.

Problem 3. Is it true that $Q_0(\Delta)$ and $Q_Z(\Delta)$ coincide?

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