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ON ABSOLUTE RIESZ SUMMABILITY FACTORS OF FOURIFR SERIES

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1. Definitions and Notations.

Let $\sum u_n$ be an infinite series and let $0=\lambda_0<\lambda_1<\cdots<\lambda_n\to\infty$. For $k\geqq 0$, we write

$$R^k(\lambda_m)=\sum_{n=1}^{m-1}\left(1-\frac{\lambda_n}{\lambda_m}\right)^k u_n .$$

The series $\sum u_n$ is said to be absolutely summable by Riesz discrete method (R^*, λ_n, k), or summable $|R^*, \lambda_n, k|$, if $\{R^k(\lambda_n)\}$ is of bounded variation. When the 'order' of summation $k=1$, the method of summation is equivalent to the usually known Riesz method $|R, \lambda_n, 1|$. In this specialcase the method is also sometimes known as the method $|R, \mu_n|$, or the method $|\bar{N}, \mu_n|$, where $\{\mu_n\}=\{\lambda_n-\lambda_{n-1}\}$. In this paper we are concerned with this special case and we will denote the method of summation by $|R, \lambda_n, 1|$. Thus the series $\sum u_n$ is summable $|R, \lambda_n, 1|$ if

$$\sum_1^\infty |\Delta R^1(\lambda_m)|=\sum_{m=1}^\infty\left(\frac{1}{\lambda_m}-\frac{1}{\lambda_{m+1}}\right)\left|\sum_{n=0}^m \lambda_n u_n\right|<\infty .$$

When $\{\lambda_n\}=\{n\}$, the method is the same as the Cesàro method $|C, 1|$, and when $\{\lambda_n\}=\{\log n\}$ the method is called the logarithmic method.

Let $f(t)$ be a Lebesgue integrable 2π-periodic function and let the Fourier series of $f(t)$ be given by

$$f(t)\sim\frac{a_0}{2}+\sum_1^\infty(a_n\cos nt+b_n\sin nt)=\sum_0^\infty A_n(t) .$$

We set

$$\phi(t)=\frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\} ,$$

and throughout the paper we write to denote

$$\Phi(t)=\int_0^t|\phi(u)|\,du ,$$

$$p_n=\int_{1/n}^\pi\frac{|\phi(u)|\,du}{u} ,$$

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$$D_n(t)=\frac{1}{2}+\cos t+\cos 2t+\cdots+\cos nt\,,$$

$$E_n(t)=\sum_0^n \lambda_\nu \cos \nu t\,,$$

$$F_n(t)=\sum_0^n \varepsilon_\nu\lambda_\nu \cos \nu t\,,$$

$$S_n(t)=\sum_0^n A_\nu(t)\,,$$

$$h_n=\begin{cases} (\log n)^{\beta+\frac{1}{2}}\,, & \text{when } \beta>-\frac{1}{2} \\ (\log\log n)^{\frac{1}{2}}\,, & \text{when } \beta=-\frac{1}{2} \\ 1\,, & \text{when } \beta<-\frac{1}{2} \end{cases}$$

$$k_n=\begin{cases} (\log n)^{\beta+1}\,, & \text{when } \beta>-1 \\ \log\log n\,, & \text{when } \beta=-1 \\ 1\,, & \text{when } \beta<-1\,. \end{cases}$$

$K, K_1 \cdots$ denote absolute constants, not necessarily the same, at different occurrences.

2. Theorems.

The object of this paper is to establish some general results concerning absolute summability factors for Fourier series at a point. Actually we prove the following theorems:

THEOREM 1. *Let $\psi(t)$ be a positive function and let*

$$\Phi(t)=O(t\psi(1/t))\,, \qquad \textit{as} \quad t\to 0\,,$$

and

$$\psi_n=\psi(n)\,.$$

If $\{\varepsilon_n\}$ is such that

(a) $\sum \frac{\mu_{n+1}}{\lambda_{n+1}}|\varepsilon_n|\psi_n<\infty$, $\sum_1^\infty |\Delta\varepsilon_\nu|\lambda_\nu \sum_{\nu+1}^\infty \psi_n \Delta\left(\frac{1}{\lambda_n}\right)<\infty$,

(b) $\sum_1^\infty |\Delta\varepsilon_\nu|\lambda_\nu \sum_{\nu+1}^\infty p_n \Delta\left(\frac{1}{\lambda_n}\right)<\infty$

and

(c) $\sum_1^\infty \Delta\left(\frac{1}{\lambda_n}\right)\left|\varepsilon_n \sum_1^n \mu_\nu \int_{1/n}^\pi \phi(t) D_{\nu-1}(t)\,dt\right|<\infty$

then $\sum \varepsilon_n A_n(t)$, at $t=x$, is summable $|R, \lambda_n, 1|$, if, and only if $\sum \frac{\mu_{n+1}}{\lambda_{n+1}}|\varepsilon_n|\,|S_n(x)-f(x)|<\infty$.

THEOREM 2. *Let β be a real number, and*

$$\Phi(t)=O\left(t\left(\log\frac{2\pi}{t}\right)^{\beta}\right), \qquad as \quad t\to 0.$$

If $\{\varepsilon_n\}$ is such that

(i) $\sum\frac{|\varepsilon_n|(\log n)^{\beta}}{n}<\infty$

and

(ii) $\sum k_n|\Delta\varepsilon_n|<\infty$

then $\sum\varepsilon_n A_n(t)$, at $t=x$, is summable $|C,1|$, if, and only if

$$\sum\frac{|\varepsilon_n|}{n}|S_n(x)-f(x)|<\infty.$$

THEOREM 3. *Let β be a real number, and*

$$\Phi(t)=O\left(t\left(\log\frac{2\pi}{t}\right)^{\beta}\right), \qquad as \quad t\to 0.$$

If $\{\varepsilon_n\}$ is such that

(i) $\sum\frac{|\varepsilon_n|}{n(\log n)^{1-\beta}}<\infty$

and

(ii) $\sum k_n|\Delta\varepsilon_n|<\infty$,

then $\sum\varepsilon_n A_n(t)$, at $t=x$, is summable $|R,\log n,1|$, if, and only if

$$\sum\frac{\varepsilon_n}{n\log n}|S_n(x)-f(x)|<\infty.$$

THEOREM 4. *Let β be any real number and*

$$\Phi(t)=O\left(t\left(\log\frac{2\pi}{t}\right)^{\beta}\right), \qquad as \quad t\to 0,$$

The series $\sum\varepsilon_n A_n(t)$, at $t=x$, is

(I) *summable $|C,1|$ if $\{\varepsilon_n\}$ is such that*

(i) $\{\varepsilon_n h_n\}\in B$, (ii) $\sum\frac{|\varepsilon_n|h_n}{n}<\infty$ *and* (iii) $\sum k_n|\Delta\varepsilon_n|<\infty$;

(II) *summable $|R,\log n,1|$, if $\{\varepsilon_n\}$ is such that*

(i) $\left\{\frac{\varepsilon_n h_n}{\log n}\right\}\in B$, (ii) $\sum\frac{|\varepsilon_n|h_n}{n\log n}<\infty$ *and* (iii) $\sum k_n|\Delta\varepsilon_n|<\infty$.

3. Remarks.

1. For some of the existing results in this direction one is referred to Cheng [2], Prasad and Bhatt [9], Matsumoto [7], Pati [8], Liu [6], Hsiang [5] and S.L. Wang [10]. The special case $\beta=0$ of Theorem 2 was discussed by Pati [8]. Even for the special case, our theorem furnishes more general results than one due to Pati [8].

   A theorem on the lines of Theorem 4 for general sequences $\{\lambda_n\}$ can be deduced from Theorem 1 and it can easily be verified that for corresponding logarithmico-exponential sequences dealt with in a known theorem due to Matsumoto [7, Theorem 2] here one gets more general results.

2. It has been proved elsewhere [3, Lemma 4] that a sufficient condition for summability $|R, \lambda_n, 1|$ of a series $\sum u_n$ is the convergence of $\sum \frac{\mu_n}{\lambda_n}|s_n|$, $s_n=\sum_0^n u_k$. From Theorem 3 we deduce (cf. Corollary to Theorem 3) that when $\lambda_n=\log n$, for the summability of the Fourier series of $f(x)$ at the point $t=x$, given by $\Phi(t)=O\left(t\left(\log \frac{1}{t}\right)^{-\eta}\right)$, as $t\to 0$, $\eta>0$, this condition is also necessary.

3. The case $\beta\geqq 0$ of Theorem 4 (I), on summability $|C, 1|$, has been studied by Prasad and Bhatt [9]. The summability factors studied in the present paper are sharper than those known before. The case $-1\leqq\beta\leqq 0$ of the same part has been discussed by Liu [6] whose results are again generalised here. Hsiang [5] rediscovered the case $\beta=-1$ and his result seems to be included in that of Liu [6].

4. Lemmas.

We shall need the following results towards the proof of our theorems.

LEMMA 1.

(i) $$D_n(t)=\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} =\begin{cases} O(n), \\ O\left(\frac{1}{t}\right), & t\in\left[\frac{1}{n}, \pi\right]; \end{cases}$$

(ii) $$E_n(t)=\lambda_n D_n(t)-\sum_1^n \mu_\nu D_{\nu-1}(t) =\begin{cases} O(n\lambda_n) \\ O(t^{-1}\lambda_n), & t\in\left[\frac{1}{n}, \pi\right]; \end{cases}$$

(iii) $$F_n(t)=\varepsilon_n\lambda_n D_n(t)-\varepsilon_n\sum_1^n \mu_\nu D_{\nu-1}(t)+\sum_0^{n-1} E_\nu(t)(\Delta\varepsilon_\nu) =O(n\lambda_n|\varepsilon_n|)+O\left(\sum_0^{n-1}\nu\lambda_\nu|\Delta\varepsilon_\nu|\right).$$

These results are rather obvious.

LEMMA 2. *Methods of summation $|R, \log n, 1|$ and $|R, (\log n)^r, 1|$, $r>0$, are equivalent.*

This result can be deduced as a very special case of well-known results on "Second Theorems of consistency for absolute Riesz summability" (cf. [1]).

LEMMA 3. *If* $\Phi(t)=O\left(t\left(\log\frac{2\pi}{t}\right)^{\beta}\right)$, *as* $t\to 0$, *then*

$$\sum_{0}^{n}|S_{\nu}(x)-f(x)|=O(nh_n).$$

The case $\beta=0$ is essentially due to Hardy and Littlewood [4] and is also known for $\beta\geqq 0$ (Cheng [2]). For the general case we proceed as follows.

Proof of Lemma 3. As

$$S_{\nu}(x)-f(x)=\frac{2}{\pi}\int_{0}^{\pi}\phi(t)\frac{\sin\nu t}{t}+O(1),$$

$$|S_{\nu}(x)-f(x)|=\frac{2\nu}{\pi}\int_{0}^{1/n}|\phi(t)|\,dt+\frac{2}{\pi}\left|\int_{1/n}^{\pi}\phi(t)\frac{\sin\nu t}{t}dt\right|+O(1)$$

$$=O(\log n)^{\beta}+\frac{2}{\pi}\left|\int_{1/n}^{\pi}\phi(t)\frac{\sin\nu t}{t}dt\right|+O(1),\qquad n\geqq\nu,$$

and therefore

$$\sum_{0}^{n}|S_{\nu}(x)-f(x)|^{2}$$

$$\leqq K\sum_{0}^{n}\left\{\int_{1/n}^{\pi}\phi(t)\frac{\sin\nu t}{t}dt\right\}^{2}+O(n(\log n)^{2\beta})+O(n)$$

$$=K\int_{1/n}^{\pi}\int_{1/n}^{\pi}\frac{\phi(t)}{t}\cdot\frac{\phi(u)}{u}\left(\sum_{0}^{n}\sin\nu t\sin\nu u\right)dt\,du+O(n(\log n)^{2\beta})+O(n)$$

$$=KT_n+O(n(\log n)^{2\beta})+O(n),\qquad\text{say},$$

where

$$2T_n=\int_{1/n}^{\pi}\int_{1/n}^{\pi}\frac{\phi(t)}{t}\frac{\phi(u)}{u}\left\{\frac{\sin\left(n+\frac{1}{2}\right)(u-t)}{2\sin\frac{1}{2}(u-t)}-\frac{\sin\left(n+\frac{1}{2}\right)(u+t)}{2\sin\frac{1}{2}(u+t)}\right\}dt\,du$$

$$=\int_{1/n}^{\pi}\int_{1/n}^{\pi}\frac{\phi(t)}{t}\frac{\phi(u)}{u}\frac{\sin n(u-t)}{(u-t)}dt\,du$$

$$-\int_{1/n}^{\pi}\int_{1/n}^{\pi}\frac{\phi(t)}{t}\frac{\phi(u)}{u}\frac{\sin n(u+t)}{(u+t)}dt\,du$$

$$+O\left(\int_{1/n}^{\pi}\int_{1/n}^{\pi}\frac{|\phi(t)|}{t}\frac{|\phi(u)|}{u}dt\,du\right)$$

$$=T_n'-T_n''+O(T_n'''),\qquad\text{say}.$$

As

$$\int_{1/n}^{\pi}\frac{|\phi(t)|}{t}dt=\left[\frac{\Phi(t)}{t}\right]_{1/n}^{\pi}+\int_{1/n}^{\pi}\frac{\Phi(t)}{t^{2}}dt$$

$$=\begin{cases} O(\log n)^{\beta+1}+O(\log\log n)\,, & \beta\geqq -1\,,\\ O(1)\,, \quad \beta<-1\,, \end{cases}$$

$$T_n'''=\begin{cases} O((\log n)^{2+2\beta})+O((\log\log n)^2)\,, & \beta\geqq -1\,,\\ O(1)\,, \quad \beta<-1\,. \end{cases}$$

Also

$$|T_n'|\leqq\left|\int_{1/n}^{\pi}\frac{\phi(t)}{t}\int_{1/n}^{t}\frac{\phi(u)}{u}\,\frac{\sin n(t-u)}{(t-u)}\,dt\,du\right|$$

$$+\left|\int_{1/n}^{\pi}\frac{\phi(t)}{t}\int_{t}^{\pi}\frac{\phi(u)}{u}\,\frac{\sin n(u-t)}{(u-t)}\,du\,dt\right|$$

$$=\left|\int_{1/n}^{\pi}\frac{\phi(t)}{t}\int_{1/n}^{t}\frac{\phi(u)}{u}\,\frac{\sin n(t-u)}{(t-u)}\,dt\,du\right|$$

$$+\left|\int_{1/n}^{\pi}\frac{\phi(u)}{u}\int_{1/n}^{u}\frac{\phi(t)}{t}\,\frac{\sin n(u-t)}{(u-t)}\,du\,dt\right|$$

$$=2\left|\int_{1/n}^{\pi}\int_{1/n}^{t}\frac{\phi(t)}{t}\,\frac{\phi(u)}{u}\,\frac{\sin n(t-u)}{(t-u)}du\,dt\right|$$

$$=2\left|\int_{1/n}^{\pi}\frac{\phi(t)}{t^2}\int_{1/n}^{t}\frac{\phi(u)}{(t-u)}\sin n(t-u)du\,dt\right.$$

$$\left.+2\int_{1/n}^{\pi}\frac{\phi(t)}{t^2}\int_{1/n}^{t}\frac{\phi(u)}{n}\sin n(t-u)du\,dt\right|$$

$$\leqq 4n\int_{1/n}^{\pi}\frac{|\phi(t)|}{t^2}\int_{1/n}^{t}|\phi(u)|\,du\,dt$$

$$\leqq 4n\int_{1/n}^{\pi}\frac{|\phi(t)|}{t^2}\Phi(t)dt\,.$$

Therefore,

$$|T_n'|=O\Big(n\int_{1/n}^{\pi}\frac{|\phi(t)|}{t}\Big(\log\frac{2\pi}{t}\Big)^{\beta}dt\Big)$$

$$=O\Big(n\Big\{\Big[\Big(\log\frac{2\pi}{t}\Big)^{\beta}\frac{\Phi(t)}{t}\Big]_{1/n}^{\pi}-\int_{1/n}^{\pi}\Phi(t)d\Big|\frac{\Big(\log\frac{2\pi}{t}\Big)^{\beta}}{t}\Big|\Big\}\Big)$$

$$=O(n)+O(n(\log n)^{2\beta})+O\Big(n\int_{1/n}^{\pi}\frac{\Big(\log\frac{2\pi}{t}\Big)^{2\beta}}{t}dt\Big)$$

$$=O(nh_n^2)\,.$$

Similarly, we have

$$|T_n''|=\left|\int_{1/n}^{\pi}\int_{1/n}^{\pi}\frac{\phi(t)}{t}\,\frac{\phi(u)}{u}\,\frac{\sin n(t+u)}{(t+u)}\,du\,dt\right|$$

$$\leqq 2\left|\int_{1/n}^{\pi} \frac{\phi(t)}{t} \int_{1/n}^{t} \frac{\phi(u)}{u} \frac{\sin n(t+u)}{(t+u)} du\, dt\right|$$

$$\leqq 4n \int_{1/n}^{\pi} \frac{|\phi(t)|}{t^2} \Phi(t) dt$$

$$= O(nh_n^2) .$$

Thus

$$\sum_0^n |S_\nu(x)-f(x)|^2 = O(nh_n^2) \Longrightarrow \sum_0^n |S_\nu(x)-f(x)| = O(nh_n) .$$

5. Proof of Theorems.

5.1. *Proof of Theorem 1.* We have

$$A_\nu(x) = \frac{2}{\pi} \int_0^\pi \phi(t) \cos \nu t\, dt ,$$

$$S_n(x)-f(x) = \frac{2}{\pi} \int_0^\pi \phi(t) D_n(t) dt ,$$

and the series $\sum \varepsilon_n A_n(x)$ is summable $|R, \lambda_n, 1|$, iff

$$\sum_1^\infty \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right) \left|\sum_0^n \lambda_\nu \varepsilon_\nu A_\nu(x)\right| = \frac{2}{\pi} \sum_1^\infty \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right) \left|\int_0^\pi \phi(t) \sum_0^n \varepsilon_\nu \lambda_\nu \cos \nu t\, dt\right|$$

$$= \frac{2}{\pi} \sum_1^\infty \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right) \left|\int_0^\pi \phi(t) F_n(t) dt\right| < \infty .$$

Therefore, if

$$(*) \qquad S \equiv \sum_1^\infty \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right) \left|\int_0^\pi \phi(t) \{F_n(t) - \varepsilon_n \lambda_n D_n(t)\} dt\right| < K$$

then

$$\sum \frac{\mu_{n+1}}{\lambda_{n+1}} |\varepsilon_n \{S_n(x) - f(x)\}| < \infty$$

is a necessary and sufficient condition for $\sum \varepsilon_n A_n(x)$ to be summable $|R, \lambda_n, 1|$.

We now proceed to prove (*).

$$S \leqq \sum_1^\infty \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right) \left\{\int_0^{1/n} |\phi(t) F_n(t)|\, dt + \int_0^{1/n} |\varepsilon_n \lambda_n D_n(t) \phi(t)|\, dt\right.$$

$$\left. + \left|\int_{1/n}^\pi \phi(t)(F_n(t) - \varepsilon_n \lambda_n D_n(t))\, dt\right|\right\}$$

$$= I_1 + I_2 + I_3 , \qquad \text{say} .$$

After Lemma 1,

$$I_1 \leqq K_1 \sum_1^\infty \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right) |\varepsilon_n| \int_0^{1/n} |\phi(t) E_n(t)|\, dt$$

$$+K_2\sum_1^\infty\Big(\frac{1}{\lambda_n}-\frac{1}{\lambda_{n+1}}\Big)\Big(\sum_0^{n-1}\Big|(\Delta\varepsilon_\nu)\int_0^{1/n}\phi(t)E_\nu(t)dt\Big|\Big)$$

$$\leqq K_1\sum_1^\infty\frac{\mu_{n+1}}{\lambda_{n+1}}n|\varepsilon_n|\Phi(1/n)+K_2\sum_{\nu=0}^\infty\nu\lambda_\nu|\Delta\varepsilon_\nu|\sum_{n=\nu+1}^\infty\Phi(1/n)\Big(\frac{1}{\lambda_n}-\frac{1}{\lambda_{n+1}}\Big)$$

$$\leqq K_3+K_4\sum_{\nu=0}^\infty\lambda_\nu|\Delta\varepsilon_\nu|\sum_{\nu+1}^\infty\psi_n\Delta(1/\lambda_n)$$

$$<K;$$

$$I_2\leqq K_1\sum_0^\infty\frac{\mu_{n+1}}{\lambda_{n+1}}|\varepsilon_n|\psi_n<K;$$

and

$$I_3\leqq K_1\sum_{n=1}^\infty\Big(\frac{1}{\lambda_n}-\frac{1}{\lambda_{n+1}}\Big)\sum_{\nu=0}^{n-1}|\Delta\varepsilon_\nu|\int_{1/n}^\pi|E_\nu(t)\phi(t)|\,dt$$

$$+K_2\sum_{n=1}^\infty\Big(\frac{1}{\lambda_n}-\frac{1}{\lambda_{n+1}}\Big)\Big|\varepsilon_n\sum_{\nu=1}^n\mu_\nu\int_{1/n}^\pi\phi(t)D_{\nu-1}(t)dt\Big|$$

$$\leqq K_1\sum_{\nu=0}^\infty|\Delta\varepsilon_\nu|\lambda_\nu\sum_{n=\nu+1}^\infty\Big(\frac{1}{\lambda_n}-\frac{1}{\lambda_{n+1}}\Big)\int_{1/n}^\pi\frac{|\phi(t)|}{t}dt+K_2$$

$$\leqq K_1\sum_0^\infty|\Delta\varepsilon_\nu|\lambda_\nu\sum_{\nu+1}^\infty p_n\Delta(1/\lambda_n)+K_2$$

$$<K.$$

This completes the proof of the theorem.

5.2. *Proof of Theorem 2.* After Theorem 1, it is enough to note that when $\psi(1/t)=\Big(\log\frac{2\pi}{t}\Big)^\beta$, and $\lambda_n=n$,

$$p_n=\int_{1/n}^\pi\frac{|\phi(t)|}{t}dt=O(k_n);$$

$$\sum_{n=\nu+1}^\infty p_n\Delta\Big(\frac{1}{\lambda_n}\Big)=O\Big(\frac{k_\nu}{\nu}\Big);$$

for $t\in\Big[\frac{1}{n},\pi\Big]$,

$$\Big|\sum_1^n\mu_\nu D_{\nu-1}(t)\Big|=O\Big(\frac{1}{t^2}\Big)$$

and therefore

$$\Big|\sum_1^n\mu_\nu\int_{1/n}^\pi\phi(t)D_{\nu-1}(t)dt\Big|\leqq K_1\int_{1/n}^\pi\frac{|\phi(t)|}{t^2}dt$$

$$=K_1\Big|\frac{\Phi(t)}{t^2}\Big|_{1/n}^\pi+2K_2\int_{1/n}^\pi\frac{\Phi(t)}{t^3}dt$$

$$=Kn(\log n)^\beta.$$

5.3. *Proof of Theorem 3.* In view of Lemma 2 we may employ summability method $|R, (\log n)^r, 1|$, $r>0$, instead of method $|R, \log n, 1|$. For the sake of convenience in further analysis we choose $r>1+|\beta|$. As in the case of Theorem 2, we again deduce this theorem from Theorem 1. Here

$$\mu_n=(\log (n+1))^r-(\log n)^r$$
$$=r\frac{(\log n)^{r-1}}{n}+O\Big(\frac{(\log n)^{r-1}}{n^2}\Big),$$

and therefore for $t\in\Big[\frac{1}{n}, \pi\Big]$

$$\Big|\sum_1^n \mu_\nu \sin\Big(\nu-\frac{1}{2}\Big)t\Big|$$
$$\leqq r\Big|\sum_1^n \frac{(\log \nu)^{r-1}\sin\Big(\nu-\frac{1}{2}\Big)t}{\nu}\Big|+K\sum_1^n (\log \nu)^{r-1}\frac{\Big|\sin\Big(\nu-\frac{1}{2}\Big)t\Big|}{\nu^2}$$
$$\leqq K+r\Big|\sum_{\nu\leqq 2\pi/t}(\log \nu)^{r-1}\frac{\sin\Big(\nu-\frac{1}{2}\Big)t}{\nu}\Big|+r\Big|\sum_{2\pi/t<\nu}\frac{(\log \nu)^{r-1}}{\nu}\sin\Big(\nu-\frac{1}{2}\Big)t\Big|$$
$$\leqq K_1\Big(\log\frac{2\pi}{t}\Big)^{r-1}+K_2\frac{(\log (2\pi/t))^{r-1}}{(2\pi/t)}(2\pi/t)$$
$$\leqq K\Big(\log\frac{2\pi}{t}\Big)^{r-1}.$$

Hence

$$\Big|\sum_1^n \mu_\nu\int_{1/n}^\pi \phi(t)D_{\nu-1}(t)dt\Big|$$
$$\leqq K\int_{1/n}^\pi \frac{|\phi(t)|(\log (2\pi/t))^{r-1}}{t}dt$$
$$\leqq K\Big[\Big(\log\frac{2\pi}{t}\Big)^{r-1}\frac{\Phi(t)}{t}\Big]_{1/n}^\pi+K_1\int_{1/n}^\pi \Phi(t)\Big|d\Big(\frac{\log (2\pi/t))^{r-1}}{t}\Big)\Big|$$
$$\leqq K_2(\log n)^{\beta+r},$$

which implies

$$\sum_1^\infty \Delta\Big(\frac{1}{\lambda_n}\Big)\Big|\varepsilon_n\sum_1^n \mu_\nu\int_{1/n}^\pi \phi(t)D_{\nu-1}(t)dt\Big|\leqq K\sum\frac{|\varepsilon_n|(\log n)^{\beta+r}}{n(\log n)^{r+1}}$$
$$\leqq K.$$

Again, as $p_n=O(k_n)$, the proof of Theorem 3 is completed.

5.31. Corollary to Theorem 3.

The following corollary is an interesting special case of Theorem 3.

COROLLARY. *If* $\Phi(t)=O\Big(t\Big(\log\frac{2\pi}{t}\Big)^{-\eta}\Big)$, $\eta>0$, *then* $\sum A_n(x)$ *is summable*

$|R, \log n, 1|$, *iff*

$$\sum \frac{1}{n \log n} |S_n(x)-f(x)| < \infty .$$

Proof of Tneorem 4. Let $\{\gamma_n\}$ be a given sequence. After Lemma 3 we have

$$\sum_0^n \gamma_\nu |S_\nu(x)-f(x)| = \sum_0^{n-1} \Delta(\gamma_\nu) \sum |S_k(x)-f(x)| + \gamma_n \sum_0^n |S_\nu(x)-f(x)|$$

$$= O\Big(\sum_0^{n-1} \nu h_n |\Delta\gamma_\nu|\Big) + O(nh_n\gamma_n) .$$

Case I· Take $\{\gamma_n\} = \left\{\frac{\varepsilon_n}{n}\right\}$ so that $n\gamma_n = \varepsilon_n$ and $(n+1)\Delta\gamma_n = \frac{\varepsilon_n}{n} + \Delta\varepsilon_n$, and the result follows from the sufficiency part of Theorem 2.

Case II: Take $\{\gamma_n\} = \left\{\frac{\varepsilon_n}{n \log n}\right\}$. Here we have

$$(n+1)\Delta\gamma_n = \Big(\frac{\Delta\varepsilon_n}{\log(n+1)}\Big) + O\Big(\frac{|\varepsilon_n|}{n \log n}\Big) .$$

Now we get the result in the light of Theorem 3.

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