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ON THE ASYMPTOTIC VALUES FOR REGULAR FUNCTIONS WITH BOUNDED CHARACTERISTIC

Dedicated to Professor Yûsaku Komatu on his 60th birthday

BY CHUJI TANAKA

1. Introduction.

Let $f(z)$ $(z=re^{i\theta})$ be regular and of bounded type in the unit disk $D=\{z;|z|<1\}$. In general, from the boundedness of the boundary cluster set at $z=1$, we can not conclude the boundedness of $f(z)$ in the neighborhood of $z=1$ inside D. Indeed, putting $f(z)=\exp\{(1+z)/(1-z)\}$, $f(z)$ is regular and of bounded type in D, because $f(z)$ is the quotient of two bounded regular functions: 1 and $\exp\{-(1+z)/(1-z)\}$. Then we have easily

$$|f(e^{i\theta})|=1 \qquad \text{for} \quad \theta\neq 0, \quad \lim_{r\to 1-0} f(r)=\infty,$$

which shows that the boundedness of the boundary cluster set at $z=1$ does not always mean the boundedness of $f(z)$ in the neighborhood of $z=1$ inside D.

The object of this note is to establish some additional conditions such that, from the boundedness of the boundary cluster set at $z=1$, we can conclude the boundedness of $f(z)$ in the neighborhood of $z=1$ inside D. As its applications, we shall establish theorems of Lindelöf or Montel-type on the asymptotic values of $f(z)$, including F. W. Gehring's theorems ([4]). Our method is based on the integral representation of $f(z)$.

Contents

2. Theorems of Lindelöf-type (I).

Let D be the unit disk: $|z|<1$, Γ its boundary, which is divided into two arcs Γ_i $(i=1,2)$ by two points z_0 and z_1 on Γ. Suppose that $f(z)$ $(z=re^{i\theta})$ is

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regular and of bounded characteristic in D. As usual ([10] 1-2), we define the boundary cluster sets $C_{\Gamma_i}(f, z_0)$ $(i=1, 2)$ along Γ_i $(i=1, 2)$ respectively. If $C_{\Gamma_i}(f, z_0)$ is bounded, we say that *$f(z)$ is bounded at z_0 along Γ_i.* Theorem 1 reads as follows:

THEOREM 1. *Let $f(z)$ $(z=re^{i\theta})$ be regular and of bounded characteristic in D. Suppose that $f(z)$ is bounded at $z_0=1$ along Γ_i $(i=1, 3)$, where Γ_1 is the upper arc and Γ_3 is the Jordan arc terminating at $z_0=1$ and contained in the domain $\mathfrak{D}_\Delta$:*

$$D\cap\{z\,;\ \pi/2<\arg(z-1)\leqq 3\pi/2-\Delta\}\,,$$

Δ being a positive constant less than π. Under these conditions, $f(z)$ is bounded in the domain bounded by Γ_i $(i=1, 3)$ and $|z-1|=\varepsilon$, ε being a sufficiently small positive constant.

As its corollaries, we obtain

COROLLARY 1. *Let $f(z)$ be regular and of bounded characteristic in D. Suppose that $f(z)$ is bounded at z_0 along Γ_i $(i=1, 2, 3)$, where Γ_3 is the Jordan arc terminating at z_0 and contained in a Stolz-domain with vertex at z_0. Under these conditions, $f(z)$ is bounded in $D\cap U(z_0, \varepsilon)$(*), ε being a sufficiently small positive constant.*

COROLLARY 2. *Let $f(z)$ be regular and of bounded characteristic in D. If $f(z)$ tends to a finite value a_i $(i=1, 2)$ as z approaches z_0 along Γ_i $(i=1, 2)$ respectively, and $f(z)$ is bounded on Γ_3 terminating at z_0 and contained in a Stolz-domain with vertex at z_0, then $a_1=a_2$ and $f(z)$ tends to $a_1=a_2$ uniformly as z tends to z_0 inside D.*

To establish theorem 1, we need some lemmas.

LEMMA 1. ([5], [11], p. 3). *There exists a domain G bounded by a part of Γ_1 and a curve L in D terminating at z_0 such that*

$$C_{\Gamma_1}(f, z_0)=C_G(f, z_0)\,,$$

$C_G(f, z_0)$ being the interior cluster set at z_0 with respect to G.

LEMMA 2. *Let us put*

$$g(z)=1/2\pi\cdot\int_{-\pi}^{+\pi}G(\varphi)P(e^{i\varphi}, z)d\varphi \qquad for \quad |z|<1\,,$$

where $G(\varphi)\in L(-\pi, +\pi)$, $P(e^{i\varphi}, z)=(1-|z|^2)/|e^{i\varphi}-z|^2$.

(1) *If $|G(\varphi)|\leqq m<+\infty$ almost everywhere in $(-\delta, +\delta)$ $(\delta>0)$, then*

$$\varlimsup_{\substack{z\to 1\\ |z|<1}}|g(z)|\leqq m\,. \qquad ([13],\ \text{p. } 48)$$

(2) *If $|G(\varphi)|\leqq m<+\infty$ almost everywhere in $(0, +\delta)$, then for a fixed Δ $(0<\Delta<\pi)$ we have uniformly*

(*) $U(z_0, \varepsilon)$ is ε-neighborhood of z_0.

$$\lim_{\substack{z\to 1\\ z\in\mathfrak{D}_\Delta}}(z-1)\cdot g(z)=0\,,$$

where $\mathfrak{D}_\Delta=D\cap\{z\,;\ \pi/2<\arg(z-1)\leqq 3\pi/2-\Delta\}$.

Proof. Put

$$g(z)=\int_0^{+\delta_1}+\int_{-\delta_1}^{0}+\int_{\delta_1}^{\pi}+\int_{-\pi}^{-\delta_1}=I_1+I_2+I_3+I_4\,,$$

where δ_1 $(0<\delta_1<\delta)$ will tend to zero later.

By the assumption: $|G(\varphi)|\leqq m$ almost everywhere in $(0,+\delta)$,

$$|I_1|\leqq m/2\pi\cdot\int_0^{\delta_1}P(e^{i\varphi},z)d\varphi<m/2\pi\cdot\int_{-\pi}^{+\pi}P(e^{i\varphi},z)d\varphi=m\,. \tag{2.1}$$

Putting $z=re^{i\theta}$, we have for $\delta_1\leqq\varphi\leqq\pi$, $|\theta|\leqq\delta_1/2$,

$$\varphi-\theta\geqq\delta_1/2\,,$$

$$|e^{i\varphi}-z|^2\geqq 1-2r\cos(\delta_1/2)+r^2\geqq\sin^2(\delta_1/2)\,,$$

so that

$$|I_3|\leqq\frac{1-r^2}{2\pi\sin^2\left(\frac{\delta_1}{2}\right)}\cdot\int_{\delta_1}^{\pi}|G(\varphi)|d\varphi<\frac{1-r^2}{2\pi\sin^2\left(\frac{\delta_1}{2}\right)}\cdot\int_{-\pi}^{+\pi}|G(\varphi)|d\varphi\,.$$

Similarly

$$|I_4|<\frac{1-r^2}{2\pi\sin^2\left(\frac{\delta_1}{2}\right)}\cdot\int_{-\pi}^{+\pi}|G(\varphi)|d\varphi\,.$$

Hence

$$|I_3+I_4|<\frac{1-r^2}{\pi\sin^2\left(\frac{\delta_1}{2}\right)}\cdot\int_{-\pi}^{+\pi}|G(\varphi)|d\varphi\,. \tag{2.2}$$

Since $\mathfrak{D}_\Delta\supset\mathfrak{D}_{\Delta'}$ for $0<\Delta<\Delta'<\pi$, we can assume that

$$0<\delta_1<\Delta<\pi/4\,.$$

By the elementary calculations, we have

$$\frac{|z-e^{i\varphi}|}{|z-1|}>\sin(\Delta/2)\qquad\text{for}\quad -\delta_1\leqq\varphi\leqq 0\,,\ z\in\mathfrak{D}_\Delta\,.$$

Hence

$$\begin{aligned}|I_2|&\leqq\frac{1-|z|^2}{|z-1|^2}\cdot\frac{1}{2\pi\sin^2\left(\frac{\Delta}{2}\right)}\cdot\int_{-\delta_1}^{0}|G(\varphi)|d\varphi\\&<\frac{1}{|z-1|}\cdot\frac{1}{\pi\sin^2\left(\frac{\Delta}{2}\right)}\cdot\int_{-\delta_1}^{0}|G(\varphi)|d\varphi\,.\end{aligned} \tag{2.3}$$

By (2.1), (2.2) and (2.3),

$$\overline{\lim_{\substack{z\to1\\ z\in\mathfrak{D}_\Delta}}}|(z-1)\cdot g(z)|\leqq\frac{1}{\pi\sin^2\left(\frac{\Delta}{2}\right)}\cdot\int_{-\delta_1}^{0}|G(\varphi)|d\varphi\,. \tag{2.4}$$

Letting $\delta_1\to+0$ in (2.4), we have

$$\overline{\lim_{\substack{z\to1\\ z\in\mathfrak{D}_\Delta}}}|(z-1)\cdot g(z)|=0\,,$$

which proves part (2) of our lemma.

LEMMA 3. *Suppose that $f(z)$ is regular in $\bar{D}$ except at $z=1$, and of bounded characteristic in D. Then $f(z)$ has the following integral representation:*

$$f(z)=B(z)\cdot\exp\left\{1/2\pi\cdot\int_{-\pi}^{+\pi}\log|f(e^{i\varphi})|\cdot Q(e^{i\varphi},z)d\varphi\right\}\cdot\exp\{C\cdot Q(1,z)+i\lambda\}\,,$$

where $B(z)$: Blaschke products extended over the zeros of $f(z)$, which may have the unique limit point: $z=1$,(*)

$$Q(e^{i\varphi},z)=(e^{i\varphi}+z)/(e^{i\varphi}-z)\,,\qquad C,\lambda:\ \textit{real constants}.$$

Proof. It is well konwn ([12], p. 79) that $f(z)$ has the following integral representation;

$$f(z)=B(z)\cdot D_1(z)\cdot D_2(z)\,, \tag{2.5}$$

where $B(z)$: Blaschke products extended over the zeros of $f(z)$,

$$D_1(z)=\exp\left\{1/2\pi\cdot\int_{-\pi}^{+\pi}\log|f(e^{i\varphi})|\cdot Q(e^{i\varphi},z)d\varphi\right\},$$

$$D_2(z)=\exp\left\{1/2\pi\cdot\int_{-\pi}^{+\pi}Q(e^{i\varphi},z)d\mu(\varphi)+i\lambda\right\},$$

$\mu(\varphi)$: the real function of bounded variation with $\mu'(\varphi)=0$ for almost all $\varphi\in[-\pi,+\pi]$, λ: a real constant..

Since $f(z)$ is regular on $|z|=1$ except at $z=1$, its zeros may have the unique limit point: $z=1$. In the neighbourhood of $z=e^{i\varphi_0}$ $(\varphi_0\neq0)$, $f(z)$ can be represented by

$$f(z)=(z-e^{i\varphi_0})^n\cdot F(z)\,, \tag{2.6}$$

where n: non negative integer, $F(z)$ is regular at $z=e^{i\varphi_0}$ and $F(e^{i\varphi_0})\neq0$. Since

$$(z-e^{i\varphi_0})^n=e^{i\lambda^*}\cdot\exp\left\{1/2\pi\cdot\int_{-\pi}^{+\pi}n\cdot\log|e^{i\varphi}-e^{i\varphi_0}|\cdot Q(e^{i\varphi},z)d\varphi\right\},$$

λ^*: a real constant, by (2.5) and (2.6)

(*) If the number of zeros is finite, $z=1$ is not the limit point of zeros.

(2.7) $$\exp\{H(z)\}$$
$$=F(z)/B(z)\cdot\exp\left\{-1/2\pi\cdot\int_{-\pi}^{+\pi}G(\varphi)\cdot Q(e^{i\varphi}, z)d\varphi\right\}\cdot\exp(i(\lambda^*-\lambda)),$$

where $H(z)=1/2\pi\cdot\int_{-\pi}^{+\pi}Q(e^{i\varphi}, z)d\mu(\varphi)$, $G(\varphi)=\log|F(e^{i\varphi})|\neq\infty$ in the neighborhood $\varphi=\varphi_0$. Since $z=e^{i\varphi_0}$ $(\varphi_0\neq 0)$ is not the limit point of zeros, $B(z)$ is regular and of modulus one on the arc: $A(e^{i\varphi}, \varphi_0-\varepsilon\leqq\varphi\leqq\varphi_0+\varepsilon)$, ε being a sufficiently small positive constant ([15], p. 410). Hence by (2.6) and lemma 2 (1), the right hand side of (2.7) is not equal to 0 or ∞ on A. In other words, $R(H(z))$ is not equal to $\pm\infty$ on A. Then we can conclude that $\mu(\varphi)\equiv$constant on A.

Indeed, if $\mu(\varphi)\not\equiv$constant on A, $\mu(\varphi)$ admits the following representation:

$$\mu(\varphi)=\mu_1(\varphi)+\mu_2(\varphi)+\mu_3(\varphi),$$

where all functions $\mu_i(\varphi)$ $(i=1,2,3)$ are of bounded variation $\mu_1(\varphi)$ is absolutely continuous, $\mu_2(\varphi)$ is continuous and $\mu_2'(\varphi)=0$ a.e., and $\mu_3(\varphi)$ is a step-function. Since $\mu'(\varphi)=0$, $\mu_i'(\varphi)=0$ $(i=2,3)$ a.e., $\mu_1'(\varphi)\equiv 0$ a.e., i.e. $\mu_1(\varphi)\equiv 0$. If $\mu_2(\varphi)$ is a constant, then $\mu_3(\varphi)$ is certainly not a constant, so that

$$H(z)=1/2\pi\cdot\int_{CA}Q(e^{i\varphi}, z)d\mu(\varphi)+\sum_{n=1}^{\infty}Q(e^{i\varphi_n}, z)\cdot J_n,$$

where $\{e^{i\varphi_n}\}\in A$, $\sum_{n=1}^{\infty}|J_n|<+\infty$. Hence $R(H(z))$ is equal to $\pm\infty$ on A, which is impossible. If $\mu_2(\varphi)$ is not a constant, then it is well known that $\mu_2'(\varphi)=\pm\infty$ at a non-denumeable set of points on A. Since $\mu_3(\varphi)$ is discontinuous at an enumerable set of points on A, there exists a non-denumerable set E of points on A such that $\mu'(\varphi)=\pm\infty$ for $e^{i\varphi}\in E$. Hence, by P. Fatou's theorem, $R(H(z))$ is equal to $\pm\infty$ on E, which is again impossible. Thus $\mu(\varphi)\equiv$constant on A.

Since $z=e^{i\varphi_0}\neq 1$ is arbitrary, $d\mu(\varphi)=0$ in $(-\pi,+\pi)$ except at $\varphi=0$. Hence $H(z)$ reduces to the form: $C\cdot Q(1,z)$ (C: real constant), which proves our lemma.

Now we can establish theorem 1.

Proof of theorem 1. By lemma 1, there exist two analytic Jordan arcs Γ_i^* $(i=1,2)$, closely near Γ_i $(i=1,2)$ and contained in D except at $z=1$, such that $f(z)$ is bounded on Γ_1^* and Γ_2^* is contained in $C(\mathfrak{D}_\Delta)$. Denote by D^* the subdomain of D bounded by a part of Γ_i^* $(i=1,2)$ and a cross-cut connecting Γ_1^* and Γ_2^*. If we map conformally D onto $|\zeta|<1$ in such a manner that $z=1$ goes into $\zeta=1$, then by Lindelöf-Carathéodory's theorem ([2], p. 92) the image of Γ_3 is also contained in the domain of the same character as $\mathfrak{D}_\Delta$. Therefore, without any loss of generality, we can assume that $f(z)$ is regular on Γ except at $z=1$.

By lemma 3, we have easily

(2.8) $$\log^+|f(z)|\leqq 1/2\pi\cdot\int_0^{2\pi}\log^+|f(e^{i\varphi})|\cdot P(e^{i\varphi}, z)d\varphi+C^+\cdot\frac{1-|z|^2}{|1-z|^2},$$

where $C^+ = \max(C, 0)$. Since $f(z)$ is bounded at $z=1$ along Γ_1, by lemma 2 (2)

$$1/2\pi \cdot \int_{-\pi}^{+\pi} \log^+ |f(e^{i\varphi})| \cdot P(e^{i\varphi}, z) d\varphi < \frac{\varepsilon_1}{|z-1|} \tag{2.9}$$

in $U(1, \delta_1) \cap \mathfrak{D}_\Delta$, where ε_1 : any given positive constant, and δ_1 : a positive constant dependent upon ε_1. By (2.8) and (2.9)

$$|f(z)| < \exp\left\{O\left(\left|\frac{1}{z-1}\right|\right)\right\} \qquad \text{for} \quad z \in U(1, \delta_1) \cap \mathfrak{D}_\Delta .$$

Hence

$$|f(z)| < \exp\left\{\varepsilon_2 \cdot \left|\frac{1}{z-1}\right|^{\frac{\pi}{\alpha}}\right\} \qquad \text{for} \quad z \in U(1, \delta_2) \cap \mathfrak{D}_\Delta , \tag{2.10}$$

where $\alpha = \pi - \Delta$, ε_2 : any given positive constant and δ_2; a positive constant dependent upon ε_2. Since $f(z)$ is bounded on Γ_i $(i=1, 3)$, by (2.10) and Phragmen-Lindelöf's theorem ([16], pp. 64-66), $f(z)$ is also bounded in the domain bounded by Γ_i $(i=1, 3)$ and $|z-1| = \varepsilon$, ε being a sufficiently small positive constant, which is to be proved.

In corollary 1, we can replace the boundedness of $f(z)$ on Γ_3 by another conditions;

THEOREM 2. *Let $f(z)$ be regular and of bounded characteristic in D. Suppose that $f(z)$ is bounded at $z=z_0$ along Γ_i $(i=1, 2)$ and that $f(z)$ is also bounded on the sequence of curves $\{C_n\}$, where C_n is the cross-cut connecting both chordal sides of a Stolz-domain with vertex at z_0, and C_n tends to z_0 as $n \to +\infty$. Under these conditions, $f(z)$ is bounded in $U(z_0, \varepsilon) \cap D$, ε being a sufficiently small positive constant.*

To establish theorem 2, we need

LEMMA 4. *Let Blaschke products $B(z)$ have zeros with the unique limit point: $z=1$. Then there exists a set E of φ with outer capacity zero contained in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ such that, for fixed $\varphi \overline{\in} E$,*

$$\lim_{\substack{z \to 1 \\ z \in C_\varphi}} |z-1| \cdot \log |B(z)| = 0 ,$$

where C_φ is the circular arc which connects $z = \pm 1$ and has the tangent: $\arg(z-1) = \varphi$ at $z=1$.

Although R. P. Boas ([1], p. 115) has proved this lemma in the case of Blaschke products in the upper half-plane: $I(z) > 0$, we get lemma 4 by a suitable linear transformation.

Proof of theorem 2. Without any loss of generality, we can assume that $z_0 = 1$. As in theorem 1, we can suppose that $f(z)$ is regular on $|z|=1$ except at $z=1$. Using $f(z)+k$ (k : a suitable constant) instead of $f(z)$, if necessary, we can further assume that there exist two constants k_i $(i=1, 2)$ such that

(2.11) $$0<k_1\leqq|f(z)|\leqq k_2<+\infty \quad \text{on} \quad \Gamma_i \ (i=1,2).$$

By lemma 3, we have

(2.12) $$|f(z)|=|B(z)|\exp\left(C\frac{1-|z|^2}{|1-z|^2}\right)\cdot\exp(D(z)),$$

where $D(z)=1/2\pi\cdot\int_{-\pi}^{+\pi}\log|f(e^{i\varphi})|\cdot P(e^{i\varphi}, z)d\varphi$. By (2.11) and lemma 2 (1), $D(z)$ is bounded in $U(1, \varepsilon)\frown D$, ε being a sufficiently small positive constant. By lemma 3 and lemma 4, there exists a circular arc C_φ contained in a Stolz-domain with vertex at $z=1$ such that

$$|B(z)|>\exp\left(-\frac{\varepsilon_1}{|1-z|}\right)$$

in the neighborhood of $z=1$ on C_φ, where ε_1: any given positive constant. Therefore, denoting by z_n the intersection point between C_n and C_φ, by (2.12) we have for $n\geqq N(\varepsilon_1)$

(2.13) $$|f(z_n)|>\exp\left\{\frac{1}{|1-z_n|}\left(C\cdot\frac{1-|z_n|^2}{|1-z_n|}-\varepsilon_1\right)\right\}\exp(D(z_n)).$$

Since z_n is contained in a Stolz-domain with vertex at $z=1$, there exists a positive constant k_3 such that

(2.14) $$\frac{1-|z_n|}{|1-z_n|}\geqq k_3>0.$$

By (2.13) and (2.14), we can conclude that $C\leqq 0$. Indeed, if $C>0$, by (2.13) and (2.14)

(2.15) $$|f(z_n)|>\exp\left\{\frac{1}{|1-z_n|}(Ck_3-\varepsilon_1)+D(z_n)\right\},$$

so that, putting $Ck_3-\varepsilon_1>0$, the right hand side of (2.15) is unbounded as $n\to+\infty$, which is contrary to the boundedness of $f(z_n)$ $(n=1, 2, \cdots)$. Since $C\leqq 0$, by (2.12)

$$|f(z)|\leqq\exp(D(z)),$$

so that, from the boundedness of $D(z)$ in $U(1, \varepsilon)\frown D$, we can conclude the boundness of $f(z)$ in $U(1, \varepsilon)\frown D$, which is to be proved.

As an immediate consequence of theorem 2, we get

COROLLARY 3. *Let $f(z)$ be regular and of bounded characteristic in D. Suppose that $f(z)$ tends to a finite value a_i $(i=1, 2)$ as $z\to z_0$ along Γ_i $(i=1, 2)$ respectively, and that $f(z)$ is bounded on the sequence of curves $\{C_n\}$, where C_n is the cross-cut connecting both chordal sides of a Stolz-domain with vertex at z_0 and C_n tends to z_0 as $n\to+\infty$. Under these conditions, $a_1=a_2$ and $f(z)$ tends to $a_1=a_2$ uniformly as $z\to z_0$ inside D.*

3. F. W. Gehring's theorem.

F. W. Gehring has proved the following interesting theorem.

F. W. Gehring's theorem ([4], p. 284). *Suppose that $f(z)$ is regular and of bounded characteristic in D. Then following propositions hold:*

(1) *For each z_0 on Γ, $A(z_0, f)$ (the set of asymptotic values at z_0) contains at most two finite values.*

(2) *If $A_\Delta(z_0, f)$ (the set of angular asymptotic values at z_0 on Γ) containes a finite value, then $A(z_0, f)$ contains only one finite value.*

He has proved this theorem using the systematic use of the harmonic majorant of $\log^+|f(z)|$ and modified A. J. Macintyre's theorem ([8], p. 38), which is, however, not familiar to us. We can establish this theorem as the application of theorem 1 and corollary 3.

Proof of F. W. Gehring's theorem. We may assume that $z_0=1$. If $A(1, f)$ contains three finite values a_i $(i=1, 2, 3)$, there exist three Jordan arcs Γ_i $(i=1, 2, 3)$ in $D\cup\Gamma$ such that $f(z)\to a_i$ $(i=1, 2, 3)$ as $z\to 1$ along Γ_i $(i=1, 2, 3)$ respectively. By the preliminary conformal mapping, from the beginning we can assume that $\Gamma_1\cup\Gamma_2$ forms a part of Γ and that Γ_3 is contained in D except at $z=1$.

For a sufficiently small positive Δ, following two cases may occur:

(A) Γ_3 intersects infinitely many times both chordal sides of a Stolz-domain with vertex at $z=1$:

$$D\cap\{z\,;\ \pi/2+\Delta\leqq\arg(z-1)\leqq 3\pi/2-\Delta\}\ .$$

(B) Γ_3 is contained in one of two domains D_i $(i=1, 2)$:

$$D_1=D\cap\{z\,;\ \pi/2<\arg(z-1)\leqq 3\pi/2-\Delta\}\ ,$$

$$D_2=D\cap\{z\,;\ \pi/2+\Delta\leqq\arg(z-1)<3\pi/2\}\ .$$

In case (A), by corollary 3 we have $a_1=a_2$, which is impossible. In case (B), if $\Gamma_3\in D_1$ or D_2, by theorem 1 we have $a_3=a_1$ or $a_3=a_2$ respectively, which is again impossible. Thus part (1) is completely established.

Now we proceed to the second part. Suppose that $f(z)\to a_i$ $(i=1, 3)$ as $z\to 1$ along Γ_i $(i=1, 3)$ respectively, where a_i $(i=1, 3)$ are finite and that Γ_3 is contained in a Stolz-domain with vertex at $z=1$. Without any loss of generality, we can assume that Γ_1 is a cross-cut of D, and that Γ_3 is contained in one of two Jordan domains into which D is split by Γ_1. By the preliminary conformal mapping and the similar arguments as in the proof of theorem 1, from the beginning we can assume that Γ_1 is a part of Γ and that Γ_3 is contained in D_1. Then, by theorem 1, we have $a_1=a_3$, which is impossible. Thus, part (2) is also established.

4. The least harmonic majorant of $\log^+|f(z)|$.

Let $f(z)$ be regular and of bounded characteristic in D. The subharmonic function: $\log^+|f(z)|$ has the harmonic majorant: $h(z)$ in D. Indeed, by (2.5) we get easily $h(z)$ as follows;

$$\log^+|f(z)|\leqq h(z) \tag{4.1}$$
$$=1/2\pi\cdot\int_{-\pi}^{+\pi}\log^+|f(e^{i\varphi})|\cdot P(e^{i\varphi}, z)d\varphi+1/2\pi\cdot\int_{-\pi}^{+\pi}P(e^{i\varphi}, z)d^+\mu(\varphi),$$

where $P(e^{i\varphi}, z)=\frac{1-|z|^2}{|e^{i\varphi}-z|^2}$, $\log^+ x=\max(\log x, 0)$, $d\mu^+(\varphi)=\max(d\mu(\varphi), 0)$.

We shall now prove the following theorem which is interesting in itself.

THEOREM 3. *$h(z)$ is the least harmonic majorant of* $\log^+|f(z)|$ *in* $|z|<1$.

Proof. We divide the proof in three parts.

(1) Let us define a positive harmonic function $u(z, r)$ $(0\leqq r<1)$ such that

$$\begin{aligned} &u(z, r)=\log^+|f(re^{i\theta})| \quad \text{on} \quad |z|=r, \\ &u(z, r): \text{harmonic in } |z|<r. \end{aligned} \tag{4.2}$$

It is well-known that $u(z, r)$ is given by Poisson integral;

$$u(z, r)=1/2\pi\cdot\int_0^{2\pi}\log^+|f(re^{i\varphi})|\cdot P(re^{i\varphi}, z)d\varphi, \tag{4.3}$$

where $P(re^{i\varphi}, z)=\frac{r^2-|z|^2}{|re^{i\varphi}-z|^2}$ $(|z|<r)$. Since $\log^+|f(z)|$ is subharmonic, by (4.2)

$$\log^+|f(z)|\leqq u(z, r) \quad \text{in} \quad |z|\leqq r. \tag{4.4}$$

Hence

$$u(z, r)=\log^+|f(z)|\leqq u(z, R) \quad \text{on} \quad |z|=r<R<1,$$

so that

$$0\leqq u(z, r)\leqq u(z, R) \quad \text{in} \quad |z|\leqq r<R<1.$$

Therefore $\{u(z, r)\}$ $(0\leqq r<1)$ is an increasing sequence of r. Because of the bounded characteristic of $f(z)$, there exists a finite constant M such that

$$u(0, r)=1/2\pi\cdot\int_0^{2\pi}\log^+|f(re^{i\theta})|d\theta<M<+\infty.$$

By A. Harnack's theorem, $u(z, r)$ converges to $u(z)$ uniformly in the wider sense in $|z|<1$, so that letting $r\to1$ in (4.3) and (4.4), we have for $|z|<1$

$$\log^+|f(z)|\leqq u(z) \tag{4.5}$$
$$=\lim_{r\to1} 1/2\pi\cdot\int_0^{2\pi}\log^+|f(re^{i\varphi})|\cdot P(re^{i\varphi}, z)d\varphi.$$

Then $u(z)$ is the least harmonic majorant of $\log^+|f(z)|$. Indeed, if $v(z)$ is any harmonic majorant of $\log^+|f(z)|$, by (4.2),

$$u(z, r)=\log^+|f(z)|\leqq v(z) \qquad \text{on} \quad |z|=r,$$

so that

$$u(z, r)\leqq v(z) \qquad \text{for} \quad |z|\leqq r.$$

Letting $r\to 1$, we have $u(z)\leqq v(z)$, which shows that $u(z)$ is the least harmonic majorant of $\log^+|f(z)|$ in $|z|<1$.

(2) Putting $F_r(\varphi)=\int_0^{\varphi}\log^+|f(re^{i\theta})|\,d\theta$, we get easily

$$(4.6)\qquad \begin{aligned} &\int_0^{2\pi}|dF_r(\varphi)|=\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta<M<+\infty, \\ &|F_r(\varphi)|\leqq\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta<M<+\infty. \end{aligned}$$

Hence, by E. Helly's first theorem ([12], p. 15), there exist a sequence $\{r_n\}$ $(0<r_1<r_2\cdots<r_n\to 1)$ and a function $F_1(\varphi)$ non-negative and non-decreasing such that

$$(4.7)\qquad F_1(\varphi)=\lim_{n\to\infty}F_{r_n}(\varphi)=\lim_{n\to+\infty}\int_0^{\varphi}\log^+|f(r_ne^{i\theta})|\,d\theta.$$

By (4.6), (4.7) and E. Helly's second theorem ([12], p. 15),

$$(4.8)\qquad \begin{aligned} \lim_{n\to+\infty}\int_0^{2\pi}\log^+|f(r_ne^{i\varphi})|\cdot P(e^{i\varphi}, z)\,d\varphi&=\lim_{n\to+\infty}\int_0^{2\pi}P(e^{i\varphi}, z)\,dF_{r_n}(\varphi) \\ &=\int_0^{2\pi}P(e^{i\varphi}, z)\,dF_1(\varphi). \end{aligned}$$

Let us put for $r_n>r$ $(z=re^{i\theta})$

$$\begin{aligned} u(z, r_n)&=I_1+I_2 \\ &=1/2\pi\cdot\int_0^{2\pi}\log^+|f(r_ne^{i\varphi})|\cdot P(e^{i\varphi}, z)\,d\varphi \\ &\quad+1/2\pi\cdot\int_0^{2\pi}\log^+|f(r_ne^{i\varphi})|\{P(r_ne^{i\varphi}, z)-P(e^{i\varphi}, z)\}\,d\varphi. \end{aligned}$$

Since

$$P(Re^{i\varphi}, re^{i\theta})=1+2\sum_{k=1}^{+\infty}(r/R)^k\cos(k(\theta-\varphi)) \qquad \text{for} \quad R>r,$$

we have

$$\begin{aligned} |I_2|&\leqq 2\sum_{k=1}^{+\infty}\{(r/r_n)^k-r^k\}1/2\pi\cdot\int_0^{2\pi}\log^+|f(r_ne^{i\varphi})|\,d\varphi \\ &<2\cdot\left\{\frac{r_n}{r_n-r}-\frac{1}{1-r}\right\}M, \end{aligned}$$

so that

$$I_2 \longrightarrow 0 \quad \text{as} \quad n \to +\infty .$$

Hence, by (4.5) and (4.8)

$$u(z)=1/2\pi \cdot \int_0^{2\pi} P(e^{i\varphi}, z)dF_1(\varphi) , \tag{4.9}$$

where $F_1(\varphi)=\lim_{n\to\infty}\int_0^{\varphi}\log^+|f(r_n e^{i\theta})|d\theta$. Since $F_1(\varphi)$ is non-decreasing, by the decomposition theorem of functions of bounded variation, we can put

$$F_1(\varphi)=\int_0^{\varphi} F(\theta)d\theta+\mu_1(\varphi) , \tag{4.10}$$

where $F(\theta)\in L(0, 2\pi)$, $\mu_1'(\varphi)=0$ a. e. and $d\mu_1(\varphi)\geqq 0$.

Since $u(z)$ is the least harmonic majorant of $\log^+|f(z)|$ in $|z|<1$, we have

$$\log^+|f(re^{i\varphi})|\leqq u(re^{i\varphi})\leqq h(re^{i\varphi}) \qquad \text{on} \quad |z|=r<1 .$$

Therefore, by P. Fatou's theorem and (4.1), (4.9),

$$\log^+|f(e^{i\varphi})|\leqq F_1'(\varphi)\leqq \log^+|f(e^{i\varphi})| \qquad \text{a. e.,}$$

i. e.

$$F_1'(\varphi)=\log^+|f(e^{i\varphi})| \qquad \text{a. e.}$$

Hence, by (4.9) and (4.10) we get next integral representation of the least harmonic majorant $u(z)$:

$$u(z)=1/2\pi \cdot \int_0^{2\pi} P(e^{i\varphi}, z)dF_1(\varphi) , \tag{4.11}$$

where $F_1(\varphi)=\lim_{n\to\infty}\int_0^{\varphi}\log^+|f(r_n e^{i\theta})|d\theta=\int_0^{\varphi}\log^+|f(e^{i\theta})|d\theta+\mu_1(\varphi)$, $\mu_1'(\varphi)=0$ a. e. and $d\mu_1(\varphi)\geqq 0$.

(3) Let us put

$$G_r(\varphi)=\int_0^{\varphi}\log^+|1/f(re^{i\theta})|d\theta .$$

Since $\log^+|1/f(re^{i\theta})|\leqq|\log|f(re^{i\theta})||$, we get

$$\int_0^{2\pi}|dG_r(\varphi)|=\int_0^{2\pi}\log^+|1/f(re^{i\theta})|d\theta\leqq\int_0^{2\pi}|\log|f(re^{i\theta})||d\theta<N<+\infty ,$$

$$|G_r(\varphi)|\leqq\int_0^{2\pi}\log^+|1/f(re^{i\theta})|d\theta\leqq\int_0^{2\pi}|\log|f(re^{i\theta})||d\theta<N<+\infty ,$$

where N is a positive finite constant, because $f(z)$ is of bounded characteristic. Therefore, by similar arguments as in (2), we can select a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ and a function $G_1(\varphi)$ non-negative and non-decreasing such that

$$G_1(\varphi)=\lim_{k\to\infty} G_{r_{n_k}}(\varphi)=\lim_{k\to\infty}\int_0^{\varphi}\log^+|1/f(r_{n_k}e^{i\theta})|d\theta . \tag{4.12}$$

Hence, by the decomposition-theorem of functions of bounded variation, we have

$$G_1(\varphi)=\int_0^{\varphi} G(\theta)d\theta+\mu_2(\varphi)\,, \tag{4.13}$$

where $G(\theta)\in L(0, 2\pi)$ and $\mu_2'(\varphi)=0$ a. e., $d\mu_2(\varphi)\geqq 0$.

By (2.5), we can put

$$f(z)=B(z)\cdot\exp\left\{1/2\pi\cdot\int_0^{2\pi}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}dF(\varphi)+i\lambda\right\},$$

where $F(\varphi)=\int_0^{\varphi}\log|f(e^{i\theta})|d\theta+\mu(\varphi)=\lim_{r\to 1-0}\int_0^{\varphi}\log|f(re^{i\theta})|d\theta$ for except perhaps an enumerable set of φ ([9], p. 198, p. 201).

By (4.11), (4.12) and (4.13)

$$\begin{aligned} F(\varphi)&=\lim_{k\to\infty}\int_0^{\varphi}\log^+|f(r_{n_k}e^{i\theta})|d\theta-\lim_{k\to\infty}\int_0^{\varphi}\log^+|1/f(r_{n_k}e^{i\theta})|d\theta \\ &=\int_0^{\varphi}\{\log^+|f(e^{i\theta})|-G(\theta)\}d\theta+\{\mu_1(\varphi)-\mu_2(\varphi)\}\,. \end{aligned}$$

By the uniqueness of the decomposition of $F(\varphi)$, we have

$$\int_0^{\varphi}\log|f(e^{i\theta})|d\theta=\int_0^{\varphi}\{\log^+|f(e^{i\theta})|-G(\theta)\}d\theta\,,$$

$$\mu(\varphi)=\mu_1(\varphi)-\mu_2(\varphi)\,,$$

so that

$$\begin{aligned} &\int_0^{\varphi}G(\theta)\,d\theta=\int_0^{\varphi}\log^+|1/f(e^{i\theta})|d\theta\,, \\ &d^+\mu(\varphi)=\max\{(d\mu_1(\varphi)-d\mu_2(\varphi)),0\}\leqq d\mu_1(\varphi)\,. \end{aligned} \tag{4.14}$$

Hence, by (4.1) and (4.11), $h(z)\leqq u(z)$. On the other hand, since $u(z)$ is the least harmonic majorant of $\log^+|f(z)|$ in $|z|<1$, it is evident that $u(z)\leqq h(z)$. Thus we have $h(z)=u(z)$, which is to be proved.

Remark. By (4.14), we have

$$\lim_{k\to\infty}\int_0^{\varphi}\log^+|1/f(r_{n_k}e^{i\theta})|d\theta=\int_0^{\varphi}\log^+|1/f(e^{i\theta})|d\theta+\mu_2(\varphi)\,.$$

5. Theorems of Lindelöf-type (II).

Using the least harmonic majorant $h(z)$, we can establish several theorems somewhat different from theorem 1-2.

THEOREM 4. *Let $f(z)$ be regular and of bounded characteristic in D. Suppose that $f(z)$ is bounded at z_0 along Γ_i $(i=1,2)$ and that $h(z_n)$ is bounded on the sequence of points $\{z_n\}$, contained in a Stolz-domain with vertex at $z=z_0$ and tending to $z=z_0$ as $n\to+\infty$. Under these conditions, $f(z)$ is bounded in $U(z_0,\varepsilon)$*

$\frown D$, *ε being a sufficiently small positive constant.*

As its consequence, we obtain next corollary which is a generalization of the corollary in the preceding paper ([14], pp. 98-99).

COROLLARY 4. *Let $f(z)$ be regular and of bounded characteristic in D. Suppose that $f(z)$ tends to a finite value a_i $(i=1, 2)$ as z approaches z_0 along Γ_i $(i=1, 2)$ respectively, and that $h(z_n)$ is bounded on the sequence of points $\{z_n\}$ contained in a Stolz-domain with vertex at $z=z_0$ and tending to $z=z_0$ as $n\to\infty$. Under these conditions, $a_1=a_2$ and $f(z)$ tends uniformly to $a_1=a_2$ as $z\to z_0$ inside D.*

Proof. We may put $z_0=1$. Considering $f(z)+k$ (k: a suitable constant) instead of $f(z)$, if necessary, we can assume that there exist two constants k_i $(i=1, 2)$ such that

$$0<k_1\leqq|f(z)|\leqq k_2<+\infty \tag{5.1}$$

in the neighborhood of the open arc: $A(e^{i\theta};\ 0<|\theta|<\delta)$. By (5.1), Blaschke product: $B(z)$ has no limit point of zeros in the neighborhood of $e^{i\theta}\in A$. Hence, by (2.5) and similar arguments as in the proof of lemma 3, we can conclude that $d\mu(\varphi)=0$ in the neighborhood of $\varphi=\theta$. Therefore, by (2.5) we can put

$$f(z)=B(z)\cdot D_1(z)\cdot D_2(z)\ , \tag{5.2}$$

where

$$D_2(z)=\exp\left\{1/2\pi\cdot\int_{CA}Q(e^{i\varphi}, z)d\mu(\varphi)+C\cdot Q(1, z)+i\lambda\right\},$$

CA: the complementary arc of A, C and λ: real constants, so that, by (4.1)

$$h(z)=h_1(z)+h_2(z)+C^+\cdot P(1, z)\ , \tag{5.3}$$

where

$$h_1(z)=1/2\pi\cdot\int_{-\pi}^{+\pi}\log^+|f(e^{i\varphi})|\cdot P(e^{i\varphi}, z)d\varphi\ ,\qquad h_2(z)=1/2\pi\cdot\int_{CA}P(e^{i\varphi}, z)d^+\mu(\varphi)\ .$$

By (5.1), (5.3) and lemma 2 (1), $h_i(z)$ $(i=1, 2)$ is bounded in the heighborhood of $z=1$, so that from the boundedness of $h(z_n)$ on $\{z_n\}$, we can conclude that $C^+\cdot P(1, z_n)$ is bounded on $\{z_n\}$. Since $\{z_n\}$ is contained in a Stolz-domain with vertex at $z=1$, there exists a positive constant k_3 such that

$$P(1, z_n)>\frac{k_3}{|1-z_n|}\ .$$

Hence, $C^+\cdot\dfrac{k_3}{|1-z_n|}$ is bounded on $\{z_n\}$, which is possible only if $C^+=0$. Thus, by (5.3) and (4.1),

$$\log^+|f(z)|\leqq h_1(z)+h_2(z)\ ,$$

from which we can conclude the boundedness of $f(z)$ in the neighborhood of $z=1$, which is to be proved.

In theorem 4, we have assumed the both-sided boundedness at z_0. If we assume only the one-sided boundedness at z_0, what we can say about the boundedness of $f(z)$ in D? We shall give an answer to this question in the following theorem.

THEOREM 5. *Let $f(z)$ be regular and of bounded characteristic in D. Suppose that $f(z)$ is bounded at $z_0=1$ along the upper arc Γ_1, and that the sequence of $\{h(a_n)\}$ is bounded: $h(a_n)<M<+\infty$ $(n=1, 2, \cdots)$, where*

(1) $|a_n|<1$, $\lim\limits_{n\to\infty} a_n=1$,

(2) $a_n\in\mathfrak{D}_\Delta=D\cap\{z\,;\,\pi/2<\arg(z-1)\leqq 3\pi/2-\Delta\}$ $(0<\Delta<\pi)$,

(3) $\lim\limits_{n\to+\infty} d(a_n, a_{n+1})<+\infty$, *$d(a, b)$ being the non-Euclidean distance between a and b.*

Under these conditions, $f(z)$ is bounded in the domain bounded by Γ_1, Γ_3, and $|z-1|=\varepsilon$ (ε: a sufficiently small positive constant), where Γ_3 is the Jordan arc terminating at $z_0=1$ and composed of the segments connecting a_n and a_{n+1}.

To establish theorem 5, we need some lemmas.

LEMMA 5. ([14], p. 99) *Let $f(z)$ be regular and of bounded characteristic in D:*

$$1/2\pi\cdot\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta<M<+\infty \qquad (0\leqq r<1)\,.$$

Then

$$\log|f(z)|\leqq 2M\frac{1+R}{1-R}-\frac{1-R}{1+R}\cdot 1/2\pi\cdot\int_0^{2\pi}|\log|f(e^{i\theta})||\,d\theta$$

for $|z|\leqq R<1$.

LEMMA 6. ([14], p. 100) *Let $\{f_n(z)\}$ $(n=1, 2, \cdots)$ be a sequence of regular functions in D. The necessary and sufficient condition for $\{f_n(z)\}$ to be of uniformly bounded characteristic:*

$$1/2\pi\cdot\int_0^{2\pi}\log^+|f_n(re^{i\theta})|\,d\theta<M<+\infty\,,$$

where $0\leqq r<1$, M: a constant independent of n, is the existence of a sequence of positive harmonic functions $\{u_n(z)\}$ such that

(i) $\log^+|f_n(z)|\leqq u_n(z)$ *in* $|z|<1$,

(ii) $u_n(0)<M<+\infty$.

Proof of theorem 5. Let us put

$$f_n(w)=f\Big(\frac{w+a_n}{1+\bar{a}_n w}\Big)\,, \qquad h_n(w)=h\Big(\frac{w+a_n}{1+\bar{a}_n w}\Big)\,.$$

Then

$$\log^+|f_n(w)|\leqq h_n(w)\,, \qquad h_n(0)=h(a_n)<M<+\infty\,. \tag{5.4}$$

By (5.4) and lemma 6, the sequence of regular functions $\{f_n(w)\}$ in $|w|<1$ is of uniformly bounded characteristic:

$$(5.5)\qquad 1/2\pi\cdot\int_0^{2\pi}\log^+|f_n(Re^{i\varphi})|\,d\varphi<M<+\infty \qquad (n=1, 2, \cdots).$$

Hence, by (5.5) and lemma 5,

$$(5.6)\qquad \log|f_n(w)|\leqq 2M\cdot\frac{1+R}{1-R}-\frac{1-R}{1+R}\cdot 1/2\pi\cdot\int_0^{2\pi}|\log|f_n(e^{i\varphi})||\,d\varphi$$

$$<2M\cdot\frac{1+R}{1-R}.$$

Since $|w|\leqq R$ is mapped onto $D(a_n, \rho)=\{z\,;\, d(a_n, z)\leqq\rho\}$ $(\rho=\tanh^{-1}R)$ by the linear transformation: $z=\dfrac{w+a_n}{1+\bar{a}_n w}$, by (5.6) $f(z)$ is uniformly bounded in the sequence of non-Euclidean disks: $\{D(a_n, \rho)\}$. Therefore, if we take ρ so large that

$$\overline{\lim_{n\to+\infty}}\, d(a_n, a_{n+1})<\rho<+\infty\,,$$

then $f(z)$ is bounded on Γ_3. Hence, by theorem 1, $f(z)$ is also bounded in the domain bounded by Γ_1, Γ_3 and $|z-1|=\varepsilon$, which is to be proved.

By what is proved above, the next theorem is also obtained;

THEOREM 5*. *Let $f(z)$ be regular and of bounded characteristic in D. If $h(a_n)$ $(n=1, 2, \cdots;\, |a_n|<1)$ is bounded, then $f(z)$ is uniformly bounded in $\bigcup_n D(a_n, \rho)$ for any positive finite ρ.*

6. Theorems of P. Montel-type.

As an application of theorem 5, we can prove the following theorem of P. Montel-type, which generalizes a theorem in the preceding paper ([14], p. 99).

THEOREM 6. *Let $f(z)$ be regular and of bounded characteristic in D. Suppose that $f(z)$ tends to a finite value α as $z\to z_0=1$ along Γ_1. If the sequence of $\{h(a_n)\}$ $(n=1, 2, \cdots)$ is bounded, where $|a_n|<1$, $\lim_{n\to+\infty} a_n=1$, $\arg(1-a_n)=\psi$, $|\psi|<\pi/2$, $(n=1, 2, \cdots)$ and $\lim_{n\to+\infty} d(a_n, a_{n+1})<+\infty$, then $f(z)$ tends uniformly to α as $z\to z_0=1$ inside $\mathfrak{D}_\Delta$, Δ being any positive constant less than $\pi/2-|\psi|$.*

To establish theorem 6, we begin with

LEMMA 7. *Let L be the chord connecting $z=1$ and $z=e^{i2\varphi}$ $(0<\varphi\leqq\pi/2)$, and l be the segment on L with end points z_1 and z_2. If the non-Euclidean distance between z_1 and z_2 is finite:*

$$d(z_1, z_2)<k<+\infty\,,$$

then there exists a constant K depending upon only k such that

$$\int_{z\in l}\frac{|dz|}{1-|z|^2}<K\cdot d(z_1, z_2).$$

Proof. Put

$$w(z)=\frac{z-z_1}{1-\bar{z}_1 z}. \tag{6.1}$$

The chord L is mapped onto the circular arc A in w-pl. passing through three points: $w(1)=e^{i(\pi+2\varphi)}$, $w(z_1)=0$, $w(e^{i2\varphi})=-1$.

Since the non-Euclidean metric is invariant under the transformation (6.1), and the inequality: $d(z_1, z_2)<k<+\infty$ is equivalent to the inequality:

$$\left|\frac{z_2-z_1}{1-\bar{z}_1 z_2}\right|<\tanh k,$$

we have

$$|w(z_2)|<\tanh k, \qquad d(z_1, z_2)=\int_C\frac{|dw|}{1-|w|^2}, \tag{6.2}$$

where C is the segment connecting $w=0$ and $w=w(z_2)$. Let us denote by l^* the circular arc on A with end points $w=0$ and $w(z_2)$, which is the image of l under the transformation (6.1). Then, by (6.2) we have

$$\begin{aligned}&\int_{z\in l}\frac{|dz|}{1-|z|^2}\\ &=\int_{w\in l^*}\frac{|dw|}{1-|w|^2}<\frac{1}{1-(\tanh k)^2}\cdot\int_{w\in l^*}|dw|\\ &<\frac{1}{1-(\tanh k)^2}\cdot\frac{\pi}{2}\cdot\int_{w\in C}|dw|^{(*)}<\frac{1}{1-(\tanh k)^2}\cdot\frac{\pi}{2}\cdot\int_{w\in C}\frac{|dw|}{1-|w|^2}.\end{aligned}$$

Putting $K=\frac{1}{1-(\tanh k)^2}\cdot\frac{\pi}{2}$, we have

$$\int_{z\in l}\frac{|dz|}{1-|z|^2}<K\cdot d(z_1, z_2),$$

which is to be proved.

LEMMA 8. *Let us define the non-Euclidean circle:*

$$d(z, a)=\rho \qquad (0<\rho<+\infty), \tag{6.3}$$

where a lines on the chord L connecting $z=1$ and $z=e^{i2\varphi}$ $\left(0<\varphi\leqq\frac{\pi}{2}\right)$. If a varies along L from $z=1$ to $z=e^{i2\varphi}$, then the envelope of (6.3) is composed of two circular arcs, which connect $z=1$ and $z=e^{i2\varphi}$ and make the angle:

$$2\cdot\tan^{-1}\left(\sin\varphi\cdot\frac{2r}{1-r^2}\right), \qquad r=\tanh\rho$$

(*) The length of the circular arc is less than its chordal length multiplied by $\pi/2$.

at $z=1$.

Proof. Put

$$w(z)=\frac{z-a}{1-\bar{a}z}. \tag{6.4}$$

(6.3) is equivalent to $\left|\frac{z-a}{1-\bar{a}z}\right|=r$ ($r=\tanh\rho$), so that (6.3) is mapped onto the fixed circle: $|w|=r$. By (6.4) L is mapped onto the fixed circular arc passing through three fixed points:

$$w(1)=e^{i(\pi+2\varphi)}, \qquad w(a)=0, \qquad w(e^{i2\varphi})=-1.$$

Hence the fixed two circular arcs A_i ($i=1,2$) which pass through $w(1)=e^{i(\pi+2\varphi)}$ and $w(e^{i2\varphi})=-1$, and touch the circle: $|w|=r$, are evidently the image of the envelope of (6.3) as a varies along L from $z=1$ to $z=e^{i2\varphi}$. By the simple calculation, we can show that two arcs A_i ($i=1,2$) make the angle: $2\tan^{-1}\left(\sin\varphi\cdot\frac{2r}{1-r^2}\right)$ at $w=e^{i(\pi+2\varphi)}$, from which our lemma easily follows.

LEMMA 9. *Denote by* $\{a_n\}$ ($|a_n|<1$) *the sequence of points such that*

$$\lim_{n\to+\infty} a_n=1, \quad \arg(1-a_n)=-\vartheta, \quad 0\leqq\vartheta<\pi/2, \quad \lim_{n\to+\infty} d(a_n, a_{n+1})<+\infty.$$

Put

$$D(\rho)=\bigcup_{n\geqq 1} D(a_n, \rho),$$

where $D(a_n, \rho)=\{z\,;\, d(z, a_n)\leqq\rho\}$. *If* ρ *is sufficiently large, then* $D(\rho)$ *covers the fixed Stolz-domain with vertex at* $z=1$ *in the neighborhood of* $z=1$.

Proof. By the assumptions, there exists a positive finite constant k such that

$$d(a_n, a_{n+1})<k \qquad (n=1, 2, \cdots). \tag{6.5}$$

Denote by a any point on the segment l_n connecting a_n and a_{n+1}, and by b any boundary point of the domain: $D(a_n, \rho)\cup D(a_{n+1}, \rho)$ respectively. By (6.5) and lemma 7,

$$\begin{aligned} d(a, b) &> \rho-\max\{d(a_n, a), d(a_{n+1}, a)\} > \rho-\int_{z\in l_n}\frac{|dz|}{1-|z|^2} > \rho-0(k) \\ &=\rho(1-o(1)), \end{aligned}$$

so that, for any given $\varepsilon>0$, there exists $\rho(\varepsilon)$ independent of n such that

$$d(a, b)>\rho(1-\varepsilon) \qquad \text{for} \qquad \rho\geqq\rho(\varepsilon).$$

Hence

$$D(\rho)\supset \bigcup_{\substack{|a-1|\leqq|a_1-1| \\ a\in L}} D(a, \rho(1-\varepsilon)), \tag{6.6}$$

where L is the chord connecting $z=1$ and $z=e^{i(\pi-2\vartheta)}$. Since

$$2\cdot\tan^{-1}\left(\cos\vartheta\cdot\frac{2r}{1-r^2}\right)\longrightarrow\pi \qquad \text{as} \quad \rho\to+\infty \ (r=\tanh(\rho(1-\varepsilon)))\,,$$

by (6.6) and lemma 8, $D(\rho)$ covers the fixed Stolz-domain with vertex at $z=1$ in the neighborhood of $z=1$, provided that ρ is sufficiently large. Thus our lemma is completely established.

Now we can prove theorem 6.

Proof of theorem 6. We first assume that $-\pi/2<\phi\leqq 0$. Putting $0<\varDelta'<\varDelta<\frac{\pi}{2}-|\phi|$, if ρ is sufficiently large, then by lemma 9, $D(\rho)$ covers the Stolz-domain with vertex at $z=1$:

$$\{z;\ |\arg(z-1)|\leqq\pi/2-\varDelta'\}\frown U(1,\varepsilon)\,,$$

ε being a small positive constant. Therefore, by theorem 5 and 5*, $f(z)$ is bounded in $\mathfrak{D}_{\varDelta'}\frown U(1,\varepsilon)$, so that by classical P. Montel's theorem $f(z)$ tends uniformly to α as $z\to 1$ inside $\mathfrak{D}_{\varDelta}\subset\mathfrak{D}_{\varDelta'}$.

The case: $0<\phi<\pi/2$ also can be treated by similar arguments.

Next we shall sharpen theorem 6 as follows.

THEOREM 7. *Let $f(z)$ be regular and of bounded characteristic in D. Suppose that there exists the measurable set E contained in $(0,+\delta)$ $(\delta>0)$ such that*

(6.7)
(1) *$f(e^{i\varphi})$ tends to a finite value α as $\varphi\to+0$ along E.*
(2) *the lower density $\lambda^{(*)}$ of E at $\varphi=0$ is positive.*

Furthermore, if the sequence of $\{h(a_n)\}$ $(n=1,2,\cdots)$ is bounded, where

$$|a_n|<1\,,\quad \lim_{n\to+\infty}a_n=1\,,\quad \arg(1-a_n)=\phi \qquad (|\phi|<\pi/2)$$

and

$$\overline{\lim_{n\to+\infty}}\, d(a_n, a_{n+1})<+\infty\,,$$

then $f(z)$ tends uniformly to α as $z\to 1$ inside the Stolz-domain with vertex at $z=1$.

Remark. (1) In the case of the bounded regular function in D, this theorem has been proved by M. L. Cartwright [3] and Y. Kawakami [6] independently. (2) Theorem 7 is a generalization of the theorem due to the author ([14], p. 98).

We begin with

LEMMA 10. *Let E be the measurable set contained in the upper arc:*

(*) $\lambda=\underline{\lim}_{\varphi\to+0}\, 1/\varphi\cdot m(E\frown 0,\varphi))$.

$A(e^{i\theta};\ 0<\theta<\delta)$ which has the positive lower density λ at $\theta=0$;

$$\varliminf_{t\to 0} 1/t\cdot mE(t)=\lambda>0\,,$$

where $E(t)=E\frown(e^{i\theta};\, 0<\theta<t)$. Let $E(w, a)$ be the image of E_a by the linear transformation: $w=\dfrac{z-a}{1-\bar{a}z}$ ($|a|<1$), where E_a is defined as follows:

$$\begin{aligned} E_a&=E(2\varphi) && \text{if} \quad \arg a=\varphi>0\,, \\ &=E(1-|a|) && \text{if} \quad \arg a=0\,, \\ &=E(\varphi) && \text{if} \quad \arg a=-\varphi<0\,. \end{aligned}$$

If a tends to 1 along the fixed chord: $\arg(1-a)=\phi$ ($|\phi|<\pi/2$), then

$$\varliminf_{\substack{a\to 1 \\ \arg(1-a)=\phi}} mE(w, a)\geqq\lambda\cdot k(\phi)\,,$$

where $k(\phi)$ is a positive constant dependent upon only ϕ.

Proof. We have easily

$$mE(w, a)=\int_{E_a}\frac{1-|a|^2}{|1-\bar{a}z|^2}|dz|\,. \tag{6.8}$$

We distinguish three cases: $-\pi/2<\phi<0$, $\phi=0$, $0<\phi<\pi/2$.

(*1*) First we assume that $-\pi/2<\phi<0$. Put

$$a=|a|\cdot e^{i\varphi} \quad (\varphi>0)\,, \qquad z=\exp(i(1+\alpha)\varphi) \quad (-1\leqq\alpha\leqq+1)\,.$$

Then

$$\begin{aligned} |dw|&=\frac{1-|a|^2}{|1-\bar{a}z|^2}|dz|=\frac{1-|a|^2}{|e^{-i\alpha\varphi}-|a||^2}\cdot\varphi|d\alpha| \\ &\geqq\frac{1-|a|^2}{|e^{i\varphi}-|a||^2}\cdot\varphi|d\alpha|\,. \end{aligned}$$

Hence, by (6.8)

$$mE(w, a)=\int_{E(2\varphi)}\frac{1-|a|^2}{|1-\bar{a}z|^2}|dz|\geqq\frac{(1-|a|^2)}{|e^{i\varphi}-|a||^2}\cdot\int_{z\in E(2\varphi)}\varphi|d\alpha|\,. \tag{6.9}$$

By the definition of λ, for any positive ε, there exists $t(\varepsilon)$ such that

$$mE(t)>t(\lambda-\varepsilon) \qquad \text{for} \quad 0<t<t(\varepsilon)\,,$$

so that

$$\int_{z\in E(2\varphi)}\varphi|d\alpha|>2\varphi(\lambda-\varepsilon) \qquad \text{for} \quad 0<\varphi<t(\varepsilon)/2\,.$$

Therefore, by (6.9)

(6.10) $$mE(w, a) \geqq \frac{(1-|a|^2)\varphi}{|e^{i\varphi}-|a||^2} \cdot 2(\lambda-\varepsilon) \qquad \text{for} \quad 0<\varphi<1/2 \cdot t(\varepsilon).$$

By the sin-rule, we have

$$\frac{\varphi}{1-|a|} = \frac{\sin(\varphi+|\phi|)}{\sin\left(\dfrac{\pi-\varphi}{2}-|\phi|\right)} \cdot \frac{\varphi}{|1-e^{i\varphi}|},$$

so that

(6.11) $$\lim_{\varphi\to+0} \frac{\varphi}{1-|a|} = \tan(|\phi|).$$

Since

$$\frac{(1-|a|^2)\varphi}{|e^{i\varphi}-|a||^2} = (1+|a|)\frac{\varphi}{1-|a|} \cdot \frac{1}{\left|1+\imath\dfrac{\varphi}{1-|a|}+o(1)\right|^2}$$

by (6.11), we have by (6.10)

$$\varliminf_{\substack{a\to1 \\ \arg(1-a)=\phi}} mE(w, a) \geqq 2\cdot\sin(2|\phi|)\cdot(\lambda-\varepsilon).$$

Letting $\varepsilon\to+0$,

(6.12) $$\varliminf_{\substack{a\to1 \\ \arg(1-a)=\phi}} mE(w, a) \geqq 2\cdot\sin(2|\phi|)\cdot\lambda,$$

(*2*) Next we assume that $\phi=0$. In this case, put

$$\varphi=(1-|a|), \quad z=\exp(\imath\alpha\varphi) \qquad (0\leqq\alpha\leqq1).$$

By the slight modification of the above arguments, we get easily

(6.13) $$\varliminf_{a\to1} mE(w, a) \geqq \lambda.$$

(*3*) Finally we assume that $0<\phi<\pi/2$. Put

$$a=|a|e^{-\imath\varphi} \quad (\varphi>0), \qquad z=\exp(i(-1+\alpha)\varphi) \quad (1\leqq\alpha\leqq2).$$

By the similar arguments as in (6.9) and (6.10), we have

(6.14) $$mE(w, a) = \int_{E(\varphi)} \frac{1-|a|^2}{|1-\bar{a}z|^2}|dz| \geqq \frac{(1-|a|^2)\varphi}{|e^{i2\varphi}-|a||^2}\cdot(\lambda-\varepsilon)$$

for $0<\varphi<t(\varepsilon)$. Since

$$\frac{\varphi}{1-|a|} \longrightarrow \tan\phi \quad \text{as} \quad a\to1 \qquad \text{along } \arg(1-a)=\phi,$$

we have

$$\frac{(1-|a|^2)\varphi}{|e^{i2\varphi}-|a||^2} = (1+|a|)\cdot\frac{\varphi}{1-|a|}\cdot\frac{1}{\left|1+2\imath\dfrac{\varphi}{1-|a|}+o(1)\right|^2} \longrightarrow \frac{2\tan\phi}{1+(2\tan\phi)^2}.$$

Hence, by (6.14) and letting $\varepsilon \to +0$,

$$\varliminf_{\substack{a\to 1\\ \arg(1-a)=\phi}} mE(w, a) \geqq \frac{2\tan\phi}{1+(2\tan\phi)^2}\cdot\lambda. \tag{6.15}$$

Thus, by (6.12), (6.13) and (6.15), our lemma is completely established.

Proof of theorem 7. By lemma 10, there exists a sufficiently large integer N such that

$$mE(w, a_n) > \lambda/2\cdot k(\phi) \qquad \text{for} \quad n \geqq N. \tag{6.16}$$

By the linear transformation: $w(z)=\dfrac{z-a_n}{1-\bar{a}_n z}$, E_{a_n} is mapped onto $E(w, a_n)$ which has the fixed limit point: $w(1)=e^{i2\phi}$. Hence, by (6.16) and (6.7) (1°), we can find a fixed set E^* ($mE^*>0$) lying on $|w|=1$ and a sequence of integers $\{n_k\}$ such that

$$\begin{aligned} &(1)\quad \varlimsup_{k\to+\infty}(n_{k+1}-n_k)<+\infty, \\ &(2)\quad E^*\subset E(w, a_{n_k})\ (k=1, 2, \cdots), \\ &(3)\quad f_{n_k}(w)\to\alpha \ \text{for}\ w\in E^* \ \text{as}\ k\to+\infty, \end{aligned} \tag{6.17}$$

where $f_n(w)=f\left(\dfrac{w+a_n}{1+\bar{a}_n w}\right)$. Putting $h_n(w)=h\left(\dfrac{w+a_n}{1+\bar{a}_n w}\right)$, by the assumptions we have

$$\log^+|f_{n_k}(w)| \leqq h_{n_k}(w) \qquad \text{for} \quad |w|<1,$$

$$h_{n_k}(0)=h(a_{n_k})<M<+\infty.$$

Hence, by lemma 6, the sequence of regular functions $\{f_{n_k}(w)\}$ is of uniformly bounded characteristic;

$$1/2\pi\cdot\int_0^{2\pi}\log^+|f_{n_k}(Re^{i\theta})|\,d\theta<M<+\infty \qquad (0\leqq R<1,\ k=1, 2, \cdots). \tag{6.18}$$

Setting $F_k(w)=f_{n_k}(w)-\alpha$, we have

$$1/2\pi\cdot\int_0^{2\pi}\log^+|F_k(Re^{i\theta})|\,d\theta \leqq M+\log^+|\alpha|+\log 2=M^*<+\infty.$$

Therefore, by lemma 5, for $|w|\leqq R<1$,

$$\begin{aligned} \log|F_k(w)| &\leqq 2M^*\cdot\frac{1+R}{1-R}-\frac{1-R}{1+R}\cdot 1/2\pi\cdot\int_0^{2\pi}|\log|F_k(e^{i\theta})||\,d\theta \\ &\leqq 2M^*\cdot\frac{1+R}{1-R}-\frac{1-R}{1+R}\cdot 1/2\pi\cdot\int_{E^*}|\log|F_k(e^{i\theta})||\,d\theta. \end{aligned} \tag{6.19}$$

By (6.17) (3) and (6.19), the sequence $\{f_{n_k}(w)\}$ tends to α uniformly in $|w|\leqq R<1$ as $k\to+\infty$. Since $|w|\leqq R$ is mapped onto $D(a_{n_k}, \rho)$ ($\rho=\tanh^{-1}R$) by $z=\dfrac{w+a_n}{1+\bar{a}_n w}$,

$f(z)$ tends uniformly to α as z tends to 1 inside the sequence of non-Euclidean circles: $D(a_{n_k}, \rho)$ $(k=1, 2, \cdots)$. By (6.17) (1) and $\overline{\lim_{n\to+\infty}} d(a_n, a_{n+1})<+\infty$, we have

$$\overline{\lim_{k\to+\infty}} d(a_{n_k}, a_{n_{k+1}})<+\infty .$$

Hence, by lemma 9, $f(z)$ tends uniformly to α as $z\to 1$ inside the Stolz-domain with vertex at $z=1$, which is to be proved.

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