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A CRITERION FOR HOLOMORPHIC EXTENSION OF PRINCIPAL BUNDLES

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Abstract

Let G be a complex affine algebraic group, and let E_G be a holomorphic principal G-bundle on the complement $M \backslash S$, where S is a closed complex analytic subset, of complex codimension at least two, of a connected complex manifold M. We give a criterion for E_G to extend to M as a holomorphic principal G-bundle. Two applications of this criterion are given.

1. Introduction

Let M be a connected complex manifold. Let $S \subset M$ be a closed complex analytic subset such that the complex codimension of S is at least two. Define $U := M \backslash S \subset M$.

We prove the following theorem (see Theorem 3.1):

Theorem 1.1. *Let G be a complex affine algebraic group and $E_G \to U$ a holomorphic principal G-bundle. If E_G admits a holomorphic connection, then it extends uniquely to M as a holomorphic principal G-bundle.*

For vector bundles, Theorem 1.1 was proved by Buchdahl and Harris [3].

Now let G be a complex reductive affine algebraic group. Fix a connected maximal compact subgroup $K \subset G$.

If $E_G \to U$ is a holomorphic principal G-bundle, and $E_K \subset E_G$ is a C^∞ reduction of structure group to K, then there is a unique C^∞ connection ∇^G on E_G which preserves E_K and is compatible with the holomorphic structure of E_G. The curvature $\mathscr{K}(\nabla^G)$ of ∇^G is a smooth section of $\Omega_U^{1,1} \otimes \mathrm{ad}(E_K)$, where $\mathrm{ad}(E_K)$ is the adjoint vector bundle of E_K.

Fix a K-invariant inner product on $\mathrm{Lie}(G)$. Fix a Hermitian structure on M. These choices enable us to define L^p-norms on the smooth sections of $\Omega_U^{1,1} \otimes \mathrm{ad}(E_K)$.

We prove the following (see Theorem 3.2):

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Theorem 1.2. *Let E_G be a holomorphic principal G-bundle over U and*

$$E_K \subset E_G$$

a C^∞ reduction of structure group of E_G to the subgroup $K \subset G$. If the curvature of the natural connection ∇^G on E_G has finite L^n-norm, where n is the complex dimension of M, then E_G extends uniquely to a holomorphic principal G-bundle over M.

For vector bundles, Theorem 1.2 was proved by Harris and Tonegawa [4].

In Proposition 2.3 we prove a criterion for a holomorphic principal G-bundle on U to extend to M as a holomorphic principal G-bundle. Both Theorem 1.1 and Theorem 1.2 are proved using this criterion.

2. Criterion for extension

Let G be a complex affine algebraic group. A holomorphic principal G-bundle over a complex manifold Y is a holomorphic fiber bundle $\phi : E_G \to Y$ equipped with a holomorphic right action of G

$$\psi : E_G \times G \to E_G$$

such that

- $\phi \circ \psi = \phi \circ p_1$, where p_1 is the natural projection of $E_G \times G$ to E_G, and
- the map to the fiber product $p_1 \times \psi : E_G \times G \to E_G \times_Y E_G$ is a holomorphic isomorphism.

(We recall that $E_G \times_Y E_G$ is the submanifold of $E_G \times E_G$ consisting of all points (z_1, z_2) such that $\phi(z_1) = \phi(z_2)$.)

Fix an algebraic embedding

$$\rho : G \hookrightarrow \mathrm{GL}(V), \tag{2.1}$$

where V is a finite dimensional complex vector space (since G is an affine algebraic group, a faithful G-module exists).

A theorem of Chevalley says that there is a finite dimensional complex vector space W, a complex line

$$\ell \subset W$$

and an algebraic homomorphism

$$\eta : \mathrm{GL}(V) \to \mathrm{GL}(W) \tag{2.2}$$

such that

$$\rho(G) = \{T \in \mathrm{GL}(V) \mid \eta(T)(\ell) = \ell\}, \tag{2.3}$$

where ρ is the homomorphism in (2.1). (See [5, p. 80, Theorem 11.2].) Fix such a triple (W, ℓ, η).

Let M be a connected complex manifold and

$$U \subset M \tag{2.4}$$

a dense open subset. Let

$$\phi : E_G \to U \tag{2.5}$$

be a holomorphic principal G-bundle. Let

$$E_V := E_G \times^{\rho} V \quad (\text{respectively, } E_W := E_G \times^{\eta\circ\rho} W) \tag{2.6}$$

be the holomorphic vector bundle over U associated to E_G for the G-module V (respectively, W) (see (2.1) and (2.2)). Therefore, by definition, E_V (respectively, E_W) is the quotient of $E_G \times V$ (respectively, $E_G \times W$) where two points (z_1, v_1) and (z_2, v_2) of $E_G \times V$ (respectively, $E_G \times W$) are identified if and only if there is an element $g \in G$ such that $(z_2, v_2) = (z_1 g, \rho(g^{-1})(v_1))$ (respectively, $(z_2, v_2) = (z_1 g, (\eta \circ \rho)(g^{-1})(v_1))$).

Let

$$E_\ell := E_G \times^{\eta\circ\rho} \ell \subset E_W \tag{2.7}$$

be the holomorphic line subbundle associated to E_G for the G-module ℓ in (2.3).

Let

$$E_{\mathrm{GL}(V)} \to U \tag{2.8}$$

be the holomorphic principal $\mathrm{GL}(V)$-bundle defined by the holomorphic vector bundle E_V in (2.6). So, $E_{\mathrm{GL}(V)}$ parametrizes all linear isomorphisms from V to the fibers of E_V. Note that $E_{\mathrm{GL}(V)}$ is a quotient of $E_G \times \mathrm{GL}(V)$; the quotient map sends any $(g, A) \in (E_G)_x \times \mathrm{GL}(V)$ to the isomorphism $V \to (E_V)_x$ that maps any v to the equivalence class of $(g, A(v))$. This also shows that we have a holomorphic embedding

$$\iota : E_G \to E_{\mathrm{GL}(V)} \tag{2.9}$$

that sends any $z \in E_G$ to the equivalence class of (z, Id_V).

The homomorphism η in (2.2) makes W a $\mathrm{GL}(V)$-module. Let

$$E'_W := E_{\mathrm{GL}(V)} \times^{\mathrm{GL}(V)} W \to U \tag{2.10}$$

be the holomorphic vector bundle associated to the principal $\mathrm{GL}(V)$-bundle $E_{\mathrm{GL}(V)}$ in (2.8) for the $\mathrm{GL}(V)$-module W. So, by definition, E'_W is the quotient of $E_{\mathrm{GL}(V)} \times W$ where two points (z_1, w_1) and (z_2, w_2) are identified if there is an element $A \in \mathrm{GL}(V)$ such that $(z_2, w_2) = (z_1 g, \eta(A^{-1})(w_1))$.

Since E_V and E_W are associated to the principal G-bundle E_G for the G-modules V and W respectively, we conclude that there is a natural holomorphic isomorphism

$$E'_W \xrightarrow{\sim} E_W, \tag{2.11}$$

where E'_W and E_W are constructed in (2.10) and (2.6) respectively. The isomorphism in (2.11) sends the equivalence class of $(z, A, w) \in E_G \times \mathrm{GL}(V) \times W$

to the equivalence class of $(z, \eta(A)(w))$ (recall that $E_{\mathrm{GL}(V)}$ is a quotient of $E_G \times \mathrm{GL}(V)$, while E'_W and E_W are quotients of $E_{\mathrm{GL}(V)} \times W$ and $E_G \times W$ respectively).

Assumption 2.1. Assume that the following two conditions hold:
(1) the holomorphic vector bundles E_V and E_W in (2.6) extend as holomorphic vector bundles $\overline{E}_V$ and $\overline{E}_W$ respectively to M, and
(2) the holomorphic line subbundle E_ℓ in (2.7) extends to M as a holomorphic line subbundle of $\overline{E}_W$.

Let $\overline{E}_{\mathrm{GL}(V)} \to M$ be the holomorphic principal $\mathrm{GL}(V)$-bundle defined by the holomorphic vector bundle $\overline{E}_V$ in Assumption 2.1(1). So, $\overline{E}_{\mathrm{GL}(V)}$ parametrizes all linear isomorphisms from V to the fibers of $\overline{E}_V$. Let

$$\overline{E}'_W := \overline{E}_{\mathrm{GL}(V)} \times^{\mathrm{GL}(V)} W \to M \tag{2.12}$$

be the holomorphic vector bundle associated to the principal $\mathrm{GL}(V)$-bundle $\overline{E}_{\mathrm{GL}(V)}$ for the $\mathrm{GL}(V)$-module W in (2.2).

Assumption 2.2. Assume that the isomorphism in (2.11) on U extends to a holomorphic isomorphism of vector bundles over M

$$\theta : \overline{E}'_W \xrightarrow{\sim} \overline{E}_W, \tag{2.13}$$

where $\overline{E}'_W$ is constructed in (2.12), and $\overline{E}_W$ is the extension in Assumption 2.1(1).

Proposition 2.3. *The holomorphic principal G-bundle $E_G \to U$ extends to a holomorphic principal G-bundle over M.*

Proof. Let $\overline{E}_{\mathrm{GL}(W)} \to M$ be the holomorphic principal $\mathrm{GL}(W)$-bundle defined by the holomorphic vector bundle $\overline{E}_W$. So, $\overline{E}_{\mathrm{GL}(W)}$ parametrizes all linear isomorphisms from W to the fibers of $\overline{E}_W$. Using the isomorphism θ in (2.13), a holomorphic map of fiber bundles

$$\varphi : \overline{E}_{\mathrm{GL}(V)} \to \overline{E}_{\mathrm{GL}(W)} \tag{2.14}$$

can be constructed as follows. Recall that for any $x \in M$, an element of the fiber $(\overline{E}_{\mathrm{GL}(V)})_x$ is a linear isomorphism $V \to (\overline{E}_V)_x$. Given any linear isomorphism

$$\alpha : V \to (\overline{E}_V)_x,$$

we have an isomorphism $\tilde{\alpha} : W \to (\overline{E}'_W)_x$ that sends any w to the equivalence class of (α, w). The map φ in (2.14) sends α to the element of $(\overline{E}_{\mathrm{GL}(W)})_x$ corresponding to the isomorphism

$$\theta(x) \circ \tilde{\alpha} : W \to (\overline{E}_W)_x.$$

For any $z \in \overline{E}_{\mathrm{GL}(V)}$ and $A \in \mathrm{GL}(V)$, clearly, $\varphi(zA) = \varphi(z)\eta(A)$, where η is the homomorphism in (2.2).

Let

$$P := \{A \in \mathrm{GL}(W) \mid A(\ell) = \ell\} \subset \mathrm{GL}(W) \tag{2.15}$$

be the maximal parabolic subgroup, where ℓ is the line in (2.3). Let

$$\overline{E}_\ell \subset \overline{E}_{\mathrm{GL}(W)}$$

be the holomorphic line subbundle over M obtained by extending the holomorphic line subbundle E_ℓ in (2.7) (see Assumption 2.1(2)). This line subbundle $\overline{E}_\ell$ gives a reduction of structure group of the holomorphic principal $\mathrm{GL}(W)$-bundle $\overline{E}_{\mathrm{GL}(W)}$

$$\overline{E}_P \subset \overline{E}_{\mathrm{GL}(W)} \tag{2.16}$$

to the subgroup P defined in (2.15). This reduction is uniquely determined by the condition that for any point $x \in M$, the submanifold

$$(\overline{E}_P)_x \subset (\overline{E}_{\mathrm{GL}(W)})_x$$

is the space of all linear isomorphisms $A : W \to (\overline{E}_W)_x$ such that $A(\ell) = (\overline{E}_\ell)_x$.

Finally, define

$$\overline{E}_G := \{z \in \overline{E}_{\mathrm{GL}(V)} \mid \varphi(z) \in \overline{E}_P\}, \tag{2.17}$$

where φ and $\overline{E}_P$ are constructed in (2.14) and (2.16) respectively. Let

$$\gamma : \overline{E}_G \to M$$

be the restriction of the natural projection of $\overline{E}_{\mathrm{GL}(V)}$ to M. For any $x \in U$, it is straight forward to check that $\gamma^{-1}(x)$ is identified with the fiber $(E_G)_x \subset (E_{\mathrm{GL}(V)})_x$; the fiber $(E_G)_x$ is identified as a submanifold of $(E_{\mathrm{GL}(V)})_x$ using ι in (2.9). For the action of the group $\mathrm{GL}(V)$ on $\overline{E}_{\mathrm{GL}(V)}$, the subgroup G clearly preserves $\overline{E}_G \subset \overline{E}_{\mathrm{GL}(V)}$, and

$$\gamma : \overline{E}_G \to M$$

is a holomorphic principal G-bundle. This completes the proof of the proposition. $\square$

3. Some applications

3.1. Holomorphic connections and extensions. Let M be a connected complex manifold. Let

$$S \subset M \tag{3.1}$$

be a closed complex analytic subset such that the complex codimension of S is at least two. Define

$$U := M \setminus S. \tag{3.2}$$

Let G be as before. See [2] for holomorphic connections on holomorphic principal G-bundles.

THEOREM 3.1. *Let $E_G \to U$ be a holomorphic principal G-bundle equipped with a holomorphic connection ∇. Then E_G extends uniquely to a holomorphic principal G-bundle over M.*

Proof. Fix ρ and η as in (2.1) and (2.2) respectively. Define the vector bundles E_V and E_W as in (2.6). The holomorphic connection ∇ induces a holomorphic connection on any fiber bundle associated to E_G. Let ∇^V and ∇^W be the holomorphic connections on E_V and E_W respectively induced by ∇. The vector bundles E_V and E_W extends uniquely to holomorphic vector bundles on M (see [3, p. 38, Corollary 2.4]). Let $\overline{E}_V$ (respectively, $\overline{E}_W$) be the holomorphic vector bundle on M obtained by extending E_V (respectively, E_W).

Consider the holomorphic line subbundle $E_\ell \subset E_W$ in (2.7). Since ℓ is a submodule of the G-module W, the induced holomorphic connection ∇^W on E_W preserves the line subbundle E_ℓ. Consequently, E_ℓ extends uniquely to a holomorphic line subbundle $\overline{E}_\ell$ of $\overline{E}_W$. Therefore, Assumption (2.1) is satisfied.

From the uniqueness of the extensions $\overline{E}_V$ and $\overline{E}_W$ it follows that the isomorphism in (2.11) extends as in (2.13) to a holomorphic isomorphism of vector bundles over M. Hence, Assumption (2.2) is also satisfied.

Therefore, from Proposition 2.3 we conclude that E_G extends to a holomorphic principal G-bundle over M. That this extension is unique follows from the uniqueness of the extensions of E_V, E_W and E_ℓ. $\square$

3.2. Bound on curvature. Let G be a reductive linear algebraic group defined over $\mathbf{C}$. Fix a maximal compact connected subgroup

$$K \subset G. \tag{3.3}$$

Let $\mathfrak{g}$ be the Lie algebra of G. The group G has the adjoint action on $\mathfrak{g}$. Fix a positive Hermitian form h on $\mathfrak{g}$ fixed by the adjoint action of K.

Let $E_G \to Y$ be a holomorphic principal G-bundle over a complex manifold Y. Let

$$E_K \subset E_G$$

be a C^∞ reduction of structure group of E_G to the subgroup K in (3.3). Then there is a unique connection ∇^K of the principal K-bundle E_K such that the connection ∇^G on E_G induced by ∇^K is compatible with the holomorphic structure of E_G; see [1, p. 220, Definition 3.1].

Let $\mathrm{ad}(E_G) := E_G \times^G \mathfrak{g}$ be the adjoint vector bundle of E_G, in other words, $\mathrm{ad}(E_G)$ is the holomorphic vector bundle over Y associated to E_G for the adjoint action of G on $\mathfrak{g}$. Let

$$\mathrm{ad}(E_K) := E_K \times^K \mathrm{Lie}(K) \subset \mathrm{ad}(E_G)$$

be the adjoint vector bundle of E_K. The K-invariant Hermitian form h on $\mathfrak{g}$ defines a Hermitian structure on $\mathrm{ad}(E_G)$. The curvature $\mathscr{K}(\nabla^G)$ of the connection ∇^G is a C^∞ section of $\Omega_Y^{1,1} \otimes \mathrm{ad}(E_K)$ over U. If we fix a Hermitian structure on Y, then combining it together with the above Hermitian structure on $\mathrm{ad}(E_G)$ we can define p-norm, $p > 0$, on the smooth sections of $\Omega_Y^{1,1} \otimes \mathrm{ad}(E_G)$ (see [4, p. 30]).

Let M be a connected complex manifold of complex dimension n. As in (3.1), let S be a closed complex analytic subset of complex codimension at least two. Define U as in (3.2).

THEOREM 3.2. *Let E_G be a holomorphic principal G-bundle over U and*

$$E_K \subset E_G$$

a C^∞ reduction of structure group of E_G to the subgroup $K \subset G$. Assume that the curvature of the natural connection ∇^G has finite L^n-norm. Then E_G extends uniquely to a holomorphic principal G-bundle over M.

Proof. Consider the G-modules V and W in (2.1) and (2.2) respectively. Fix K-invariant Hermitian structures h_V and h_W on V and W respectively. Since h_V (respectively, h_W) is K-invariant, it induces a Hermitian structure on the associated vector bundle E_V (respectively, E_W) in (2.6).

Consider the unique connection ∇^G on E_G associated to the reduction $E_K \subset E_G$. Let ∇^V and ∇^W be the holomorphic connections on E_V and E_W respectively induced by ∇^G. Note that ∇^V (respectively, ∇^W) coincides with the unique Hermitian connection on the holomorphic vector bundle E_V (respectively, E_W).

Since the curvature of the connection ∇^G has a finite L^n-norm, it follows immediately that the curvatures of the induced connections ∇^V and ∇^W are also of finite L^n-norm. Therefore, E_V (respectively, E_W) extends uniquely to a holomorphic vector bundle $\overline{E}_V$ (respectively, $\overline{E}_W$) over M (see [4, p. 29, Theorem 1]).

The line subbundle $E_\ell \subset E_W$ in (2.7) is preserved by the connection ∇^W. Therefore, E_ℓ extends uniquely to a holomorphic line subbundle of $\overline{E}_W$. Hence the theorem follows from Proposition 2.3. □

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