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MEAN CURVATURES FOR ANTIHOLOMORPHIC p-PLANES IN SOME ALMOST HERMITIAN MANIFOLDS

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1. Let (M, g) be an n-dimensional Riemannian manifold with (positive definite) metric tensor g. We denote by $K(x, y)$ the sectional curvature for a 2-plane spanned by x and y. Let m be a point of M and π a q-plane at m. An orthonormal basis $\{e_i;\ i=1, 2, \cdots. n\}$ such that $e_1, e_2, \cdots, e_q$ span π is called an adapted basis for π. Then

$$\rho(\pi)=\frac{1}{q(n-q)} \sum_{a=q+1}^{n} \sum_{\alpha=1}^{q} K(e_\alpha, e_a) \tag{1}$$

is independent of the choice of an adapted basis for π and is called by S. Tachibana [5] the *mean curvature* $\rho(\pi)$ for π.

Before formulating the main theorem of this paper, we give some propositions for the mean curvature.

PROPOSITION A (S. Tachibana [5]). *In an $n(>2)$-dimensional Riemannian manifold $(M. g)$, if the mean curvature for a q-plane is independent of the choice of q-planes at each point, then*

(i) *for $q=1$ or $n-1$, (M, g) is an Einstein space;*
(ii) *for $1<q<n-1$ and $2q\neq n$, (M, g) is of constant curvature;*
(iii) *for $2q=n$, (M, g) is conformally flat.*

The converse is true.

Taking holomorphic $2p$-planes instead of q-planes, an analogous result in Kähler manifolds is obtained:

PROPOSITION B (S. Tachibana [6] and S. Tanno [7]). *In a Kähler manifold (M, g, J), $n=2k\geqq 4$, if the mean curvature for a holomorphic $2p$-plane is independent of the choice of holomorphic $2p$-planes at a point m, then*

(i) *for $1\leqq p\leqq k-1$ and $2p\neq k$ (M, g, J) is of constant holomorphic sectional curvature at m;*
(ii) *for $2p=k$, the Bochner curvature tensor vanishes at m.*

The converse is true.

Remark that the case $n=2$ is trivial and that Proposition B can be formulated *globally*. In this case, the converse of (ii) is true if and only if the scalar cur-

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vature is constant.

This proposition has been generalized by the author [9]. To state the obtained result we need some definitions.

Let M be a C^∞ differentiable manifold which is *almost Hermitian*, that is, the tangent bundle has an almost complex structure J and a Riemannian metric g such that $g(JX, JY)=g(X, Y)$ for all $X, Y\in \mathfrak{X}(M)$ where $\mathfrak{X}(M)$ is the Lie algebra of C^∞ vector fields on M. We suppose that $\dim M=n=2k$ and we denote by ∇ the Riemannian connexion on M.

Let now $X, Y\in \mathfrak{X}(M)$ such that $g(X, Y)=g(JX, Y)=0$. They defined a field of 2-planes called *antiholomorphic planes*. The sectional curvature of M restricted to such fields is the *antiholomorphic sectional curvature*. More generally, every subspace N_m of the tangent space $T_m(M)$ at $m\in M$ is called an *antiholomorphic space* if $JN_m\subset N_m^\perp$.

We say that an almost Hermitian manifold is of *constant type* at $m\in M$ provided that for $x\in T_m(M)$ we have

$$\lambda(x, y)=\lambda(x, z) \tag{2}$$

with

$$\lambda(x, y)=R(x, y, x, y)-R(x, y, Jx, Jy) \tag{2'}$$

(R is the Riemann curvature tensor) whenever the planes defined by x, y and x, z are antiholomorphic and $g(y, y)=g(z, z)$. If this holds for all $m\in M$, we say that M has *(pointwise) constant type*. Finally, if M has pointwise constant type and for $X, Y\in \mathfrak{X}(M)$ with $g(Y, X)=g(JX, Y)=0$, $\lambda(X, Y)$ is constant whenever $g(X, X)=g(Y, Y)=1$, then M is said to have *global constant type*. Remark that these definitions coincide with those of A. Gray for nearly Kähler manifolds [2].

An almost Hermitian manifold M such that

$$(\nabla_X J)Y+(\nabla_{JX}J)JY=0 \qquad \text{for all} \quad X, Y\in \mathfrak{X}(M) \tag{3}$$

is called a *quasi-Kähler manifold* [1] and if for all $X\in \mathfrak{X}(M)$ we have

$$(\nabla_X J)X=0\,, \tag{4}$$

the manifold is said to be *nearly Kähler* [2]. Such a manifold is necessarily quasi-Kähler. In [4] G. B. Rizza defined a *para-Kähler manifold* as an almost Hermitian manifold such that

$$R(x, y, z, w)=R(x, y, Jz, Jw) \tag{5}$$

for all x, y, z, w. All these manifolds satisfy

$$R(x, y, z, w)=R(Jx, Jy, Jz, Jw) \tag{6}$$

(see [2], [3], [4]) (except some quasi-Kähler manifolds which we exclude in the following) and are evidently generalirations of Kähler manifolds. Remark that it follows at once from (6) that

$$K(x, y)=K(Jx, Jy), \qquad K(x, Jy)=K(Jx, y), \tag{7}$$

$$k(x, y)=k(Jx, Jy), \qquad k(x, Jy)+k(Jx, y)=0. \tag{8}$$

k is the Ricci tensor defined by

$$k(x, y)=\sum_{i=1}^{n} R(x, e_i, y, e_i) \tag{9}$$

where $\{e_i\}$ is an orthonormal local frame field.

Now we have

PROPOSITION C (L. Vanhecke [9]). *Let M be an $n(=2k)$-dimensional almost Hermitian manifold which is quasi-Kähler with pointwise constant type or para-Kähler. If the mean curvature for holomorphic $2p$-planes is independent of the choice of holomorphic $2p$-planes at each point m and $1\leqq p\leqq k-1$, $2p\neq k$, then M is an Einstein manifold. The converse is true.*

Remark that in this case the mean curvature $\rho(\pi)$ of a holomorphic $2p$-plane equals the antiholomorphic sectional curvature.

The main purpose of this paper is to prove an analogous result considering now the mean curvature of an antiholomorphic p-plane.

MAIN THEOREM. *Let M be an $n(=2k)$-dimensional almost Hermitian manifold which is quasi-Kähler with pointwise constant type or para-Kähler. If the mean curvature for antiholomorphic p-planes is independent of the choice of antiholomorphic p-planes at each point m and $1\leqq p\leqq k-1$, then M is an Einstein manifold. The converse is true.*

We prove first the case $p=1$. To prove the other cases we shall prove the following theorem:

THEOREM. *Let M be an $n(=2k)$-dimensional almost Hermitian manifold which is quasi-Kähler with constant type at a point $m\in M$ or para-Kähler. If the mean curvature for antiholomorphic p-planes is independent of the choice of antiholomorphic p-planes at m and $1<p\leqq k-1$, then M has constant holomorphic sectional curvature at m. The converse is true.*

The main theorem follows then immediately from the two following propositions.

PROPOSITION D (L. Vanhecke [8]). *Let M be a quasi-Kähler manifold with pointwise constant holomorphic sectional curvature μ and pointwise constant type λ. Then M is an Einstein manifold with*

$$2k(x, x)=(k+1)\mu+3(k-1)\lambda \tag{10}$$

for $g(x, x)=1$, where $\dim M=n=2k$ *and*

$$4\nu=\mu+3\lambda, \tag{11}$$

ν denoting the constant antiholomorphic sectional curvature.

This proposition is a generalization of an analogous one for nearly Kähler manifolds [2].

PROPOSITION E (G. B. Rizza [4]). *Let M be a para-Kähler manifold with pointwise constant holomorphic sectional curvature μ. Then M is an Einstein manifold with $4\nu=\mu$, ν denoting the constant antiholomorphic sectional curvature and*

$$2k(x, x)=(k+1)\mu \tag{12}$$

where $\dim M=2k$.

Remark that the same theorem can be proved for the almost Hermitian manifolds such that they satisfy (6) and which are of constant type at a point $m\in M$.

2. Case $p=1$.

Let

$$(e_1, e_2, \cdots, e_p, Je_1, Je_2, \cdots, Je_p, e_{p+1}, e_{p+2}, \cdots, e_k, Je_{p+1}, Je_{p+2}, \cdots, Je_k)$$

be an adapted basis such that $e_1, e_2, \cdots, e_p$ span the antiholomorphic p-plane. Then, the antiholomorphic mean curvature $\rho(\pi)$ for π is

$$\rho(\pi)=\frac{1}{p(n-p)}\{\sum_{a=p+1}^{k}\sum_{\alpha=1}^{p}(K(e_\alpha, e_a)+K(e_\alpha, Je_a))+\sum_{\beta=1}^{p}\sum_{\alpha=1}^{p}K(e_\alpha, Je_\beta)\}\,. \tag{13}$$

This can be written as follows:

$$p(n-p)\rho(\pi)=2p(k-p)\sigma(\pi')+\sum_{\beta=1}^{p}\sum_{\alpha=1}^{p}K(e_\alpha, Je_\beta) \tag{14}$$

where $\sigma(\pi')$ is the holomorphic mean curvature of the $2p$-plane π' spanned by $e_1, e_2, \cdots, e_p, Je_1, Je_2, \cdots, Je_p$. Since

$$k(e_\alpha, e_\alpha)=\sum_{i=1}^{k}\{K(e_\alpha, e_i)+K(e_\alpha, Je_i)\}\,, \tag{15}$$

we have

$$2p(k-p)\sigma(\pi')=\sum_{\alpha=1}^{p}k(e_\alpha, e_\alpha)-\sum_{\beta=1}^{p}\sum_{\alpha=1}^{p}\{K(e_\alpha, e_\beta)+K(e_\alpha, Je_\beta)\} \tag{16}$$

and then it follows

$$p(n-p)\rho(\pi)=\sum_{\alpha=1}^{p}k(e_\alpha, e_\alpha)-\sum_{\beta=1}^{p}\sum_{\alpha=1}^{p}K(e_\alpha, e_\beta)\,. \tag{17}$$

For $p=1$ we obtain

$$\rho(\pi)=\frac{1}{n-1}k(e_1, e_1) \tag{18}$$

and with our hypotheses we have

$$k(x, x)=(n-1)\rho \tag{19}$$

for all x such that $g(x, x)=1$. This proves the assertion for $p=1$.

3. Prove of the Theorem.

First we write (17) as follows:

$$p(n-p)\rho(\pi)=\sum_{\alpha=1}^{p-1} k(e_\alpha, e_\alpha)+k(e_p, e_p)-\sum_{\beta=1}^{p-1}\sum_{\alpha=1}^{p-1} K(e_\alpha, e_\beta)-2\sum_{\alpha=1}^{p-1} K(e_\alpha, e_p). \quad (20)$$

Considering now the antiholomorphic p-plane π_1 spanned by $e_1, e_2, \cdots, e_{p-1}, Je_p$ and writing the analogous expression for π_1 we obtain by substraction

$$\sum_{\alpha=1}^{p-1} K(e_\alpha, e_p)=\sum_{\alpha=1}^{p-1} K(e_\alpha, Je_p) \quad (21)$$

or

$$\sum_{\alpha=1}^{p} K(e_\alpha, e_p)=\sum_{\alpha=1}^{p} K(e_\alpha, Je_p)-H(e_p), \quad (22)$$

where $H(e_p)$ denotes the holomorphic sectional curvature for the 2-plane spanned by e_p and Je_p. We obtain so in general for $1\leqq\beta\leqq p$

$$\sum_{\alpha=1}^{p} K(e_\alpha, e_\beta)=\sum_{\alpha=1}^{\beta} K(e_\alpha, Je_\beta)-H(e_\beta). \quad (23)$$

It follows then from (17):

$$p(n-p)\rho(\pi)=\sum_{\alpha=1}^{p} k(e_\alpha, e_\alpha)+\sum_{\alpha=1}^{p} H(e_\alpha)-\sum_{\beta=1}^{p}\sum_{\alpha=1}^{p} K(e_\alpha, Je_\beta) \quad (24)$$

and with (14) and (16) we get

$$2p(n-p)\rho(\pi)=2\sum_{\alpha=1}^{p} k(e_\alpha, e_\alpha)+\sum_{\alpha=1}^{p} H(e_\alpha)-\sum_{\beta=1}^{p}\sum_{\alpha=1}^{p} \{K(e_\alpha, e_\beta)+K(e_\alpha, Je_\beta)\}. \quad (25)$$

Since $p\leqq k-1$, we can consider the analogous formula for the antiholomorphic p-plane π_2 spanned by $e_1, e_2, \cdots e_{p-1}$ and e_{p+1}. We get by substraction and $\rho(\pi)=\rho(\pi_2)$:

$$\begin{aligned} &k(e_p, e_q)-\sum_{\alpha=1}^{p-1} \{K(e_\alpha, e_p)+K(e_\alpha, Je_p)\} \\ &\qquad =k(e_{p+1}, e_{p+1})-\sum_{\alpha=1}^{p-1} \{K(e_\alpha, e_{p+1})+K(e_\alpha, Je_{p+1})\} \end{aligned} \quad (26)$$

or in general

$$\begin{aligned} &k(e_\beta, e_\beta)+H(e_\beta)-\sum_{\alpha=1}^{p} \{K(e_\alpha, e_\beta)+K(e_\alpha, Je_\beta)\} \\ &\qquad =k(e_a, e_a)-\sum_{\alpha=1}^{p} \{K(e_\alpha, e_a)+K(e_\alpha, Je_a)\}+K(e_\beta, e_a)+K(e_\beta, Je_a) \end{aligned} \quad (27)$$

where $1\leqq\beta\leqq p$ and $p+1\leqq a\leqq k$. Addition with respect to β gives

$$\sum_{\beta=1}^{p} k(e_\beta, e_\beta)+\sum_{\beta=1}^{p} H(e_\beta)-A=pk(e_a, e_a)-(p-1)\sum_{\alpha=1}^{p} \{K(e_\alpha, e_a)+K(e_\alpha, Je_a)\} \quad (28)$$

where

$$A=\sum_{\beta=1}^{p}\sum_{\alpha=1}^{p}\{K(e_\alpha, e_\beta)+K(e_\alpha, Je_\beta)\}\,. \tag{29}$$

Substituting A with (25) in (28) we obtain

$$-\sum_{\beta=1}^{p}k(e_\beta, e_\beta)+2p(n-p)\rho(\pi)$$
$$=p\,k(e_a, e_a)-(p-1)\sum_{\alpha=1}^{p}\{K(e_\alpha, e_a)+K(e_\alpha, Je_a)\}\,. \tag{30}$$

Furthermore, by addition with respect to a we get

$$2p(n-p)(k-p)\rho(\pi)-(k-p)\sum_{\beta=1}^{p}k(e_\beta, e_\beta)=p\sum_{a=p+1}^{k}k(e_a, e_a)-(p-1)B \tag{31}$$

where

$$B=\sum_{a=p+1}^{k}\sum_{\alpha=1}^{p}\{K(e_\alpha, e_a)+K(e_\alpha, Je_a)\,. \tag{32}$$

It follows now easily with the formulas of above that

$$B=2p(n-p)\rho(\pi)-\sum_{\beta=1}^{p}k(e_\beta, e_\beta)-\sum_{\beta=1}^{p}H(e_\beta) \tag{32'}$$

and so we obtain from (31)

$$(k-p-1)\sum_{\beta=1}^{p}k(e_\alpha, e_\alpha)+(p-1)\sum_{\beta=1}^{p}H(e_\beta)$$
$$=2p(n-p)(k-1)\rho(\pi)-p\sum_{\iota=1}^{k}k(e_\iota, e_\iota)\,. \tag{33}$$

Considering again π and π_2 it follows finally from (33) that

$$(k-p-1)k(x, x)+(p-1)H(x) \tag{34}$$

is independent of the unit vector x. This proves the theorem for $p=k-1$ and $k\neq 2$.

Since k satisfies (8) we have a J-basis (e_ι, Je_ι) such that k is diagonal with respect to (e_ι, Je_ι). So it follows from (34) and $H(e_\iota)=H(Je_\iota)$ for

$$x=\sum_{\iota=1}^{k}(A_i e_i+B_\iota Je_\iota)\,,\qquad \sum_{\iota=1}^{k}(A_\iota^2+B_i^2)=1\,, \tag{35}$$

that, for $p\neq 1$,

$$H(x)=\sum_{\alpha=1}^{k}(A_\iota^2+B_\iota^2)H(e_\iota)\,. \tag{36}$$

We have for example

$$H(e_\alpha+Je_a)=\frac{1}{2}H(e)+\frac{1}{2}H(e_a)\,. \tag{37}$$

We need now the following formula for quasi-Kähler manifolds of constant type (see [3]):

$$K(x, y)+K(x, Jy)$$
$$=\frac{1}{4}\{H(x+Jy)+H(x-Jy)+H(x+y)+H(x-y)-H(x)-H(y)\}+\frac{3}{2}\lambda \quad (38)$$

where

$$\lambda=\lambda(x, y)=\lambda(x, Jy) \quad (39)$$

is the constant type and $g(x, x)=g(y, y)=1$, $g(Jx, y)=0$. The same formula is valid for para-Kähler manifolds putting $\lambda=0$.

With the help of (38) we have

$$K(e_\alpha, e_a)+K(e_\alpha, Je_a)=\frac{1}{4}H(e_\alpha)+\frac{1}{4}H(e_a)+\frac{3}{2}\lambda \quad (40)$$

for $\alpha\neq a$ and then it follows from (32′):

$$2p(n-p)\rho(\pi)$$
$$=\sum_{\alpha=1}^{p}k(e_\alpha, e_\alpha)+\frac{k-2p+4}{4}\sum_{\alpha=1}^{p}H(e_\alpha)+\frac{p}{4}\sum_{i=1}^{k}H(e_i)+\frac{3}{2}p(k-p)\lambda\,. \quad (41)$$

Further we have

$$k(e_\alpha, e_\alpha)=H(e_\alpha)+\sum_{\substack{i=1\\ i\neq\alpha}}^{k}\{K(e_\alpha, e_i)+K(e_\alpha, Je_i)\} \quad (42)$$

and it follows with (40):

$$k(e_\alpha, e_\alpha)=\frac{k+2}{4}H(e_\alpha)+\frac{1}{4}\sum_{i=1}^{k}H(e_i)+\frac{3}{2}(k-1)\lambda\,. \quad (43)$$

Finally, substituting this expression in (41) we get

$$4p(n-p)\rho(\pi)=(k-p+3)\sum_{\alpha=1}^{p}H(e_\alpha)+p\sum_{i=1}^{k}H(e_i)+\frac{3}{2}k(2k-p-1)\lambda\,. \quad (44)$$

Considering again π and π_2 and remarking that $k+3\neq p$ we obtain finally

$$H(e_i)=H(e_j) \quad (45)$$

and this proves the theorem.

4. Proof of the converse.

Let M be a para-Kähler manifold or a quasi-Kähler manifold with (pointwise) constant type and suppose that the holomorphic sectional curvature is constant at a point $m\in M$.

In [8] we proved the following formula for $g(x. x)=g(y, y)=1$ and $g(x, y)=0$:

$$K(x, y)=\frac{\mu}{4}\{1+3g^2(Jx, y)\}+\frac{5}{8}\lambda(x, y)+\frac{1}{8}\lambda(x, Jy)\,. \quad (46)$$

The same formula, with $\lambda=0$, is proved in [4] for para-Kähler manifolds with (pointwise) constant holomorphic sectional curvature. It follows for $\alpha\neq a$:

$$K(e_\alpha, e_a)+K(e_\alpha, Je_a)=\frac{1}{2}(\mu+3\lambda)=2\nu \tag{47}$$

where ν is the antiholomorphic sectional curvature at m. So we have for (13):

$$2(n-p)\rho(\pi)=(k+1)\mu+(k-1)3\lambda-2(p-1)\nu\,. \tag{48}$$

This proves the converse.

It is interesting to remark that it follows from (43) and (19) that

$$k(x, x)=(n-p)\rho+(p-1)\nu \tag{49}$$

for $g(x, x)=1$.

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