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KŌDAI MATH. SEM. REP.
28 (1976), 85—89

KAEHLERIAN MANIFOLDS WITH VANISHING BOCHNER CURVATURE TENSOR

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§1. Introduction.

Let (M, F, g) be a Kaehlerian manifold of real dimension n with almost complex structure F and Kaehlerian metric g. We cover M by a system of coordinate neighborhoods $\{U; x^h\}$, where here and in the sequel the indices $h, i, j, k \cdots$ run over the range $\{1, 2, \cdots, n\}$ and denote by g_{ji}, ∇_i, $K_{kji}{}^h$, K_{ji}, K and $F_j{}^i$ local components of g, the operator of covariant differentiation with respect to the Levi-Civita connection, the curvature tensor, the Ricci tensor, the scalar curvature and F of M respectively.

The Bochner curvature tensor of M is defined to be [6]

$$B_{kji}{}^h = K_{kji}{}^h + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k{}^h g_{ji} - L_j{}^h g_{ki}$$
$$+ F_k{}^h M_{ji} - F_j{}^h M_{ki} + M_k{}^h F_{ji} - M_j{}^h F_{ki} - 2(M_{kj} F_i{}^h + F_{kj} M_i{}^h),$$

where

$$L_{ji} = -\frac{1}{n+4} K_{ji} + \frac{1}{2(n+2)(n+4)} K g_{ji}, \quad L_k{}^h = L_{kt} g^{th},$$

$$M_{ji} = -L_{jt} F_i{}^t = -\frac{1}{n+4} H_{ji} + \frac{1}{2(n+2)(n+4)} K F_{ji}, \quad M_k{}^h = M_{kt} g^{th},$$
$$H_{ji} = -K_{jt} F_i{}^t.$$

It is known that a Kaehlerian manifold with vanishing Bochner curvature tensor is a complex analogue to a conformally flat Riemannian manifold and that the Bochner curvature tensor has properties quite similar to those of Weyl conformal curvature tensor.

Recently, S. I. Goldbreg [1] proved

THEOREM A. *Let M be an n-dimensional $(n \geqq 3)$ compact conformally flat Riemannian manifold with constant scalar curvature. If the length of the Ricci tensor is less than $K/\sqrt{n-1}$, then M is a space of constant curvature.*

Also, S. I. Goldberg and M. Okumura [2] proved

THEOREM B. *Let M be an n-dimensional $(n \geqq 3)$ compact conformally flat*

Received July 7, 1975.

Riemannian manifold. If the length of the Ricci tensor is constant and less than $K/\sqrt{n-1}$, *then* M *is a space of constant curvature.*

The purpose of the present paper is to prove the following theorems corresponding to those of Goldberg-Okumura, replacing the vanishing of the Weyl conformal curvature tensor of a Riemannian manifold by that of the Bochner curvature tensor of a Kaehlerian manifold.

THEOREM 1. *Let* M *be a Kaehlerian manifold of real dimension* n $(n\geqq 4)$ *with constant scalar curvature whose Bochner curvature tensor vanishes. If the length of the Ricci tensor is not greater than* $K/\sqrt{n-2}$, *then* M *is a space of constant holomorphic sectional curvature.*

THEOREM 2. *Let* M *be a Kaehlerian manifold of real dimension* n $(n\geqq 4)$ *whose Bochner curvature tensor vanishes. If the length of the Ricci tensor is constant and not greater than* $K/\sqrt{n-2}$, *then* M *is a space of constant holomorphic sectional curvature.*

§ 2. Preliminaries.

In a Kaehlerian manifold, we have

$$F_j{}^t F_i{}^s g_{ts} = g_{ji}, \qquad \nabla_j F_i{}^h = 0, \qquad F_j{}^t F_i{}^s K_{ts} = K_{ji}.$$

Under the assumption that the Bochner curvature tensor vanishes, we can prove [7]

$$K_{kji}{}^h K_h{}^k K^{ji} = \frac{1}{n+4}\left[4K_t{}^s K_s{}^r K_r{}^t + \frac{2(n+1)}{n+2} K K_{ji} K^{ji} - \frac{1}{n+2} K^3\right]. \tag{2.1}$$

M. Matsumoto [3] proved

LEMMA A. *If a Kaehlerian space with vanishing Bochner curvature tensor has constant scalar curvature, then its Ricci tensor is parallel.*

The Ricci formula for Ricci tensor K_{ji} is given by

$$\nabla_l \nabla_k K_{ji} - \nabla_k \nabla_l K_{ji} = -K_{lkj}{}^r K_{ri} - K_{lki}{}^r K_{jr}. \tag{2.2}$$

If M is a Kaehlerian manifold with constant scalar curvature whose Bochner curvature tensor vanishes, we get, from Lemma A and (2.2),

$$K_{lkj}{}^r K_{ri} + K_{lki}{}^r K_{jr} = 0. \tag{2.3}$$

Transvecting (2.3) with $g^{ki} K^{jl}$, we find

$$-K_{klj}{}^r K_r{}^k K^{jl} + K_l{}^r K_{jr} K^{jl} = 0. \tag{2.4}$$

Hence, using (2.1) we get, from (2.4),

$$n K_t{}^s K_s{}^r K_r{}^t - \frac{2(n+1)}{n+2} K K_{ji} K^{ji} + \frac{1}{n+2} K^3 = 0. \tag{2.5}$$

K. Yano and S. Ishihara [7] prove

LEMMA B. *In a Riemannian manifold of dimension n, for Q defined by*

$$Q = nK_t{}^s K_s{}^r K_r{}^t - \frac{2(n+1)}{n+2} K K_{ji} K^{ji} + \frac{1}{n+2} K^3,$$

we have

$$Q = P + \frac{3n}{(n-1)(n+2)} K\Big(K_{ji} - \frac{K}{n} g_{ji}\Big)\Big(K^{ji} - \frac{K}{n} g^{ji}\Big),$$

where,

$$P = nK_t{}^s K_s{}^r K_r{}^t - \frac{2n-1}{n-1} K K_{ji} K^{ji} + \frac{1}{n-1} K^3.$$

S. Tachibana [6] proved

LEMMA C. *If the Bochner curvature tensor of a Kaehlerian manifold vanishes and the manifold is an Einstein manifold, then the Kaehlerian manifold is of constant holomorphic sectional curvature.*

On the other hand, by a straightforward computation, we obtain

$$\begin{aligned} \frac{1}{2}\Delta(K_{ji}K^{ji}) &= (\nabla_h K_{ji})(\nabla^h K^{ji}) + \frac{1}{2} K^{ji}\nabla_j \nabla_i K + K^{ji} K_{rj} K_i{}^r \\ &\quad - K_{kjih} K^{kh} K^{ji} + K^{ji} \nabla^r (\nabla_r K_{ji} - \nabla_i K_{rj}). \end{aligned} \tag{2.6}$$

In a Kaehlerian manifold M, put

$$\begin{aligned} B_{kji} = \nabla_k K_{ji} - \nabla_j K_{ki} + \frac{1}{2(n+2)} (g_{ki}\delta_j^a - g_{ji}\delta_k^a + F_{ki}F_j{}^a - F_{ji}F_k{}^a \\ + 2F_{kj}F_i{}^a)\nabla_a K. \end{aligned} \tag{2.7}$$

Then, by a straightforward computation, we find [6]

$$\nabla_a B_{kji}{}^a = \frac{n}{n+4} B_{kji}. \tag{2.8}$$

Using (2.7), (2.8) and the assumption that the Bochner curvature tensor vanishes, we have, from (2.6),

$$\frac{1}{2}\Delta(K_{ji}K^{ji}) = Q + (\nabla_h K_{ji})(\nabla^h K^{ji}) + \frac{1}{2(n+2)} \{(n+4)K^{ji} + Kg^{ji}\}\nabla_j \nabla_i K. \tag{2.9}$$

M. Okumura proved

LEMMA D. [5] *Let a_i, $i=1, 2, \cdots, n$ be n real numbers satisfying*

$$\sum_{i=1}^n a_i = 0 \quad \text{and} \quad \sum_{i=1}^n a_i{}^2 = k^2,$$

for a certain k. Then we have

$$-\frac{n-2}{\sqrt{n(n-1)}} k^3 \leqq \sum_{i=1}^n a_i{}^3 \leqq \frac{n-2}{\sqrt{n(n-1)}} k^3.$$

LEMMA E. [4] *If a given set of* $n+1$ $(n \geqq 2)$ *real numbers* $a_1, \cdots, a_n$ *and* k *satisfies the inequality*

$$\sum_{i=1}^{n} a_i^2 + k < \frac{1}{n-1} \left(\sum_{i=1}^{n} a_i \right)^2$$

then, for any pair of distinct i *and* j $(i, j=1, 2, \cdots, n)$, *we have*

$$k < 2a_i a_j .$$

§ 3. Proof of Theorem 1.

We put

$$S_{ji} = K_{ji} - \frac{K}{n} g_{ji} ,$$

then we have $S_j{}^i F_i{}^l = F_j{}^i S_i{}^l$, since $K_j{}^i F_i{}^l = F_j{}^i K_i{}^l$. Moreover we see that

$$\begin{aligned}
&\operatorname{trace} S = S_i{}^i = 0 , \\
&\operatorname{trace} S^2 = S_{ji} S^{ji} = K_{ji} K^{ji} - \frac{K}{n} \geqq 0 , \\
&\operatorname{trace} S^3 = S_j{}^i S_i{}^h S_h{}^j = K_j{}^i K_i{}^h K_h{}^j - \frac{3}{n} K S_{ji} S^{ji} - \frac{K^3}{n^2} .
\end{aligned}$$

In the second inequality, the equality holds if and only if M is an Einstein space. We put trace $S^2 = f^2$. From the comutativity of $S_j{}^i$ and $F_j{}^i$, we can see that every characteristic root of $S_j{}^i$ is multiple one. Combining this fact with Lemma D, we have

LEMMA 1. *Put* $S_{ji} = K_{ji} - \frac{K}{n} g_{ji}$, $f^2 = S_{ji} S^{ji}$ *and let* a_i, $i=1, 2, \cdots, n$ *be eigenvalues of* $S_j{}^i$. *Then we have*

$$-\frac{n-4}{\sqrt{2n(n-2)}} f^3 \leqq \sum_{i=1}^{n} a_i^3 \leqq \frac{n-4}{\sqrt{2n(n-2)}} f^3 .$$

Using Lemma 1, we have

$$\begin{aligned}
P &= nK_t{}^s K_s{}^r K_r{}^t - \frac{2n-1}{n-1} K K_{ji} K^{ji} + \frac{1}{n-1} K^3 \\
&= n\Big(S_t{}^s S_s{}^r S_r{}^t + \frac{3}{n} K S_{ji} S^{ji} + \frac{K^3}{n^2} \Big) - \frac{2n-1}{n-1} K\Big(S_{ji} S^{ji} + \frac{K^2}{n} \Big) + \frac{1}{n-1} K^3 \\
&= n S_t{}^s S_s{}^r S_r{}^t + \frac{n-2}{n-1} K S_{ji} S^{ji} \\
&\geqq -\frac{n(n-4)}{\sqrt{2n(n-2)}} f^3 + \frac{n-2}{n-1} K f^2 = f^2 \Big\{ \frac{n-2}{n-1} K - \frac{n(n-4)}{\sqrt{2n(n-2)}} f \Big\} .
\end{aligned}$$

By the assumption of Theorem 1, that is, $K_{ji} K^{ji} \leqq \frac{K^2}{n-2}$, if $n \geqq 4$, we see that

$P\geqq 0$. This, together with (2.5) and Lemma B, implies $P=0$ and

$$K\Big(K_{ji}-\frac{K}{n}g_{ji}\Big)\Big(K^{ji}-\frac{K}{n}g^{ji}\Big)=0\,.$$

So, if $K\neq 0$, we have

$$K_{ji}=\frac{K}{n}g_{ji}$$

that is, M is an Einstein manifold. From Lemma C, we conclude that the Theorem 1 holds if $K\neq 0$. If $K=0$, since $K_{ji}K^{ji}=S_{ji}S^{ji}=0$, we have $K_{ji}=0$ and consequently $K_{kji}{}^{h}=0$, that is, the Kaehlerian manifold M is of zero curvature. Thus Theorem 1 has been proved.

§4. Proof of Theorem 2.

Under the assumption of Theorem 2, that is, $K_{ji}K^{ji}=$constant, we have, from (2.9),

$$\frac{1}{2(n+2)}\{(n+4)K_{ji}+Kg_{ji}\}\nabla^{j}\nabla^{i}K=-(Q+\nabla_{h}K_{ji}\nabla^{h}K^{ji})\,. \tag{4.1}$$

Let a_{i} $(i=1,\cdots,n)$ be the eigenvalues of $K_{j}{}^{i}$. From the commutativity $K_{j}{}^{i}$ and $F_{j}{}^{i}$, we see that every characteristic root of $K_{j}{}^{i}$ is multiple one. Therefore, the assumption $K_{ji}K^{ji}=\text{constant}\leqq\frac{K^{2}}{n-2}$ implies the inequality $\sum_{\lambda=1}^{m}a_{\lambda}<\frac{1}{m-1}(\sum_{\lambda=1}^{m}a_{\lambda})^{2}$. Using Lemma E for $m=\frac{n}{2}$ real numbers, we find $2a_{\lambda}a_{\mu}=0$ $(\lambda,\mu=1,\cdots,m)$, that is, $K_{j}{}^{i}$ is definite. Since g_{ji} is positive definite and in the proof of Theorem 1, we already found $Q\geqq 0$, because of $P\geqq 0$, (4.1) implies $Q=0$, consequently $P=0$. So we can prove Theorem 2 by an argument similar to that used in the proof of Theorem 1.

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