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POSITIVE TOEPLITZ OPERATORS OF FINITE RANK ON THE PARABOLIC BERGMAN SPACES

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Abstract

We define the Toeplitz operators on the parabolic Bergman spaces by using a positive bilinear form. In this setting we characterize finite rank Toeplitz operators. A relation with the Carleson inclusion is also discussed.

§1. Introduction

We consider the α-parabolic operator

$$L^{(\alpha)} := \frac{\partial}{\partial t} + (-\Delta_x)^\alpha$$

on the upper half space $\boldsymbol{R}_+^{n+1}$, where $\Delta_x := \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ is the Laplacian on the x-space $\boldsymbol{R}^n$ and $0 < \alpha \le 1$. Here we denote by $X = (x, t)$, $Y = (y, s)$ and $Z = (z, r)$ points in $\boldsymbol{R}_+^{n+1} = \boldsymbol{R}^n \times (0, \infty)$. We denote by $(\boldsymbol{b}_\alpha^2(\lambda), \langle\cdot,\cdot\rangle)$ the Hilbert space

$$\boldsymbol{b}_\alpha^2(\lambda) := \{u \in L^2(\boldsymbol{R}_+^{n+1}, V^\lambda); L^{(\alpha)}\text{-harmonic on } \boldsymbol{R}_+^{n+1}\},$$

where $\lambda > -1$ and V^λ is the $(n+1)$-dimensional weighted Lebesgue measure $t^\lambda\, dxdt$ on $\boldsymbol{R}_+^{n+1}$. Note that if $\lambda \le -1$, then $\boldsymbol{b}_\alpha^2(\lambda) = \{0\}$. Since for $X \in \boldsymbol{R}_+^{n+1}$ the point evaluation $u \mapsto u(X) : \boldsymbol{b}_\alpha^2(\lambda) \to \boldsymbol{R}$ is bounded (see [5, Proposition 4.1]), the orthogonal projection from $L^2(V^\lambda) := L^2(\boldsymbol{R}_+^{n+1}, V^\lambda)$ to $\boldsymbol{b}_\alpha^2(\lambda)$ is represented as an integral operator by a kernel $R_{\alpha,\lambda}$, which is called the α-parabolic Bergman kernel.

For a positive Radon measure μ on $\boldsymbol{R}_+^{n+1}$, put

$$\mathrm{Dom}(T_\mu^\lambda) := \left\{u \in \boldsymbol{b}_\alpha^2(\lambda); \iint |R_{\alpha,\lambda}(\cdot, Y)u(Y)|\, d\mu(Y) \in L^2(\boldsymbol{R}_+^{n+1}, V^\lambda)\right\}$$

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and for $u \in \mathrm{Dom}(T_\mu^\lambda)$ we set

$$(T_\mu^\lambda u)(X) := \iint R_{\alpha,\lambda}(X, Y) u(Y)\, d\mu(Y). \tag{1}$$

We call T_μ^λ a positive Toeplitz operator with symbol μ and weight t^λ. For the case $\lambda = 0$, under the assumption on μ that

$$\iint \frac{1}{(1 + t + |x|^{2\alpha})^\tau}\, d\mu(x,t) < \infty \tag{2}$$

for some $\tau > 0$, we proved in [9] that $\mathrm{Dom}(T_\mu^\lambda) = \boldsymbol{b}_\alpha^2(0)$ and $T_\mu^\lambda : \boldsymbol{b}_\alpha^2(0) \to \boldsymbol{b}_\alpha^2(0)$ is bounded if and only if μ is an α-parabolic Carleson measure. Furthermore, we have already discussed its compactness ([10]) and Schatten class ([12] and [14]).

In this note, we shall study the rank of positive Toeplitz operators on $\boldsymbol{b}_\alpha^2(\lambda)$ for $\lambda > -1$. In order to discuss without the assumption (2), we give an alternative definition of Toeplitz operator. We recall the following general theory (see, for example, [3] or [4]): Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and $\mathscr{E}$ be a bilinear form defined on a subspace $\mathscr{D}$ of $\mathscr{H}$. We denote by $\overline{\mathscr{D}}$ the closure of $\mathscr{D}$ in $\mathscr{H}$. If $\mathscr{E}$ is positive, i.e., $\mathscr{E}(u,u) \geq 0$ for all $u \in \mathscr{D}$, and if $\mathscr{E}$ is closed, i.e., complete with respect to the inner product $\langle\cdot,\cdot\rangle + \mathscr{E}(\cdot,\cdot)$, then there exists a unique positive self-adjoint operator $\tilde{T}$ on a dense subset $\mathrm{Dom}(\tilde{T})$ in $\overline{\mathscr{D}}$ such that

$$\mathscr{E}(u,v) = \langle \tilde{T}u, v\rangle$$

for every $u \in \mathrm{Dom}(\tilde{T})$ and every $v \in \mathscr{D}$. Note that the domain of $\sqrt{\tilde{T}}$ coincides with $\mathscr{D}$, and $\mathscr{E}(u,v) = \langle\sqrt{\tilde{T}}u, \sqrt{\tilde{T}}v\rangle$ holds for $u, v \in \mathscr{D}$.

Let $\mu \geq 0$ be a Radon measure on $\boldsymbol{R}_+^{n+1}$ and $\lambda > -1$. Applying the above general theory to $\mathscr{H} = \boldsymbol{b}_\alpha^2(\lambda)$, $\mathscr{D} = \boldsymbol{b}_\alpha^2(\lambda) \cap L^2(\boldsymbol{R}_+^{n+1}, \mu)$ and a bilinear form

$$\mathscr{E}(u,v) := \iint u(X)v(X)\, d\mu(X),$$

we have a positive self-adjoint operator $\tilde{T}_\mu^\lambda$ on $\mathrm{Dom}(\tilde{T}_\mu^\lambda) \subset \overline{\mathscr{D}}$ such that

$$\iint (\tilde{T}_\mu^\lambda u)(X) v(X)\, dV^\lambda(X) = \iint u(X)v(X)\, d\mu(X) \tag{3}$$

for every $u \in \mathrm{Dom}(\tilde{T}_\mu^\lambda)$ and $v \in \mathscr{D}$. Then we also define the rank of $\tilde{T}_\mu^\lambda$ by

$$\mathrm{rank}(\tilde{T}_\mu^\lambda) := \dim(\tilde{T}_\mu^\lambda(\mathrm{Dom}(\tilde{T}_\mu^\lambda))).$$

Now, we shall state our main theorem.

THEOREM 1. *Let $\lambda > -1$ and μ be a positive Radon measure on $\boldsymbol{R}_+^{n+1}$. If there exists a dense subspace $\mathscr{D}_0$ in $\boldsymbol{b}_\alpha^2(\lambda)$ such that $\mathscr{D}_0 \subset \mathrm{Dom}(\tilde{T}_\mu^\lambda)$ and $\dim(\tilde{T}_\mu^\lambda(\mathscr{D}_0)) < \infty$, then μ is a finite linear combination of point masses and*

$$\mathrm{rank}(\tilde{T}_\mu^\lambda) = \#\mathrm{supp}(\mu) = \dim(\tilde{T}_\mu^\lambda(\mathscr{D}_0))$$

holds, where $\#A$ denotes the cardinal number of a set A.

We note that if μ satisfies (2), then $\tilde{T}_\mu^\lambda$ is a self-adjoint extension of T_μ^λ (see Remark 1 below). Moreover, if $\mathrm{supp}(\mu)$ is compact, then $\tilde{T}_\mu^\lambda = T_\mu^\lambda$ on $\boldsymbol{b}_\alpha^2(\lambda)$. Hence denoting $\mathrm{rank}(T_\mu^\lambda) := \dim(T_\mu^\lambda(\boldsymbol{b}_\alpha^2(\lambda)))$, we have the following

THEOREM 2. *Let $\lambda > -1$, and let $\mu \geq 0$ be a Radon measure on $\boldsymbol{R}_+^{n+1}$ with compact support. If the corresponding Toeplitz operator T_μ^λ is of finite rank, then the support of μ is a finite set, and moreover we have* $\mathrm{rank}(T_\mu^\lambda) = \#\mathrm{supp}(\mu)$.

Theorem 1 implies Theorem 2, but we give a direct proof of Theorem 2 in section 4. Using Theorem 2 we give a proof of Theorem 1 in section 5. In section 6, we make a relation to Carleson inclusions.

In the theory of classical holomorphic Bergman space on the unit disc in the complex plane, Luecking [7] solved the finite rank problem for complex measures with compact support. A generalization to higher dimensions is given by Choe [1].

§2. Preliminaries

We recall some basic properties of a fundamental solution of $L^{(\alpha)}$, of fractional derivatives of Riemann-Liouville type and of the parabolic Bergman kernel, which we use later. For proofs and more information about them, see [8], [5] and [6].

Let $0 < \alpha \leq 1$. A measurable function u on $\boldsymbol{R}_+^{n+1}$ is said to be $L^{(\alpha)}$-harmonic, if u is continuous on $\boldsymbol{R}_+^{n+1}$ and if $L^{(\alpha)}u = 0$ in the sense of distribution, i.e.,

$$\iint u(X) \cdot ((L^{(\alpha)})^* \varphi(X)) \, dV(X) = 0$$

for every $\varphi \in C_c^\infty(\boldsymbol{R}_+^{n+1})$, where

$$(L^{(\alpha)})^* \varphi(x,t) := -\frac{\partial}{\partial t}\varphi(x,t) - c_{n,\alpha} \lim_{\delta \downarrow 0} \int_{|y|>\delta} (\varphi(x+y,t) - \varphi(x,t))|y|^{-n-2\alpha} \, dy,$$

$c_{n,\alpha} = -4^\alpha \pi^{-n/2}\Gamma((n+2\alpha)/2)/\Gamma(-\alpha) > 0$ and $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$.

We put

$$W^{(\alpha)}(x,t) := \begin{cases} (2\pi)^{-n} \displaystyle\int_{\mathbf{R}^n} \exp(-t|\xi|^{2\alpha} + \sqrt{-1}x \cdot \xi) \, d\xi & t > 0 \\ 0 & t \leq 0. \end{cases}$$

This is a fundamental solution of $L^{(\alpha)}u = 0$ so that

$$L^{(\alpha)} W^{(\alpha)} = \delta_{(0,0)} \quad \text{(in the sense of distributions)}$$

holds, where $\delta_{(x,t)}$ denotes the point mass (Dirac measure) at $(x,t) \in \boldsymbol{R}^{n+1}$. Note also that $W^{(\alpha)}(x,t) \geq 0$ and for every $0 < s < t$,

$$W^{(\alpha)}(x,t) = \int_{\mathbf{R}^n} W^{(\alpha)}(x-y, t-s) W^{(\alpha)}(y,s) \, dy \tag{4}$$

holds. When $\alpha = 1$ or $\alpha = 1/2$, we see the explicit closed form: for $t > 0$,

$$W^{(1)}(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t} \quad \text{and} \quad W^{(1/2)}(x,t) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{t}{(t^2 + |x|^2)^{(n+1)/2}}.$$

To describe the α-parabolic Bergman kernel with a weight, we use the fractional derivatives. For $\kappa \in \boldsymbol{R}$ and $\varphi \in C_c^\infty((0,\infty))$, we put

$$\partial_t^{-\kappa}\varphi(t) := \frac{1}{\Gamma(\kappa)} \int_0^t (t-\tau)^{\kappa-1} \varphi(\tau)\, d\tau$$

when $\kappa > 0$, and in general, taking $m \in \boldsymbol{N}$ with $\kappa - m < 0$, we put

$$\partial_t^\kappa \varphi(t) := \partial_t^{\kappa - m} \partial_t^m \varphi(t).$$

We define $\mathscr{D}_t$ and its fractional power $\mathscr{D}_t^\kappa$ as the dual of ∂_t in the sense of distributions:

$$\mathscr{D}_t := (\partial_t)^* = -\partial_t \quad \text{and} \quad \mathscr{D}_t^\kappa := (\partial_t^\kappa)^*.$$

Then for $\kappa > 0$ and $\kappa - m < 0$ with $m \in \boldsymbol{N}$, if a function f on $(0,\infty)$ satisfies

$$\int_1^\infty |(\mathscr{D}_t^m f)(\tau)| \tau^{\kappa - m}\, d\tau < \infty,$$

then

$$\mathscr{D}_t^\kappa f(t) = \frac{1}{\Gamma(m-\kappa)} \int_t^\infty (\tau - t)^{m-\kappa-1} (\mathscr{D}_t^m f)(\tau)\, d\tau.$$

Now let $\lambda > -1$. The reproducing kernel $R_{\alpha,\lambda}$ of $\boldsymbol{b}_\alpha^2(\lambda)$ is given by a fractional derivative of $W^{(\alpha)}$:

$$R_{\alpha,\lambda}(X,Y) = R_{\alpha,\lambda}(x,t,y,s) := \frac{2^{\lambda+1}}{\Gamma(\lambda+1)} \mathscr{D}_t^{\lambda+1} W^{(\alpha)}(x-y, t+s).$$

In fact, it is shown in [5, theorem 5.2] that $R_{\alpha,\lambda}$ has a reproducing property on $\boldsymbol{b}_\alpha^p(\lambda)$:

$$\boldsymbol{b}_\alpha^p(\lambda) := \{u \in L^p(\boldsymbol{R}_+^{n+1}, V^\lambda); L^{(\alpha)}\text{-harmonic on } \boldsymbol{R}_+^{n+1}\},$$

where $1 \le p < \infty$, i.e., for any $u \in \boldsymbol{b}_\alpha^p(\lambda)$,

$$u(X) = \iint R_{\alpha,\lambda}(X,Y) u(Y)\, dV^\lambda(Y) := R_{\alpha,\lambda} u(X) \tag{5}$$

holds true. Also, there exist constants $C_1, C_2 > 0$ such that

$$|R_{\alpha,\lambda}(X,Y)| \le C_1 (t + s + |x-y|^{2\alpha})^{-(n/2\alpha+1)-\lambda} \quad \text{for every } X, Y \in \boldsymbol{R}_+^{n+1}$$

and

$$\iint R_{\alpha,\lambda}(X,Y)^2\, dV^\lambda(Y) = C_2 t^{-(n/2\alpha+1+\lambda)/2} \quad \text{for } X \in \boldsymbol{R}_+^{n+1}. \tag{6}$$

Moreover if we define

$$R^{\nu}_{\alpha,\lambda}(X,Y) := \frac{2^{\nu+\lambda+1}}{\Gamma(\nu+\lambda+1)} s^{\nu}\mathscr{D}_t^{\nu+\lambda+1} W^{(\alpha)}(x-y,t+s),$$

then for $\nu > -(\lambda+1)(1-1/p)$,

$$D^{\nu}(\lambda) := \{R^{\nu}_{\alpha,\lambda}f; f \in L^p(\boldsymbol{R}^{n+1}_+, V^{\lambda}), \operatorname{supp}(f) \text{ is compact}\} \tag{7}$$

is a dense subspace of $\boldsymbol{b}^p_{\alpha}(\lambda)$, and for every $u \in D^{\nu}(\lambda)$, there exists a constant $C > 0$ such that

$$|u(x,t)| \le C(1+t+|x|^{2\alpha})^{-(n/2\alpha+1)-(\nu+\lambda)} \tag{8}$$

on $\boldsymbol{R}^{n+1}_+$.

§3. Linear independence of the parabolic Bergman kernels

We begin with the following lemmas.

LEMMA 1. *Let $t_0 > 0$. Then the bounded linear operator $P^{(\alpha)}_{t_0} : L^2(\boldsymbol{R}^n, dx) \to L^2(\boldsymbol{R}^n, dx)$, defined by*

$$P^{(\alpha)}_{t_0} f(x) := \int_{\boldsymbol{R}^n} W^{(\alpha)}(x-y,t_0) f(y)\, dy, \tag{9}$$

is injective.

Proof. Using the spectral decomposition of the Laplacian on $L^2(\boldsymbol{R}^n, dx)$,

$$-\Delta = \int_0^{\infty} \lambda \, dE(\lambda),$$

we have

$$P^{(\alpha)}_{t_0} = \int_0^{\infty} e^{-t_0\lambda}\, dE(\lambda),$$

because the fundamental solution $W^{(\alpha)}$ we use here is defined by the Fourier transform, which is equivalent to the spectral decomposition. Hence, if $f \in L^2(\boldsymbol{R}^n, dx)$ satisfies $P^{(\alpha)}_{t_0} f = 0$, then we have

$$\int_0^{\infty} e^{-t_0\lambda} d\|E(\lambda)f\|^2 = 0.$$

Since $d\|E(\lambda)f\|^2$ is a positive measure on $[0,\infty)$, we see

$$\|f\|^2 = \int_0^{\infty} d\|E(\lambda)f\|^2 = 0,$$

which implies $f = 0$. $\square$

Lemma 2. *Let μ be a signed measure on $\boldsymbol{R}_-^{n+1} := \{(x,t) \in \boldsymbol{R}^n \times \boldsymbol{R} \mid t < 0\}$. Suppose that μ is a finite linear combination of point masses. If $W^{(\alpha)} * \mu = 0$ on $\boldsymbol{R}_+^{n+1}$, then $\mu = 0$, where*

$$W^{(\alpha)} * \mu(X) := \int_{\boldsymbol{R}^{n+1}} W^{(\alpha)}(X - Y)\, d\mu(Y).$$

Proof. Suppose that $\mu \neq 0$ and write $\mu = \sum_{k=1}^N c_k \delta_{(x_k, -t_k)}$ with $c_k \neq 0$ $(k = 1, \dots, N)$. Let $t_0 := \min\{t_k; 1 \le k \le N\} > 0$ and put

$$u_t(x) := W^{(\alpha)} * \mu(x, t - t_0) = \sum_{k=1}^N c_k W^{(\alpha)}(x - x_k, t + t_k - t_0).$$

Then u_t belongs to $L^2(\boldsymbol{R}^n, dx)$ for all $t > 0$ and by (4) and our assumption $W^{(\alpha)} * \mu = 0$, we have

$$P_{t_0} u_t(x) = \int W^{(\alpha)}(x - y, t_0)(W^{(\alpha)} * \mu(y, t - t_0))\, dy = W^{(\alpha)} * \mu(x, t) = 0.$$

Hence Lemma 1 shows $u_t = u(\cdot, t) = 0$ for all $t > 0$. However this contradicts the fact that

$$\lim_{t \to 0} |u(x_j, t)| = |c_j| \lim_{t \to 0} W^{(\alpha)}(0, t) = \infty,$$

where we take j such that $t_j = t_0$. This implies $\mu = 0$. □

Now, we shall show the linear independence of some families related with the fundamental solution, which is a key in the proof of our main theorems.

Proposition 1. *Let $\lambda > -1$. Then the family $(R_{\alpha,\lambda}^X)_{X \in \boldsymbol{R}_+^{n+1}}$ is linearly independent, where $R_{\alpha,\lambda}^X(Y) = R_{\alpha,\lambda}(X, Y)$.*

Proof. In the proof, we write $W_\alpha^X(Y) = W^{(\alpha)}(x - y, t + s)$. Then for every $X \in \boldsymbol{R}_+^{n+1}$, $W_\alpha^X \in \boldsymbol{b}_\alpha^p(\lambda)$ if $p > 2\alpha(\lambda + 1)/n + 1$ (see [5, Theorem 1 (2)]). Hence by (5), we have

$$\iint R_{\alpha,\lambda}^Y(Z) W_\alpha^X(Z)\, dV^\lambda(Z) = W_\alpha^X(Y) = W_\alpha^Y(X)$$

for every $X, Y \in \boldsymbol{R}_+^{n+1}$, so that any finite linear relation $\sum_{k=1}^N c_k R_{\alpha,\lambda}^{X_k}(X) \equiv 0$ implies the relation $\sum_{k=1}^N c_k W_\alpha^{X_k}(X) \equiv 0$. Writing $\mu := \sum_{k=1}^N c_k \delta_{(x_k, -t_k)}$, where $X_k = (x_k, t_k)$, we have

$$W^{(\alpha)} * \mu(X) = \sum_{k=1}^N c_k W_\alpha^{X_k}(X) = 0 \quad \text{on } \boldsymbol{R}_+^{n+1},$$

and hence Lemma 2 gives us $\mu = 0$, which implies $c_1 = c_2 = \cdots = c_N = 0$. □

§4. Proof of Theorem 2

Let $\mu \geq 0$ be a measure on $\boldsymbol{R}_+^{n+1}$ with compact support. Then as in the case that $\lambda = 0$, the Toeplitz operator T_μ^λ is bounded on $\boldsymbol{b}_\alpha^2(\lambda)$ (in fact, it is compact, see [10]). Moreover for every $u, v \in \boldsymbol{b}_\alpha^2(\lambda)$

$$(10) \qquad \langle T_\mu^\lambda u, v\rangle = \iint (T_\mu^\lambda u)(X)v(X)\, dV^\lambda(X) = \iint u(X)v(X)\, d\mu(X).$$

Note that $\mathrm{Dom}(\tilde{T}_\mu^\lambda) = \mathrm{Dom}(T_\mu^\lambda) = \boldsymbol{b}_\alpha^2(\lambda)$ and $\tilde{T}_\mu^\lambda = T_\mu^\lambda$.

Now we return to a proof of Theorem 2. Let $\mathscr{R}_\mu^\lambda := T_\mu^\lambda(\boldsymbol{b}_\alpha^2(\lambda))$ be the range of T_μ^λ and assume that $\dim(\mathscr{R}_\mu^\lambda) < \infty$. Put

$$M := \{X \in \boldsymbol{R}_+^{n+1}; u(X) = 0 \text{ for every } u \in (\mathscr{R}_\mu^\lambda)^\perp\},$$

where $\mathscr{R}^\perp$ is the orthogonal complement of a subset $\mathscr{R}$ in $\boldsymbol{b}_\alpha^2(\lambda)$. If $X \in M$ and $u \in (\mathscr{R}_\mu^\lambda)^\perp$ then by (5),

$$\langle R_{\alpha,\lambda}^X, u\rangle = \iint R_{\alpha,\lambda}(X, Y)u(Y)\, dV^\lambda(Y) = u(X) = 0.$$

This implies that $\{R_{\alpha,\lambda}^X; X \in M\} \subset ((\mathscr{R}_\mu^\lambda)^\perp)^\perp = \mathscr{R}_\mu^\lambda$, and hence Proposition 1 shows $\#M \leq \dim(\mathscr{R}_\mu^\lambda) < \infty$. Moreover, for each $u \in (\mathscr{R}_\mu^\lambda)^\perp$, we have

$$0 \leq \iint u^2(X)\, d\mu(X) = \langle T_u^\lambda u, u\rangle = 0,$$

by (10). This implies $\mu(\{X \in \boldsymbol{R}_+^{n+1}; u(X) \neq 0\}) = 0$, i.e., $\mathrm{supp}(\mu) \subset \{X \in \boldsymbol{R}_+^{n+1}; u(X) = 0\}$. Hence

$$\mathrm{supp}(\mu) \subset \bigcap_{u \in (\mathscr{R}_\mu^\lambda)^\perp} \{X \in \boldsymbol{R}_+^{n+1}; u(X) = 0\} = M,$$

which shows $\#\mathrm{supp}(\mu) \leq \#M \leq \dim(\mathscr{R}_\mu^\lambda) = \mathrm{rank}(T_\mu^\lambda) < \infty$. Since $\mathrm{rank}(T_\mu^\lambda) \leq \#\mathrm{supp}(\mu)$ is trivially true, we complete the proof of Theorem 2. □

§5. Proof of Theorem 1

Let K be an arbitrary compact set in $\boldsymbol{R}_+^{n+1}$, and consider the restricted measure $\mu|_K$ and the corresponding Toeplitz operator $T_{\mu|_K}^\lambda$. Then $T_{\mu|_K}^\lambda$ is a bounded operator on $\boldsymbol{b}_\alpha^2(\lambda)$ and

$$(11) \qquad \langle \tilde{T}_\mu^\lambda u, u\rangle = \int_{\mathbf{R}^n} u(X)^2\, d\mu(X) \geq \int_{\mathbf{R}^n} u(X)^2\, d\mu|_K = \langle T_{\mu|_K}^\lambda u, u\rangle \geq 0$$

for every $u \in \mathrm{Dom}(\tilde{T}_\mu^\lambda)$. Since $\dim(\tilde{T}_\mu^\lambda(\mathscr{D}_0)) < \infty$, $T_{\mu|_K}^\lambda$ is of finite rank and

$$\mathrm{rank}(T_{\mu|_K}^\lambda) \leq \dim(\tilde{T}_\mu^\lambda(\mathscr{D}_0))$$

holds true. To show this, let $u_1, \dots, u_m$ be any finite elements in $\boldsymbol{b}_\alpha^2(\lambda)$ such that their images $T_{\mu|_K}^\lambda u_1, \dots, T_{\mu|_K}^\lambda u_m$ are linearly independent. Denote by $\mathscr{H}$ the linear hull of $\{u_1, \dots, u_m, T_{\mu|_K}^\lambda u_1, \dots, T_{\mu|_K}^\lambda u_m\}$. Let $1_{\mathscr{H}}$ be the projection map from $\boldsymbol{b}_\alpha^2(\lambda)$ onto $\mathscr{H}$. Then $1_{\mathscr{H}} \circ T_{\mu|_K}^\lambda$ gives a symmetric linear map on a finite dimensional Hilbert space $\mathscr{H}$, so that there exist an orthonormal system $w_1, \dots, w_m \in \mathscr{H}$ and real numbers $\lambda_1 \geq \cdots \geq \lambda_m > 0$ such that

$$\langle T_{\mu|_K}^\lambda w_i, w_j \rangle = \lambda_j \delta_{ij},$$

where δ_{ij} stands for the Kronecker delta. Take $0 < \varepsilon < 1/(2m)$ with

$$\varepsilon < \frac{4}{27} \frac{\lambda_m}{\|T_{\mu|_K}^\lambda\|},$$

where $\|T_{\mu|_K}^\lambda\|$ is the operator norm of $T_{\mu|_K}^\lambda : \boldsymbol{b}_\alpha^2(\lambda) \to \boldsymbol{b}_\alpha^2(\lambda)$. Since $\mathscr{D}_0$ is a dense subspace of $\boldsymbol{b}_\alpha^2(\lambda)$, we can choose $\tilde{w}_1, \dots, \tilde{w}_m \in \mathscr{D}_0$ such that

$$\|\tilde{w}_j - w_j\| < \varepsilon, \quad j = 1, \dots, m.$$

Then the family $(\tilde{w}_j)_{j=1}^m$ is also linearly independent, if $\varepsilon > 0$ is small enough (which is easily seen by considering their Grammians). Denoting by H and $\tilde{H}$ the linear hull of $\{w_1, \dots, w_m\}$ and $\{\tilde{w}_1, \dots, \tilde{w}_m\}$, respectively, and considering a natural correspondence of $\tilde{w} = \alpha_1 \tilde{w}_1 + \cdots + \alpha_m \tilde{w}_m$ with $w = \alpha_1 w_1 + \cdots + \alpha_m w_m$ between $\tilde{H}$ and H, we have for any $\tilde{w} \in \tilde{H}$ with $\|\tilde{w}\| = 1$,

$$\frac{2}{3} \leq \frac{1}{1 + m\varepsilon} \leq \|w\| \leq \frac{1}{1 - m\varepsilon} \leq 2,$$

because $\|w - \tilde{w}\| \leq m\|w\|\varepsilon$. Then we also have

$$|\langle T_{\mu|_K}^\lambda \tilde{w}, \tilde{w} \rangle - \langle T_{\mu|_K}^\lambda w, w \rangle| \leq \varepsilon \|T_{\mu|_K}^\lambda\| (\|\tilde{w}\| + \|w\|) \leq 3\varepsilon \|T_{\mu|_K}^\lambda\|,$$

and hence by (11),

$$\begin{aligned} \langle \tilde{T}_\mu^\lambda \tilde{w}, \tilde{w} \rangle &\geq \langle T_{\mu|_K}^\lambda \tilde{w}, \tilde{w} \rangle \geq \langle T_{\mu|_K}^\lambda w, w \rangle - 3\varepsilon \|T_{\mu|_K}^\lambda\| \\ &= \sum_{j=1}^m \alpha_j^2 \lambda_j - 3\varepsilon \|T_{\mu|_K}^\lambda\| \geq \frac{2^2}{3^2} \lambda_m - 3\varepsilon \|T_{\mu|_K}^\lambda\| > 0. \end{aligned}$$

This implies $\dim(\tilde{T}_\mu^\lambda(\mathscr{D}_0)) \geq m$, because $\dim \tilde{H} = m$, and hence $\dim(\tilde{T}_\mu^\lambda(\mathscr{D}_0)) \geq \operatorname{rank}(T_{\mu|_K}^\lambda)$ follows. Since K is arbitrary, Theorem 2 shows that μ is a finite linear combination of point masses and

$$\operatorname{rank}(\tilde{T}_\mu^\lambda) \geq \dim(\tilde{T}_\mu^\lambda(\mathscr{D}_0)) \geq \#\operatorname{supp}(\mu)$$

holds. Since $\#\operatorname{supp}(\mu) \geq \operatorname{rank}(\tilde{T}_\mu^\lambda)$ is trivially true, we have $\operatorname{rank}(\tilde{T}_\mu^\lambda) = \dim(\tilde{T}_\mu^\lambda(\mathscr{D}_0)) = \#\operatorname{supp}(\mu)$. This completes the proof of Theorem 1. $\square$

We close this section by making the following remark.

Remark 1. Let $\lambda > -1$ and $\mu \geq 0$ be a Radon measure on $\boldsymbol{R}_+^{n+1}$. If μ satisfies a growth condition (2) with some constant $\tau > 0$, then $\mathrm{Dom}(\tilde{T}_\mu^\lambda)$ is dense in $\boldsymbol{b}_\alpha^2(\lambda)$ and $\tilde{T}_\mu^\lambda$ is a self-adjoint extension of T_μ^λ. In particular, $\tilde{T}_\mu^\lambda = T_\mu^\lambda$ on $\boldsymbol{b}_\alpha^2(\lambda)$ if T_μ^λ is bounded.

In fact, by (2), (7) and (8), if $\nu > -(\lambda+1)/2$, then $D^\nu(\lambda)$ is included in $L^2(\boldsymbol{R}_+^{n+1}, \mu)$ and dense in $\boldsymbol{b}_\alpha^2(\lambda)$, and hence $\mathscr{D} := \boldsymbol{b}_\alpha^2(\lambda) \cap L^2(\boldsymbol{R}_+^{n+1}, \mu)$ is dense in $\boldsymbol{b}_\alpha^2(\lambda)$. This shows that the domain $\mathrm{Dom}(\tilde{T}_\mu^\lambda)$ is also dense in $\boldsymbol{b}_\alpha^2(\lambda)$. Next, take $u \in \mathrm{Dom}(T_\mu^\lambda)$ arbitrarily. Then by the Fubini theorem, we see that for every $v \in \boldsymbol{b}_\alpha^2(\lambda)$,

$$\begin{aligned}
\langle T_\mu^\lambda u, v\rangle &= \iint (T_\mu^\lambda u)(X) v(X)\, dV^\lambda(X) \\
&= \iint \left(\iint R_{\alpha,\lambda}(X, Y) u(Y)\, d\mu(Y) \right) v(X)\, dV^\lambda(X) \\
&= \iint \left(\iint R_{\alpha,\lambda}(X, Y) v(X)\, dV^\lambda(X) \right) u(Y)\, d\mu(Y) = \iint u(Y) v(Y)\, d\mu(Y).
\end{aligned}$$

This shows that $u \in L^2(\boldsymbol{R}_+^{n+1}, \mu)$, and hence $u \in \mathscr{D}$. Thus, for evry $v \in \mathrm{Dom}(\tilde{T}_\mu^\lambda)$, we have

$$\langle u, \tilde{T}_\mu^\lambda v\rangle = \mathscr{E}(u, v) = \iint uv\, d\mu = \langle T_\mu^\lambda u, v\rangle, \tag{12}$$

which shows $u \in \mathrm{Dom}((\tilde{T}_\mu^\lambda)^*)$ and $T_\mu^\lambda u = (\tilde{T}_\mu^\lambda)^* u$. Since $\tilde{T}_\mu^\lambda$ is self-adjoint, $u \in \mathrm{Dom}(\tilde{T}_\mu^\lambda)$ and $\tilde{T}_\mu^\lambda u = T_\mu^\lambda u$ follows.

Above argument explains that the assumption (2) for symbol measures of Toeplitz operators is very natural in a sense.

§6. Relation to the Carleson inclusion

If a measure $\mu \geq 0$ on $\boldsymbol{R}_+^{n+1}$ satisfies the growth condition (2) for some $\tau > 0$, then the corresponding Carleson inclusion

$$\iota_\mu^\lambda : \boldsymbol{b}_\alpha^2(\lambda) \to L^2(\boldsymbol{R}_+^{n+1}, \mu) : u \mapsto u,$$

whose domain is $\mathrm{Dom}(\iota_\mu^\lambda) := \boldsymbol{b}_\alpha^2(\lambda) \cap L^2(\boldsymbol{R}_+^{n+1}, \mu) = \mathscr{D}$, is densely defined and is a closed operator (see Remark 1). In this section, we discuss some relations between operators $\tilde{T}_\mu^\lambda$ and ι_μ^λ.

Hereafter, for two linear operators T and S on a Hilbert space, we write $T \subset S$ if $\mathrm{Dom}(T) \subset \mathrm{Dom}(S)$ and $T = S$ on $\mathrm{Dom}(T)$ hold. Then we have

PROPOSITION 2. *Let $\lambda > -1$. If a measure $\mu \geq 0$ satisfies* (2) *for some $\tau > 0$, then $\tilde{T}_\mu^\lambda = (\iota_\mu^\lambda)^* \iota_\mu^\lambda$ holds.*

Proof. We remark that $\mathrm{Dom}((\iota_\mu^\lambda)^*\iota_\mu^\lambda) = \{u \in \boldsymbol{b}_\alpha^2(\lambda); u \in \mathrm{Dom}(\iota_\mu^\lambda), \iota_\mu^\lambda u \in \mathrm{Dom}((\iota_\mu^\lambda)^*)\}$. Now we take $u \in \mathrm{Dom}(\tilde{T}_\mu^\lambda)$ arbitrarily. Then by (12), for every $v \in \mathrm{Dom}(\iota_\mu^\lambda)$, we have

$$\langle \tilde{T}_\mu^\lambda u, v\rangle = \langle \iota_\mu^\lambda u, \iota_\mu^\lambda v\rangle_{L^2(\boldsymbol{R}_+^{n+1},\mu)} \left(= \iint u(X)v(X)\, d\mu(X)\right), \tag{13}$$

which implies $\iota_\mu^\lambda u \in \mathrm{Dom}((\iota_\mu^\lambda)^*)$ and $(\iota_\mu^\lambda)^*\iota_\mu^\lambda u = \tilde{T}_\mu^\lambda u$, i.e., $\tilde{T}_\mu^\lambda \subset (\iota_\mu^\lambda)^*\iota_\mu^\lambda$ holds. Next, since $(\iota_\mu^\lambda)^*\iota_\mu^\lambda$ is clearly symmetric and $\tilde{T}_\mu^\lambda$ is self-adjoint, we have

$$\tilde{T}_\mu^\lambda = (\tilde{T}_\mu^\lambda)^* \supset ((\iota_\mu^\lambda)^*\iota_\mu^\lambda)^* \supset (\iota_\mu^\lambda)^*\iota_\mu^\lambda,$$

which shows the proposition. □

If the Carleson inclusion ι_μ^λ is bounded on $\boldsymbol{b}_\alpha^2(\lambda)$, then the corresponding Toeplitz operator is bounded. More precisely, we have

Proposition 3. *Let $\lambda > -1$ and μ be a positive Radon measure on $\boldsymbol{R}_+^{n+1}$. If ι_μ^λ is bounded, then the measure $\mu \ge 0$ satisfies the growth condition* (2) *with $\tau > (n/2\alpha + 1) + \lambda$ and $\tilde{T}_\mu^\lambda$ is bounded. Moreover, $\|\tilde{T}_\mu^\lambda\| \le \|\iota_\mu^\lambda\|^2$ and $\tilde{T}_\mu^\lambda = T_\mu^\lambda$ holds on $\boldsymbol{b}_\alpha^2(\lambda)$.*

Proof. We assume that ι_μ^λ is bounded. Then as in the proof of Proposition 1 in [9], we see $\mu(Q^\alpha(X)) \le CV^\lambda(Q^\alpha(X))$ for all $X \in \boldsymbol{R}_+^{n+1}$ with some constant $C > 0$ (use also [5, Proposition 3.2]), where $Q^\alpha(X)$ is the α-parabolic Carleson box centered at $X \in \boldsymbol{R}_+^{n+1}$. By a similar argument to [9, Proposition 2], we have

$$\iint \frac{1}{(1 + t + |x|^{2\alpha})^\tau}\, d\mu(X) \le C \iint \frac{t^\lambda}{(1 + t + |x|^{2\alpha})^\tau}\, dxdt.$$

Hence if we take $\tau > (n/2\alpha + 1) + \lambda$, then μ satisfies (2). Thus we can use Proposition 2, which gives $\|\tilde{T}_\mu^\lambda\| \le \|\iota_\mu^\lambda\|^2$. Moreover, by (13), we have

$$\tilde{T}_\mu^\lambda u(X) = \langle \tilde{T}_\mu^\lambda u, R_{\alpha,\lambda}^X\rangle = \iint R_{\alpha,\lambda}(X, Y)u(Y)\, d\mu(Y) = T_\mu^\lambda u(X).$$

This completes the proof. □

Conversely, we have

Proposition 4. *Let $\lambda > -1$ and $\mu \ge 0$ satisfy the growth condition* (2) *for some $\tau > 0$. If $\tilde{T}_\mu^\lambda$ is bounded, then ι_μ^λ is bounded and $\|\iota_\mu^\lambda\| \le \sqrt{\|\tilde{T}_\mu^\lambda\|}$.*

Proof. We assume that $\tilde{T}_\mu^\lambda$ is bounded. Then $\mathrm{Dom}(\tilde{T}_\mu^\lambda) = \boldsymbol{b}_\alpha^2(\lambda)$ so that $L^2(\boldsymbol{R}_+^{n+1}, \mu) \subset \boldsymbol{b}_\alpha^2(\lambda)$. Hence by (12),

$$\|u\|_{L^2(\boldsymbol{R}_+^{n+1},\mu)}^2 = \langle \tilde{T}_\mu^\lambda u, u\rangle \le \|\tilde{T}_\mu^\lambda u\|_{\boldsymbol{b}_\alpha^2(\lambda)} \cdot \|u\|_{\boldsymbol{b}_\alpha^2(\lambda)} \le \|\tilde{T}_\mu^\lambda\| \cdot \|u\|_{\boldsymbol{b}_\alpha^2(\lambda)}^2.$$

This shows the proposition. □

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