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ON ANALYTIC FUNCTIONS IN A NEIGHBOURHOOD OF BOUNDARY POINTS OF RIEMANN SURFACES

Dedicated to Professor Yûsaku KOMATU on his 60th Birthday

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THEOREM 1[1]. *Let $R\notin O_g$ be a Riemann surface and let p be a singular point relative to Martin topology (i.e. p is minimal and* $\sup K(z,p)<\infty$). *Then $G\in O_{AB}$ for a domain G such that CG is thin at p.*

Analogous theorems[1] are obtained relative to N-Martin topology.

THEOREM 2[2]. *Let G be an end (domain G of R with compact relative boundary ∂G) of $R\in O_g$. Let $\mathfrak{p}$ be an ideal boundary component of G. Let $f(t)$: $t\in G$ be an analytic function. If $|f(t)|\leqq M<\infty$ in G, then $f(t)\to a$ limit as $t\to\mathfrak{p}$, $f(G)$ is a covering surface over the w-plane of a finite number N sheets and the harmonic dimension of $\mathfrak{p}$ is $\leqq N$.*

THEOREM 2′[3]. *Let $\mathfrak{p}$ be a one in Theorem 2. Let F be a completely thin set at $\mathfrak{p}$. If $G-F$ is represented as a covering surface of N number of sheets, the harmonic dimention of $\mathfrak{p}\leqq N$.*

These theorems mean a singular point p (or boundary component of harmonic dimension ∞) is so much complicated as $G-F\in O_{AB}$ (or O_{AF}) and the complicacy of p (or $\mathfrak{p}$) is not disturbed by extracting a small set F from G, where F is thin at p (or F is completely thin at $\mathfrak{p}$) and O_{AF} means a class of Riemann surface R on which there exists no non constant analytic function $f(t)$ such that $f(R)$ is at most a finite number of sheets. From these points of view we propose the following

PROBLEM 1. *About Theorem* 1, *is there a non singular point p such that $v(p)-F\in O_{AB}$? In other words, is the existence of a singular point necessary for $v(p)-F\in O_{AB}$? where $v(p)$ is a neighbourhood of p and F is thin at p.*

PROBLEM 2. *About Theorem 2 and 2′, is it true that there exists a boundary point p, instead of $\mathfrak{p}$ such that $G-F\in O_{AB}$? where F is thin at p.*

But these problems are difficult and in this paper we can only show examples as follows: *Example* 1. *There exists a point p of a Riemann surface $R\in O_g$ such that $v(p)\in O_{AB}$. Example* 2 *and* 3. *There exists a boundary point p of $R\in O_g$*

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such that $v(p)-F\in O_{ADF}$ or O_{ABF}, where F is a small set in a sense and $O_{ADF}(O_{ABF})$ means a class of Riemann surface on which there exists no non constant Drichlet bounded (bounded) analytic function such that $f(R)$ is a covering surface of a finite number of sheets. Clearly $O_{ABF}\subset O_{ADF}$. Example 4. There exists a non singular point p of a Riemann surface $\notin O_g$ such that $v(p)-F\in O_{ABF}$, where F is a small set. At first we shall construct an example using P. J. Myrberg's idea[4].

EXAMPLE 1. Let $\mathfrak{F}_0$ be a unit disc: $|z|<1$ with slits $J_n: n=1,2,\cdots$ and $I_n^i: n=1,2,\cdots,\ i=1,2,\cdots$ as follow

$$J_n=\{a_{2n+2}\leqq \operatorname{Re} z\leqq a_{2n+1},\ \operatorname{Im} z=0\}\,,$$

where $1>a_1>a_2\cdots\downarrow 0$ and $\sum_{n=1}^{\infty}\frac{1}{n+1}\log\frac{a_{2n+1}}{a_{2n+2}}=\infty$.

$$I_n^i=\left\{\arg z=\frac{\pi}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}\right),\ b_{n,2i+1}\leqq|z|\leqq b_{n,2i-1}\right\},$$

where $a_{2n+2}>b_{n,1}>b_{n,2}\cdots\downarrow 0$, $\sum_{i=1}^{\infty}(b_{n,i})^{2^n}=\infty$ for any n and $\sum_n\sum_i I_n^i\cap\sum R_z(a_{2n+1},a_{2n+2})=0$, where $R_z(a_{2n+1},a_{2n+2})=\{a_{2n+2}\leqq|z|\leqq a_{2n+1}\}$. Let $\mathfrak{F}_n: n\geqq 1$ be a leaf of the whole z-plane with slits $\sum_{m=n}^{\infty}J_m+\sum_{i=1}^{\infty}I_n^i$. Connect $\mathfrak{F}_0$ with $\mathfrak{F}_n: n=1,2,\cdots$ cross-wise on $\sum_{i=1}^{\infty}I_n^i$ so that endpoints of I_n^i are branch points of order 1. Connect $\mathfrak{F}_0,\mathfrak{F}_1,\cdots,\mathfrak{F}_n$ on $J_n: n=1,2,\cdots$ so that endpoints of J_n are branch points of order n. Then we have a Riemann surface $\mathfrak{R}$ over the z-plane with compact relative boundary $\partial\mathfrak{R}=\{|z|=1$ of $\mathfrak{F}_0\}$. It is evident $\mathfrak{R}$ has only one boundary component $\mathfrak{p}$. Let $R(a_{2n+1},a_{2n+2})$ be the part of $\mathfrak{F}_0+\mathfrak{F}_1+\cdots+\mathfrak{F}_n$ over $R_z(a_{2n+1},a_{2n+2})$. Then $R(a_{2n+1},a_{2n+2})$ is a ring domain with two boundary components over $|z|=a_{2n+1}$ and $|z|=a_{2n+2}$ with module $=\frac{1}{n+1}\log\frac{a_{2n+1}}{a_{2n+2}}$ and $R(a_{2n+1},a_{2n+2})$ separates $\mathfrak{p}$ from $\partial\mathfrak{R}$. By $\sum_n^{\infty}\operatorname{mod}R(a_{2n+1},a_{2n+2})=\infty$ $\mathfrak{R}$ is an end of another Riemann surface $\in O_g$ and $\mathfrak{p}$ is of harmonic dimension[5]$=1$. Therefore there exists only one Martin point p on $\mathfrak{p}$. Clearly p is minimal. Let C_n be the boundary component lying over $|z|=a_{2n+2}$ of $R(a_{2n+1},a_{2n+2})$ and let G_n be the domain of $\mathfrak{R}$ divided by C_n such that G_n is a neighbourhood of $\mathfrak{p}$. Put $\mathfrak{R}_n=\mathfrak{R}-G_n$. Then $\mathfrak{R}_n$ is an $(n+1)$ sheeted covering surface and $\mathfrak{R}=\sum_{n=1}^{\infty}\mathfrak{R}_n$. Let $v(p)$ be a neighbourhood of p relative to Martin topology. Then there exists a number n_0 such that $v(p)\supset\mathfrak{R}-\mathfrak{R}_{n_0}$. Assume there exists a bounded analytic function $f(t): t\in v(p)$. Let $A_n=\{0<|z|<r,\ \theta_{1,n}<\arg z<\theta_{2,n}\}: r=a_{2n_0+2},\ \theta_{1,n}=\frac{\pi}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}-\frac{1}{2^n}\right)$, $\theta_{2,n}=\frac{\pi}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}+\frac{1}{2^n}\right)$. Let Δ_n be the part of $\mathfrak{F}_0+\mathfrak{F}_n$ over A_n. Map A_n by $\zeta=\left(\frac{ze^{-i\theta_{1,n}}}{r}\right)^{2^n}$ onto $A_n^\zeta=\{0<|\zeta|<1,\ 0<\arg\zeta<\pi\}$. Then $I_n^j\to{}^\zeta I_n^j$, $b_{n,j}\to{}^\zeta b_{n,j}$

$=\left(\frac{b_{n,j}}{r}\right)^{2n}$. Let ${}^{\zeta}\varDelta_n$ be the surface consisting of two leaves (which are the same as $A_{n,\zeta}$) connected crosswise on $\sum_j {}^{\zeta}I_n^j$. Then $\varDelta_n$ and ${}^{\zeta}\varDelta_n$ are conformally equivalent. Hence $f(t)$ in $\varDelta_n$ is transformed to $f(s)$ in ${}^{\zeta}\varDelta_n$. Let s_1 and s_2 be two points in ${}^{\zeta}\varDelta_n$ such that $s_1 \neq s_2$ (except branch points) with $\text{proj.}\, s_1 = \text{proj.}\, s_2 = \zeta$. Then $(f(s_1)-f(s_2))^2$ is a bounded analytic function $g(\zeta)$ and $g(\zeta)=0$ at $\sum_j b_{n,j}^{\zeta}$. Let $G(\zeta, b_{n,j}^{\zeta})$ be a Green's function of A_n^{ζ}. Then by brief computation $G(\zeta, b_{n,j}^{\zeta}) \geqq A(\zeta)(b_{n,j}^{\zeta})^{2n}$: $A(\zeta)>0$. Hence $g(\zeta)=0$ by $\sum_j G(\zeta, b_{n,j}^{\zeta})=\infty$, whence $f(s_1)=f(s_2)$ and $f(t)=f(z)$: $z=\text{proj.}\, t$ in $\varDelta_n$. By identity theorem $f(t_1)=f(t_2)$ so far as $f(t_1)$ and $f(t_2)$ can be continued analytically, where $\text{proj.}\, t_1=\text{proj.}\, t_2$. We denote by $f_n(z)$: $n=0,1,2,\cdots$ the branch of $f(t)$ in $\mathfrak{F}_n$. Then $f_0(z)=f_n(z)$ in A_n for any n and $f_0(z)$ is analytic in $\{0<|z|<r\}-\sum_n^{\infty} J_n$. On the other hand, $\mathfrak{F}_n$ has no branch points for $|z|>a_{2n+1}$ and $f_n(z)$ is analytic in a neighbourhood of J_m: $m<n$. Hence $f_0(z)(=f_n(z))$ is analytic on $\sum_{n_0}^{\infty} J_n$ and $f_0(z)$ is analytic in $0<|z|<a_{2n_0+1}=r$ and in $|z|>a_{4n_0+1}$ (by putting $f_0(z)=f_{2n_0}(z)$). Thus $f_0(z)$ is analytic in $0<|z|\leqq\infty$. This implies $f(z)=\text{const.}$ and $v(p)\in O_{AB}$.

REMARK 1. *By the method of the proof we see at once following. Let F be a closed set in $\mathfrak{R}$ such that $F\cap\sum\varDelta_n=0$ and* proj. $(\mathfrak{R}-F)$ *covers the z-plane except a set $\in N_{AB}$, then $v(p)-F\in O_{AB}$, where N_{AB} means a class of set F such that $\{0<|z|\leqq\infty\}-F\in O_{AB}$.*

REMARK 2. *Suppose F contains branch points on $z=b_{n,i}$: $n=1,2,\cdots$, $i=1,2,3,\cdots$, Then we cannot prove $v(p)-F\in O_{AB}$, however thinly F may be distributed. On the other hand, we shall show examples of a point p such that there exists no analytic functions of some class in $v(p)-F$, if F is small in a sense. We proved*

LEMMA 1[6]. *Let G be a ring domain with radial slits s_i such that $\partial G=\Gamma_1+\Gamma_2+\sum_{i=1}^{i_0} s_i$: $\Gamma_1=\{|z|=1\}$, $\Gamma_2=\{|z|=\exp\mathfrak{M}\}$ and s_i is a radial slit in $1\leqq|z|\leqq\exp\mathfrak{M}$ and s_i may touch $\Gamma_1+\Gamma_2$. Let $U(z)$ be a harmonic function in G with continuous value. Then*

$$D(U(z))\geqq\frac{1}{\mathfrak{M}}\int_0^{2\pi}|U(e^{i\theta})-U(e^{\mathfrak{M}+i\theta})|^2 d\theta\,.$$

By the same method we have at once

LEMMA 1′. *Let G be a circular trapezoid $1<|z|<e^{\mathfrak{M}}$, $\theta_1<\arg z<\theta_2$ with a finite number of radial slits. Then*

$$D(U(z))\geqq\frac{1}{\mathfrak{M}}\int_{\theta_1}^{\theta_2}|U(e^{i\theta})-U(e^{\mathfrak{M}+i\theta})|^2 d\theta\,.$$

LEMMA 2. *Let G be a rectangle with vertices $-a$, a, $a+ih$, $-a+ih$ and $U(z)$*

be H. M. (harmonic measure) of vertical sides. Then for any $0<\delta<a$ *and for any* $\varepsilon>0$, *there exists an* h *such that*

$$U(z)<\varepsilon \qquad \text{for} \quad |\operatorname{Re} z|<a-\delta .$$

Proof. Let G_s be a rectangle with vertices, $s+\delta$, $s+\delta+ih$, $s-\delta+ih$, $s-\delta$. Then $G_s\subset G: |s|<a-\delta$. Let $U_s(z)$ be H. M. of vertical sides of G_s. Then $U(z) \leqq U_s(z)$. Now $\max_{\operatorname{Re} z=s} U_s(z)=\alpha(h, \delta)\to 0$ as $h\to 0$[7]. Hence

$$\underset{|\operatorname{Re} z|<a-\delta}{U(z)}<\alpha(h, \delta) ,$$

and we have Lemma 2.

LEMMA 3. *Let* G_n *be a domain with* $\partial G_n=\Gamma_1+\Gamma_2+\sum_i I_n^i : \Gamma_1=\{|z|=1\}$, $\Gamma_2=\{|z|=\exp(\mathfrak{M}+\alpha)\}$. $I_n^i=\left\{\arg z=\frac{2\pi i}{n},\ e^{\mathfrak{M}}\leqq|z|\leqq e^{\mathfrak{M}+\alpha}\right\}: \alpha>0,\ \mathfrak{M}>0$. *Let* J_n^i *be an arc on* $\Gamma_2 : J_n^i=\left\{\frac{2\pi i}{n}\leqq\arg z\leqq\frac{2\pi(1+i)}{n},\ |z|=e^{\mathfrak{M}+\alpha}\right\}$. *Map* G_n *by* $\zeta=f_n(z)$ *onto a domain* G_n^ζ *so that* $\Gamma_1\to\Gamma_1^\zeta=\{|\zeta|=1\}$, $I_n^i\to$*an arc on* $|\zeta|=e^{\mathfrak{M}'}$ *and* $J_n^i\to$*a radial slit*$=\left\{\arg\zeta=\frac{2\pi i}{n},\ e^{\mathfrak{M}_n'}\leqq|\zeta|\leqq e^{\mathfrak{M}_n''}\right\}$, *where* $\mathfrak{M}_n'$ *and* $\mathfrak{M}_n''$ *are suitable constants. Let* $n\to\infty$. *Then* $\mathfrak{M}_n''\to\mathfrak{M}$ *and* $f_n(z)\to z$. *Let* $U_n(z)$ *be a harmonic function in* G_n, *continuous on* $G_n+\Gamma_1+\Gamma_2+\sum_i I_n^i$ *such that* $U_n(z)=0$ *on* $\sum_i I_n^i$ *and* $D(U_n(z))\leqq 1$. *Then there exists a number* n_0 *such that*

$$\int_{\Gamma_1} U_n(z)^2 d\theta\leqq 2\mathfrak{M} \qquad \text{for} \quad n\geqq n_0 .$$

Proof. Let $\omega_n(z)$ be a harmonic function in G_n such that $\omega_n(z)=0$ on Γ_1, $\omega_n(z)=1$ on $\sum_i I_n^i$ and $\frac{\partial}{\partial n}\omega_n(z)=0$ on $\sum_i J_n^i$. Then

$$f_n(z)=\exp(\gamma_n(\omega_n(z)+i\tilde{\omega}_n(z))) ,$$

where $\gamma_n=2\pi\Big/\int_{\Gamma_1}\frac{\partial}{\partial n}\omega_n(z)ds$, $\tilde{\omega}_n(z)$ is the conjugate of $\omega_n(z)$ and $\mathfrak{M}_n''=\gamma_n$. Consider $\omega_n(z)$ in a circular trapezoid$=\left\{\frac{2\pi i}{n}<\arg z<\frac{2\pi(i+1)}{n},\ e^{\mathfrak{M}}<|z|<e^{\mathfrak{M}+\alpha}\right\}$. Since $\omega_n(z)>0$ and $\omega_n(z)=1$ on $\arg z=\frac{2\pi i}{n}$ and $=\frac{2\pi(1+i)}{n}$, there exists a number n' by Lemma 2 such that $\omega_n(z)\geqq 1-\varepsilon$ on $|z|=e^{\mathfrak{M}+\varepsilon}$ for $n\geqq n'$ for any given $\varepsilon>0$. Hence by the maximum principle $\omega_n(z)\geqq(1-\varepsilon)\frac{\log|z|}{\mathfrak{M}+\varepsilon}$ on $1<|z|<e^{\mathfrak{M}+\varepsilon}$. On the other hand, clearly $\omega_n(z)\leqq\frac{\log|z|}{\mathfrak{M}}$ in $1<|z|<e^{\mathfrak{M}}$, whence $\mathfrak{M}<\mathfrak{M}_n''<\frac{\mathfrak{M}+\varepsilon}{1-\varepsilon}$ and $\omega_n(z)\to\frac{\log|z|}{\mathfrak{M}}$ as $n\to\infty$. Since $\omega_n(z)=0$ on Γ_1, $\omega_n(z)\to\frac{\log|z|}{\mathfrak{M}}$ implies $\frac{\partial}{\partial n}\omega_n(z)\to\frac{\partial}{\partial n}\left(\frac{\log|z|}{\mathfrak{M}}\right)$ on Γ_1, $\mathfrak{M}_n''\to\mathfrak{M}$, $f_n(z)\to z$ and $f_n'(z)\to 1$

on Γ_1 uniformly as $n\to\infty$. Consider $U_n(z)$ in the ζ-plane. Then by for Lemma 1

$$D(U_n(z))=D(U_n(f_n^{-1}(z)))\geqq\frac{1}{\mathfrak{M}_n''}\int_{\Gamma_1}U_n(f_n^{-1}(z))^2d\theta\,.$$

By $f_n'(z)\to 1$ and $\mathfrak{M}_n''\to\mathfrak{M}$ as $n\to\infty$, there exists a number n_0 such that

$$D(U_n(z))\geqq\frac{1}{2\mathfrak{M}}\int_{\Gamma_1}U_n(z)^2d\theta \qquad \text{for} \quad n\geqq n_0\,.$$

LEMMA 4. *Let G_n be a domain with $\partial G_n={}_-\Gamma+{}_+\Gamma+\sum_{i=1}^{n}({}_-I_n^i+{}_+I_n^i)$: ${}_-\Gamma=\{|z|=e^{-a}\}$, ${}_+\Gamma=\{|z|=e^{a}\}$, ${}_-I_n^i=\left\{\arg z=\frac{2\pi i}{n},\ e^{-a}\leqq|z|\leqq e^{-\frac{5a}{6}}\right\}$, ${}_+I_n^i=\left\{\arg z=\frac{2\pi i}{n},\ e^{-\frac{5a}{6}}\leqq|z|\leqq e^{a}\right\}$. Let $U_n(z)$ be a harmonic function in G_n continuous on $\bar{G}_n$ such that $U_n(z)=0$ on $\sum({}_+I_n^i+{}_-I_n^i)$ and $D(U_n(z))\leqq 1$. Then for any $\varepsilon>0$ there exists a number n_0 such that*

$$|\operatorname{grad} U_n(z)|<\varepsilon \quad in \quad \{e^{-\frac{a}{2}}<|z|<e^{\frac{a}{2}}\}\,.$$

We call such G_n a ring with deviation ε.

Proof. Let $G_c=\{e^{-c}<|z|<e^{c}\}$: $\frac{2a}{3}\leqq c\leqq\frac{5a}{3}$. Let $G_c(z, z_0)$ be a Green's function of G_c. Since $\underset{z_0}{\operatorname{grad}}\frac{\partial}{\partial n}G_c(z, z_0)$ is finite and continuous relative to z, z_0 and c for $e^{-\frac{a}{2}}\leqq|z_0|\leqq e^{\frac{a}{2}}$, $z\in\partial G_c$ and $\frac{2a}{3}\leqq c\leqq\frac{5a}{6}$, there exists a const. M such that $\left|\underset{z_0}{\operatorname{grad}}\frac{\partial}{\partial n_z}G(z, z_0)\right|\leqq M$. Let $\delta=\min\left(\frac{a}{6},\ \frac{\varepsilon^2}{4\pi M^2e^{\frac{10a}{6}}}\right)$ and consider $U_n(z)$ in $e^{\frac{5a}{6}-\delta}<|z|<e^{a}\left(\frac{2a}{3}<\frac{5a}{6}-\delta<a\right)$. Then by Lemma 3, there exists a number n_0 such that

$$\int U(e^{\frac{5a}{6}-\delta+i\theta})^2d\theta\leqq 2\delta\leqq\frac{\varepsilon^2}{2\pi M^2e^{\frac{10a}{6}}} \qquad \text{for} \quad n\geqq n_0\,.$$

By Schwarz's inequality $\int_{+\Gamma}|U(e^{\frac{5a}{6}+\delta+i\theta})|d\theta<\frac{\varepsilon}{Me^{\frac{5a}{6}}}$, similarly $\int_{-\Gamma}|U(e^{-\frac{5a}{6}+\delta+i\theta})|\,d\theta\leqq\frac{\varepsilon}{Me^{\frac{5a}{6}}}$. Consider $U_n(z)$ in $\{e^{-\frac{5a}{6}+\delta}<|z|<e^{\frac{5a}{6}+\delta}\}$. Then

$$|\operatorname{grad} U_n(z)|\leqq\frac{1}{2\pi}\int_{-\Gamma_c}U_n(t)\left|\operatorname{grad}\frac{\partial}{\partial n}G(t, z)\right|e^{-\frac{5a}{6}+\delta}\,d\theta$$
$$+\frac{1}{2\pi}\int_{+\Gamma_c}|U_n(t)|\left|\operatorname{grad}\frac{\partial}{\partial n}G(t, z)\right|e^{\frac{5a}{6}-\delta}\,d\theta<\frac{\varepsilon}{\pi}\,,$$

for $z\in G_{\frac{a}{2}}$, where

$${}_-\Gamma_c=\{|z|=e^{-c}\}\,,\quad {}_+\Gamma_c=\{|z|=e^{c}\}: \qquad c=\frac{5a}{6}-\delta\,,\ \frac{2a}{3}\leqq c\leqq\frac{5a}{6}\,.$$

LEMMA 5. *Let G_n be the domain in Lemma 4 with $n \geqq n_0$. Let $\hat{G}_n$ be the same leaf as G_n (with ${}_{-}I_n^i + {}_{+}I_n^i$) of G_n. We identify each side of ${}_{-}I_n^i({}_{+}I_n^i)$ on G_n with the same side of ${}_{-}I_n^i({}_{+}I_n^i)$ of $\hat{G}_n$. Then we have a Riemann surface $\tilde{G}_n$ of planar character with connectivity $2n_0-1$. Let $f(t): t \in \tilde{G}_n$ be an analytic function in $\tilde{G}_n$ with $D(f(t)) \leqq \frac{1}{4}$. Then $\left|\frac{df(t)}{dt}\right| < 2\sqrt{2}\varepsilon$ in the part of $\tilde{G}_n$ over $e^{-\frac{a}{2}} < |z| < e^{\frac{a}{2}}$. We call such $\tilde{G}_n$ a ring surface with deviation $2\sqrt{2}\varepsilon$.*

Proof. Let t and $\hat{t}$ be points in G_n and $\hat{G}_n$ respectively such that $\text{proj.}\, t = \text{proj.}\, \hat{t} = z$. Let $\hat{t} = \hat{x} + i\hat{y}$ and $t = x + iy$. Put for simplicity $U(\hat{t}) = \hat{U}(z)$, $V(\hat{t}) = \hat{V}(z)$: $t \in \hat{G}$, $U(t) = U(z)$, $V(t) = V(z)$: $t \in G$. Then by C. R. equality

$$U_x = V_y\,, \qquad U_y = -V_x\,, \qquad \hat{U}_x = -\hat{V}_y\,, \qquad \hat{U}_y = \hat{V}_x\,. \tag{1}$$

Now $D(U(z)) = D(V(z)) \leqq \frac{1}{4}$, $D(U(z) - \hat{U}(z)) \leqq 1$ and $U(z) - \hat{U}(z) = 0$ on ${}_{+}I_n^i + {}_{-}I_n^i$. We have by Lemma

$$|U_x - \hat{U}_x| < \varepsilon\,, \qquad |V_y - \hat{V}_y| < \varepsilon\,. \tag{2}$$

By (1) and (2)

$$\left|\frac{d}{dt} f(t)\right| < 2\sqrt{2}\,\varepsilon \qquad \text{for} \quad e^{-\frac{a}{2}} < |\text{proj}\, t| < e^{\frac{a}{2}}\,.$$

Let D be a domain and let F be a compact set in D. Let $\omega(F, z, D)$ be H.M. of F, i.e. $\omega(F, z, D) = 0$ on ∂D, $=1$ on F we defind $\text{Cap}(F)$ by $\int_{\partial D} \frac{\partial}{\partial n} \omega(F, z, D) ds/2\pi$ and denote it by $\gamma(F)$. Then it is clear $\gamma(F) = 0$ if and only if F is a set of logarithmic capacity zero.

LEMMA 6. 1) (*An upper bound for Dirichlet bounded harmonic functions*). *Let D be a domain of finite connectivity and let F be a compact set in the interior of a compact set $A \subset D$. Let $H(z)$ be a harmonic function in $D - F$ such that $H(z) = 0$ on ∂D and $D(H(z)) \leqq 1$. Then $|H(z)| \leqq C(z)\sqrt{\gamma(F)}$ in $D - A$, where $C(z)$ is a constant depending only on A, D and z.*

2) *Let D_0 be a compact set in $D - A$. Let F_n be a sequence of compact sets such that $F_n \subset A$ and $\gamma(F_n) \downarrow 0$. Let $U(z)$ and $U_n(z)$ be a harmonic functions in D and $D - F_n$ respectively such that $U(z) = U_n(z)$ on ∂D, $D(U(z)) \leqq 1$ and $D(U_n(z)) \leqq 1$. Then*

$$\text{grad}\, U_n(z) \to \text{grad}\, U(z) \ \textit{in}\ D_0\ \textit{uniformly as}\ n \to \infty\,.$$

3) *Let F^* be a compact set in D with $\gamma(F^*) = 0$. Then for any $\varepsilon > 0$ and for any compact set D_0 in $D - F^*$ we can find a compact set $F \supset F^*$ such that*

$$|\text{grad}\, U(z) - \text{grad}\, U^F(z)| < \varepsilon \ \textit{on}\ D_0$$

where $U(z)$ is a harmonic function in (2) and $U^F(z)$ is a harmonic function in $D - F$ such that $U(z) = U^F(z)$ on ∂D and $D(U^F(z)) \leqq 1$.

Proof of 1) Let F_m be a decreasing sequence of compact sets such that $F_m \downarrow F$, $F_m \subset A^0$, the conncetivity of $D-F_m$ is finite, every point of ∂F_m is regular with respect to Dirichlet problem in $D-F_m$ and $H(z)$ is continuous on ∂F_m. Let $\omega(z)=\dfrac{\omega(F_m, z, D)}{\gamma(F_m)}$ and $\tilde{\omega}(z)$ be the conjugate of $\omega(z)$. Put $\zeta(z)=\exp(\omega(z)+i\tilde{\omega}(z))$ $=re^{i\theta}$. Then $\zeta(z)$ maps $D-F_m$ onto a ring $R_\zeta=\left\{1<|\zeta|<\exp\left(\frac{1}{\gamma(F_m)}\right)\right\}$ with a finite number of radial slits. Consider $H(\zeta)=H(\zeta(z))$ in R_ζ. Then by Lemma 1 and Schwarz's inequality

$$\int_{|\zeta|=\exp(1/\gamma(F_m))} |H(z)|\,d\theta \leqq \sqrt{\frac{2\pi}{\gamma(F_m)}}\,.$$

Let $V_m(z)$ be a harmonic function in $D-F_m$ such that $V_m(z)=|H(z)|$ on ∂F_m, $V_m(z)=0$ on ∂D. Then since $|H(z)|$ is subharmonic $V_m(z)\leqq V_{m+1}(z)$ and $|H(z)|$ $\leqq V_m(z)$. $\int_{\partial F_m}\frac{\partial}{\partial n}V_m(z)\omega(z)ds=\int_{\partial F_m}V_m(z)\frac{\partial}{\partial n}\omega(z)ds$, let $ds=rd\theta$ and $\partial n=\partial r$ on $\partial D+\partial F_m$. Then $\frac{1}{\gamma(F_m)}\int_{\partial D}\frac{\partial}{\partial n}V_m(z)ds=\frac{1}{\gamma(F_m)}\int_{\partial F_m}\frac{\partial}{\partial n}V_m(z)ds=\int_{\partial F_m}V_m(z)\,d\theta\leqq$ $\sqrt{\frac{2\pi}{\gamma(F_m)}}$. Hence

$$\int_{\partial D}\frac{\partial}{\partial n}V_m(z)ds\leqq\sqrt{2\pi\gamma(F_m)}\,.$$

We can find a compact set $A'\supset A$ with dist $(\partial A', A)>0$. By Harnack's theorem there exists a constant K depending on A', z, D such that $V_m(t)\geqq\frac{V(z)}{K}$, whence $V_m(t)\geqq\frac{V_m(t)}{K}\omega(A', t, D)$ on $\partial A'$. Hence by $\int_{\partial D}\frac{\partial}{\partial n}V_m(t)ds\geqq\frac{V_m(z)}{K}$ $\geqq\int_{\partial D}\frac{\partial\omega}{\partial n}(A, t, D)ds$ we have $V_m(z)\leqq\frac{K\sqrt{\gamma(F_n)}}{\sqrt{2\pi}\gamma(A')}$. Let $m\to\infty$. Then $|H(z)|$ $\leqq\lim_m V_m(z)\leqq\frac{K\sqrt{\gamma(F)}}{\sqrt{2\pi}\gamma(A')}$ and $\frac{K}{\sqrt{2\pi}\gamma(A')}$ is a required constant.

Proof of 2) Let $H_n(z)=U(z)-U_n(z)$. Then $D\left(\frac{H_n(z)}{2}\right)\leqq 1$. By (1) $|H_n(z)|$ $\leqq 2C(z)\sqrt{\gamma(F_n)}$ in $D-A$. Let D_0^* be a closed domain in $D-A$ such that $D_0^*\supset D_0$, dist $(\partial D_0^*, D_0)>0$. Let $G(z,q)$ be a Green's function of D_0^*. Then since there exists a constant $M<\infty$ such that grad $\frac{\partial}{\partial n}G(z,q)<M$: $q\in D_0$, $z\in\partial D_0^*$. Now $\max_{z\in D_0^*}|H_n(z)|\to 0$ uniformly as $n\to\infty$. We have $|\text{grad}\,U(z)-\text{grad}\,U_n(z)|\leqq$ $\int_{\partial D_0^*}2M|H_n(\zeta)|\left|\text{grad}\frac{\partial}{\partial n}G(\zeta, z)\right|ds\to 0$ as $n\to\infty$. 3) is obtained at once.

Let $\tilde{G}_n$ be a surface in Lemma 5 with $n\geqq n_0$. Suppose a sufficiently small closed set F in $\tilde{G}_n$. Then we see by Lemma 6 the property of $\tilde{G}_n$ does not change so much by extracting F from $\tilde{G}_n$.

α, β-thin set. Let F be a closed set in $\tilde{G}_n$ in Lemma 5 with deviation $2\sqrt{2}\varepsilon$.

If we can find a closed Jordan curve Γ in G_n-F (and in $\hat{G}_n-F$) such that proj Γ separates $|z|=e^{-a}$ from $|z|=e^{a}$, length of $\Gamma\leqq\alpha e^{-a}$ and $\left|\frac{df(t)}{dt}\right|<\beta\varepsilon$ for any analytic function $f(t)$ in $\tilde{G}_n-F$ with $D(F(t))\leqq\frac{1}{4}$, we call F an α,β-thin set in $\tilde{G}_n$.

EXAMPLE 2. Let

$$1>a_1>a_2,\cdots\downarrow 0 \quad\text{and}\quad \sum_n^{\infty}\log\frac{a_{2n+1}}{a_{2n+2}}=\infty\,. \tag{3}$$

Let J_n be a slit: $J_n=\{\arg z=\pi,\ a_{2n+2}\leqq|z|\leqq a_{2n+1}\}$: $n=1,2,3,\cdots$. Let

$$1>b_1>b_1'>b_2>b_2',\cdots\downarrow 0\,,\qquad \lim_n\frac{\log b_n'}{\log b_n}=1\,, \tag{3'}$$

$$\sum_n\{a_{2n+2}\leqq|z|\leqq a_{2n+1}\}\cap\sum_n^{\infty}\{b_n'\leqq|z|\leqq b_n\}=0\,.$$

Let ${}_-I_n^i$ and ${}_+I_n^i$ are slits: $n=1,2,3,\cdots$, $i=1,2,\cdots,j(n)$ as follow:

$$\begin{aligned}{}_-I_n^i&=\left\{\arg z=\frac{2\pi i}{j(n)},\ b_n'\leqq|z|\leqq b_n'e^{\frac{d_n}{6}}\right\}\\ &\qquad\qquad\qquad\qquad d_n=\frac{1}{2}\log\frac{b_n}{b_n'}\\ {}_+I_n^i&=\left\{\arg z=\frac{2\pi i}{j(n)},\ b_ne^{-\frac{d_n}{6}}\leqq|z|\leqq b_n\right\}\end{aligned}$$

where the number $j(n)$ of slits ${}_-I_n^i$ (or ${}_+I_n^i$) is so large that we can obtain a ring surface $\tilde{G}_n$ (from two leaves by identifying slits of the leaves) with deviation c_n over $\{b_n'\leqq|z|\leqq b_n\}$, where

$$\lim_n\frac{\log c_n}{\log b_n}=\infty\,. \tag{3''}$$

Let $\mathfrak{F}$ be a unit circle $|z|<1$ with slits $\sum_n J_n+\sum_n\sum_i({}_-I_n^i+{}_+I_n^i)$ and $\hat{\mathfrak{F}}$ be the same leaf as $\mathfrak{F}$. We identify $\sum_n J_n+\sum_n\sum_i({}_-I_n^i+{}_+I_n^i)$ of $\mathfrak{F}$ and $\hat{\mathfrak{F}}$. Then we have a Riemann surface $\tilde{\mathfrak{F}}$ with compact relative boundary $\partial\tilde{\mathfrak{F}}$ consisting of two components over $|z|=1$ and has one ideal boundary component $\mathfrak{p}$. The part of $\tilde{\mathfrak{F}}$ over $\{a_{2n+1}<|z|<a_{2n+2}\}$ is a ring with two boundary components of module $=\frac{1}{2}\log\frac{a_{2n+1}}{a_{2n+2}}$ separating $\mathfrak{p}$ from $\partial\tilde{\mathfrak{F}}$. Hence by (3) $\tilde{\mathfrak{F}}$ is an end of another Riemann surface $\in O_g$ and $\mathfrak{p}$ is of harmonic dimension 1. There exists only one Martin point p on $\mathfrak{p}$. Let $v(p)$ be a neighbourhood. Then $\partial v(p)$ is compact.

PROPOSITION. *Let F_1 be set of radial slits in $\tilde{\mathfrak{F}}$ such that $F_1\cap\sum_n\tilde{G}_n=0$.*

1) *Let F_2 be a closed set in $\tilde{\mathfrak{F}}$ such that F_2 is α,β-thin set in every $\tilde{G}_n$ and $v(p)-F_2$ is connected.*

2) *Let* $U_n(z)$ *be a harmonic function in* $\mathfrak{F}-F_1-F_2$ *or* $\hat{\mathfrak{F}}-F_1=F_2$ *over* $\{\theta_1<\arg z<\theta_2,\ b_n<|z|<1\}$ *such that* $U_n(z)=0$ *on* $|z|=1$ *and* $U_n(z)=1$ *on* $|z|=b_n$ *and* $U_n(z)$ *has M. D. I. (minimal Dirichlet integral). Then* $D(U_n(z))\geqq\frac{\gamma(\theta_2-\theta_1)}{-\log b'_n}$: $\gamma>0$ *for any* θ_1 *and* θ_2 *and* b'_n $\left(\textit{if } F_2=0,\ D(U_n(z))=\frac{\theta_2-\theta_1}{-\log b'_n}\right)$.
If F_2 *is so thinly distributed in* $\tilde{\mathfrak{F}}$ *that* F_2 *may satisfy condition* 1) *and* 2),

$$v(p)-F_1-F_2\in O_{ADF}.$$

Proof. Assume $w=f(t)$: $t\in v(p)-F_1-F_2$ is non const and $D(f(t))<\infty$ and $f(v(p)-F_1-F_2)$ is an L number of sheets over the w-plane. We can suppose without loss of generality $D(f(t))\leqq\frac{1}{4}$. By condition 2) there exists a Jordan curve Γ_n in $\mathfrak{F}\cap G_n$ such that $\left|\frac{df(t)}{dt}\right|<\beta c_n$ on Γ_n and length of $\Gamma_n<\alpha b_n$. Hence we can find a subsequence $\{n'\}$ of $\{n\}$ such that $f(\Gamma_{n'})\to w_0$ as $n'\to\infty$. By choosing suitable $v_l(p)\subset v(p)$ we can suppose $\partial v_l(p)\cap(\mathfrak{F}-F_1-F_2)$ has an arc λ such that $\mathrm{dist}\,(f(\lambda),w_0)=d_0>0$, $\mathrm{proj}\,\lambda$ is contained in ${}_{\theta_1}\Delta_{\theta_2}=\{\theta_1<\arg z<\theta_2\}$ and $\mathrm{proj}\,\lambda$ is connecting $e^{a+i\theta_1}$ with $e^{a'+i\theta_2}$. Let ${}_{\lambda}\Delta_{\Gamma n'}$ be the part of $\mathfrak{F}-F_1-F_2$ over ${}_{\theta_1}\Delta_{\theta_2}$ bounded by λ, F_1, F_2, $\Gamma'_{n'}$ and two segments $\arg z=\theta_1$ and θ_2, where $\Gamma'_{n'}$ is the part of $\Gamma_{n'}$ lying over ${}_{\theta_1}\Delta_{\theta_2}$. Let $U_n(z)$ be a harmonic function in ${}_{\lambda}\Delta_{\Gamma n'}$ such that $U_{n'}(z)=0$ on λ, $U_{n'}(z)=1$ on $\Gamma'_{n'}$ and $U_{n'}(z)$ has M. D. I (has minimal Dirichlet integral among all functions with the same value as $U_{n'}(z)$ on $\lambda+\Gamma'_{n'}$). Then by the Dirichlet principle and by condition 2)

$$D(U_{n'}(z))\geqq D(U'_{n'}(z))\geqq\frac{\gamma(\theta_2-\theta_1)}{-\log b'_{n'}}, \tag{4}$$

where $U'_{n'}(z)$ is a harmonic function in $\mathfrak{F}-F_1-F_2$ over $\{b'_{n'}<|z|<1,\ \theta_1<\arg z<\theta_2\}$ such that $U'_{n'}(z)=0$ on $|z|=1$, $U'_{n'}(z)=1$ on $|z|=b'_{n'}$ and $U'_{n'}(z)$ has M. D. I. Consider $f(\Gamma_{n'})$. Then by $\left|\frac{df(t)}{dt}\right|\leqq\beta c_{n'}$, diameter of $f(\Gamma_{n'})\leqq2\pi b_{n'}c_{n'}\alpha\beta$. Since diameter $(w_0+f(\Gamma_{n'}))\to0$ as $n'\to\infty$, we can find a number n_0 and a point $p_{n'}$: $n'\geqq n_0$ in $f(\Gamma_{n'})$ such that $|p_{n'}-w_0|<\frac{d_0}{4}$: $d_0<1$, $\{|w-p_{n'}|<4\pi b_{n'}c_{n'}\alpha\beta\}\supset f(\Gamma_{n'})$ and $\left\{|w-p_{n'}|>\frac{d_0}{2}\right\}\supset f(\lambda)$ for $n'\geqq n_0$. Let $V_{n'}(w)$ be a continuous function in the w-plane such that $V_{n'}(w)=1$ in $|w-p_{n'}|<4\pi b_{n'}c_{n'}\alpha\beta$, $V_{n'}(w)$ is harmonic in $\left\{4\pi b_{n'}c_{n'}\alpha\beta<|w-p_{n'}|<\frac{d_0}{2}\right\}$ and $=0$ in $|w-p_{n'}|\geqq\frac{d_0}{2}$. Then $f^{-1}(V_{n'}(w))\geqq1$ on $\Gamma_{n'}$ and $=0$ on λ and

$$D(f^{-1}(V_{n'}(w)))\leqq\frac{2\pi L}{\log\dfrac{\frac{d_0}{2}}{4\pi b'_n c'_n\alpha\beta}}. \tag{5}$$

Clearly $D(U_{n'}(z))\leqq D(V_n(f^{-1}(w)))$. By $\lim_n\frac{\log c_n}{\log b_n}=\infty$, $\lim_n\frac{\log b'_n}{\log b_n}=1$ we have by

(4) and (5) a contradiction. Hence $v(p)-F_1-F_2\in O_{ADF}$.

LEMMA 7. *Let T be a circular trapezoid with radial slits $I_n^i: i=1,2,\cdots,n-1$ such that $T=\{1<|z|<e^{\mathfrak{M}+\alpha},\ 0<\arg z<\theta\}-\sum I_n^i: I_n^i=\{\arg z=\frac{i\theta}{n},\ e^{\mathfrak{M}}\leqq|z|\leqq e^{\mathfrak{M}+\alpha}\}$. Map T onto a circular trapezoid T_ζ with slits by $\zeta=f_n(z)$ so that $\{0<\arg z<\theta,\ |z|=1\}\rightarrow\{0<\arg z<\theta,\ |\zeta|=1\}$. $\{\arg z=\theta,\ 1\leqq|z|\leqq e^{\mathfrak{M}+\alpha}\}+\left\{\left(\frac{n-1}{n}\right)\theta\leqq\arg z\leqq\theta,\ |z|=e^{\mathfrak{M}+\alpha}\right\}=\Lambda_1\rightarrow\{\arg\zeta=\theta,\ 0\leqq|\zeta|\leqq e^{\mathfrak{M}'_n}\}$. $I_n^i\rightarrow$an arc on $|\zeta|=e^{\mathfrak{M}'_n}: i=1,3,\cdots,n-1$. A circular arc $J_n^i=\left\{\frac{i\theta}{n}\leqq\arg z\leqq\frac{(i+1)\theta}{n}\right\}\rightarrow$a radial slit in T_ζ connecting $|\zeta|=e^{\mathfrak{M}'_n}: i=1,2,3,\cdots,n-2$. $\{\arg z=0,\ 0\leqq|z|\leqq e^{\mathfrak{M}+\zeta}\}+\{0\leqq\arg z\leqq\frac{\theta}{n},\ |z|=e^{\mathfrak{M}+\alpha}\}=\Lambda_2\rightarrow\{\arg\zeta=0,\ 0\leqq|\zeta|\leqq e^{\mathfrak{M}'_n}\}$, where $\mathfrak{M}'_n$ is a suitable const. Let $n\rightarrow\infty$. Then $\mathfrak{M}'_n\rightarrow\mathfrak{M}$ and $f_n(z)\rightarrow z$. Let $U_n(z)$ be a harmonic function in T such that $U_n(z)=0$ on $\sum I_n^i$ and $D(U_n(z))\leqq1$. Then*

$$\int_{|z|=1}U^2(z)d\theta\leqq2\mathfrak{M}\qquad for\quad n\geqq n_0\,.$$

Proof. Let $\omega_n(z)$ be a harmonic function in $T-\sum_i I_n^i$ such that $\omega_n(z)=0$ on $|z|=1$, $\frac{\partial}{\partial n}\omega_n(z)=0$ on $\Lambda_1+\Lambda_2+\sum_i J_n^i$, $\omega_n(z)=1$ on $\sum_i I_n^i$. Then

$$f_n(z)=\exp\left(\gamma_n(\omega_n(z_n)+i\tilde{\omega}_n(z))\right),$$

where $\gamma_n=\theta\Big/\int_{|z|=1}\frac{\partial}{\partial n}\omega_n(z)ds$ and $\tilde{\omega}_n(z)$ is the conjugate of $\omega_n(z)$. Consider $\omega_n(z)$ in $\left\{0<\arg z<\frac{\theta}{n},\ e^{\mathfrak{M}}<|z|<e^{\mathfrak{M}+\alpha}\right\}$. Then $\omega_n(z)=1$ on $\left\{\arg z=\frac{\theta}{n},\ e^{\mathfrak{M}}<|z|<e^{\mathfrak{M}+\alpha}\right\}$, $\frac{\partial}{\partial n}\omega_n(z)=0$ on $\{\arg z=0,\ e^{\mathfrak{M}}<|z|<e^{\mathfrak{M}+\alpha}\}$. By putting $\omega_n(\hat{z})=\omega_n(z)$, $\omega_n(z)$ can be continued harmonically into $\left\{-\frac{\theta}{n}<\arg z<\frac{\theta}{n},\ e^{\mathfrak{M}}<|z|<e^{\mathfrak{M}+\alpha}\right\}$, where $\hat{z}$ is the symmetric point of z with respect to $0=\arg z$. Hence by Lemma 2, for any $\varepsilon>0$, there exists a number n_0 such that $\omega_n(z)>1-\varepsilon$ on $|z|=e^{\mathfrak{M}+\varepsilon}$ in T for $n>n_0$. Clearly for the same number $\omega_n(z)>1-\varepsilon$ on $|z|=e^{\mathfrak{M}+\varepsilon}$ in T. Hence we have Lemma 7 similarly as Lemma 3.

In the following we investigate the behaviour of a ring (or a rectangle) as its module $\mathfrak{M}\rightarrow0$. Let $0<k<1$. The upper half plane: $\mathrm{Im}\,z>0$ is mapped onto a rectangle $\{-K<\mathrm{Re}\,\zeta<K,\ 0<\mathrm{Im}\,\zeta<K'\}$ by

$$\eta(z)=\int_0^z\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

so that $-\frac{1}{k},-1,1,\frac{1}{k}\rightarrow-K+iK',-K,K,K+iK'$ respectively, where K and K' are given by $K=\int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$, $K'=\int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}$: $k'^2=1-k^2$.

We denote the above rectangle in the η-plane by $R(K, K', \eta)$. Since we investigate the case k is near to 1, we put $k=1-\varepsilon^2$ and suppose $k>\frac{5}{6}$. Then the properties of $R(K, K', \eta)$ depends mostly on ε. We shall prove

LEMMA 8. 1) *Put* $k=1-\varepsilon^2\left(>\frac{5}{6}\right)$. *Then* K *and* K' *are given as follows*

$$\left|\frac{2}{k}\log\sqrt{\frac{5\delta}{6}}\right|+\frac{-2}{\sqrt{(2-\delta)(1-k-k\delta)}}\log\frac{\sqrt{1-k}}{1+\sqrt{k}}\geqq K$$

$$\geqq\frac{-2}{\sqrt{2(1+k)k}}\log\frac{\sqrt{1-k}}{1+\sqrt{k}}$$

$$=\frac{-2}{\sqrt{2(1+k)k}}\left(\log\frac{\varepsilon}{2}-\frac{\varepsilon^2}{8}+0(\varepsilon^3)\right),\qquad 0<\delta<\frac{1}{6}. \tag{6}$$

$$\frac{\pi}{2}\leqq K'\leqq\frac{\pi}{2\sqrt{1-\varepsilon^2}}. \tag{7}$$

2) *Let* $\omega(\eta)$ *be the H.M. of vertical sides of* $R(K, K', \eta)$. *Then*

$$\omega\left(\frac{iK'}{2}\right)=\frac{1}{\pi}\left(\varepsilon^2+\frac{\varepsilon^4}{2}+O(\varepsilon^6)\right).$$

3) *On the behaviour of the mapping* $\eta(z)$. *Let* ${}_{\varepsilon}V(1)=\{\operatorname{Im} z>0,\ |z-1|<\varepsilon\}$. *Then the image of* ${}_{\varepsilon}V(1)$ *falls in* $\{|\eta-K|<L\}$, *where*

$$L\leqq\frac{1}{\sqrt{k(2-k)(2-2\varepsilon)}}\left(-\log\varepsilon+\pi+\log 2+\frac{5\varepsilon^2}{16}+0(\varepsilon^3)\right)$$

and $\frac{L}{K}\to c<\frac{1}{2-\delta'}$ *as* $\varepsilon\to 0$ *for any* $\delta'\to 0$. *Hence there exists a const.* ε^* *such that the inverse image of the subrectangle* $R\left(\frac{K}{3}, K', \eta\right)$ *does not touch* ${}_{\varepsilon}V(1)+{}_{\varepsilon}V(-1)$ *for* $\varepsilon<\varepsilon^*$.

4) *Map* $R(K, K', \eta)$ *by* $\zeta=\frac{K}{\eta}$ *onto a rectangle* $R\left(1, \frac{K'}{K}, \zeta\right)$. *Then* $R\left(\frac{K}{3}, K', \eta\right)\to R\left(\frac{1}{3}, \frac{K'}{K}, \zeta\right)$. *Let* $U(\zeta)$ *be a harmonic function in* $R\left(1, \frac{K'}{K}, \zeta\right)$ *such that* $|U(\zeta)|\leqq M$ *on the vertical sides and* $U(\zeta)=0$ *on the horizontal sides. Then*

$$|\operatorname{grad} U(\zeta)|\leqq MKC\varepsilon \text{ in } R\left(\frac{1}{3}, \frac{K'}{K}, \eta\right)\qquad \text{for}\quad \varepsilon<\varepsilon^*:\ C=\frac{6\sqrt[4]{19}}{\pi}.$$

By noting $K\varepsilon\to 0$ *as* $\varepsilon\to 0$, *we see for any given* $\gamma>0$ *there exists a const.* ε_0 *such that* $|\operatorname{grad} U(\zeta)|<\gamma$ *in* $R\left(\frac{1}{3}, \frac{K'}{K}, \zeta\right)$ *for* $\varepsilon<\varepsilon_0$.

Proof of 1)

$$\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \leqq \int_0^{1-\delta} \frac{dt}{\sqrt{(1-t)(1-kt)}}$$
$$+\frac{1}{\sqrt{(2-\delta)(1+k-k\delta)}} \int_{1-\delta}^1 \frac{dt}{\sqrt{(1-t)(1-kt)}}.$$

Now by $(\sqrt{k\delta}+\sqrt{1-k\delta+k})<1$, we have

$$\frac{-2}{k} \log(\sqrt{k\delta}+\sqrt{1+k\delta-k}) \leqq \left| \frac{2}{k} \log \sqrt{\frac{5\delta}{6}} \right|.$$

Hence

$$K \leqq \frac{2}{k} \left| \log \frac{5\delta}{6} \right| + \frac{-2}{\sqrt{(2-\delta)(1+k-k\delta)k}} \log \frac{\sqrt{1-k}}{1+\sqrt{k}}.$$

On the other hand,

$$K=\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \geqq \frac{1}{\sqrt{2(1+k)}} \int_0^1 \frac{dt}{\sqrt{(1-t)(1-kt)}}$$
$$=\frac{-2}{\sqrt{2k(1+k)}} \log \frac{\sqrt{1-k}}{1+\sqrt{k}}.$$

Put $k=1-\varepsilon^2$. Then

$$\log \frac{\sqrt{1-k}}{1+\sqrt{k}} = \log \varepsilon - \log(1+\sqrt{k}) = \log \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + O(\varepsilon^4).$$

Hence we have (6). Clearly $\frac{\pi}{2} \leqq K' \leqq \frac{\pi}{2\sqrt{1-\varepsilon^2}}$. Thus we have 1).

Proof of 2) Map the upper half z-plane by $\xi = i\left(\dfrac{z-\frac{i}{\sqrt{k}}}{z+\frac{i}{\sqrt{k}}}\right)$ to $|\xi|<1$. Then by the mapping $\eta \to z \to \xi$, $\eta = \frac{K'i}{2} \to z = \frac{i}{\sqrt{k}} \to \xi = 0$ and the vertical sides of $R(K, K', \eta)$ are mapped onto arcs on $|\xi|=1$ with length $=4\tan^{-1}\frac{1-k}{1+k}=\frac{1}{\pi}\left(\varepsilon^2+\frac{\varepsilon^4}{2}+O(\varepsilon^6)\right)$. Hence we have 2).

Proof of 3) Let $z \in V_\varepsilon(1)$. Then $z=1+re^{i\theta}$, $r<\varepsilon$. Since $\int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$ does not depend on the integration path, we can suppose it is a straight connecting 1 with $1+re^{i\theta}$. We estimate the integration. Let $t=1+re^{i\theta}$. Then

$$\sqrt{1+t} \geqq \sqrt{2-r} \geqq \sqrt{2-\varepsilon} \quad \text{and} \quad |\sqrt{1+kt}| \geqq \sqrt{1+k-kr} \geqq \sqrt{2-2\varepsilon}.$$

Now $|(1-t)(1-kt)| = r|(1-kre^{i\theta}-k)| \geqq r|1-k-kr|$ and $|1-k-kr| \geqq 1-k-kr$ or $\geqq kr+k-1$ according as $r \leqq \frac{1-k}{k}$ or $r \geqq \frac{1-k}{k}$. Hence by $\frac{1-k}{k} = \frac{\varepsilon^2}{1-\varepsilon^2}$ we have

$$\int_0^{\varepsilon e^{i\theta}} \frac{dr}{\sqrt{re^{i\theta}(1-k-kre^{i\theta})}}$$

$$\leqq \int_0^{\frac{\varepsilon^2}{1-\varepsilon^2}} \frac{dr}{\sqrt{r(1-kr-k)}} + \int_{\frac{\varepsilon^2}{1-\varepsilon^2}}^{\varepsilon} \frac{dr}{\sqrt{r(kr-1+k)}}$$

$$= \frac{-2}{\sqrt{1-\varepsilon^2}} \tan^{-1}\left[\frac{\frac{\varepsilon^2}{1-\varepsilon^2}-r}{r}\right]_0^{\frac{\varepsilon^2}{1-\varepsilon^2}} + \frac{2}{\sqrt{1-\varepsilon^2}}\left[\log\left(\sqrt{r}+\sqrt{r-\frac{\varepsilon^2}{1-\varepsilon^2}}\right)\right]_{\frac{\varepsilon^2}{1-\varepsilon^2}}^{\varepsilon}$$

$$= \frac{\pi}{k} + \frac{2}{k}\left(\log\frac{1}{\sqrt{\varepsilon}} + \frac{1}{2}\log(1-\varepsilon-\varepsilon^2)\right)$$

$$= \frac{\pi}{\sqrt{k}} + \frac{-\log\varepsilon}{\sqrt{k}} + \frac{1}{\sqrt{k}}\left(\log 2 + \frac{5\varepsilon^2}{16} - \frac{3\varepsilon^3}{8} + O(\varepsilon^4)\right).$$

Thus

$$\left|\int_0^{1+re^{i\theta}} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\right| \leqq \frac{1}{\sqrt{(2-\varepsilon)(2-2\varepsilon)k}}$$

$$\left(-\log\varepsilon + \pi + \log 2 + \frac{5\varepsilon^2}{16} + O(\varepsilon)^3\right) = L : r<\varepsilon .$$

The same fact occurs for $V_\varepsilon(-1)$. By (6) we have

$$\lim \frac{L}{K} = c < \frac{1}{2-\delta'} \quad \text{for any } \delta'>0 \text{ as } \varepsilon\to 0 .$$

Hence we have 3).

Proof of 4). The mapping $z\to\eta\to\zeta$ is denoted by $\zeta=f(z)$, where $\zeta=\frac{\eta}{K}$. Then by 3) there exists a const. ε^* such that $R\left(\frac{1}{3}, \frac{K'}{K}, \zeta\right)$ is mapped onto a domain G in $\{\mathrm{Im}\, z>0\}$ such that G does not touch $V_\varepsilon(1)+V_\varepsilon(-1)$ for $\varepsilon<\varepsilon^*$. In the following we suppose $\varepsilon<\varepsilon^*<\frac{1}{4}$. A harmonic function $U(\zeta)$ in $R\left(1, \frac{K'}{K}, \zeta\right)$ is transformed to $U(z)$ such that $U(z)=0$ on $\left\{\mathrm{Im}\, z=0,\ -\infty<\mathrm{Re}\, z<-\frac{1}{k}\right\}$ $\left\{\mathrm{Im}\, z=0, \frac{1}{k}\leqq \mathrm{Re}\, z<\infty\right\}$ and $|U(z)|\leqq M$ on ${}_-I+{}_+I$, where ${}_-I=\left\{\mathrm{Im}\, z=0,\ -\frac{1}{k}\leqq \mathrm{Re}\, z\leqq 1\right\}$ and ${}_+I=\left\{\mathrm{Im}\, z=0,\ 1\leqq \mathrm{Re}\, z\leqq \frac{1}{k}\right\}$. Then

$$U(z)=\frac{1}{\pi}\int_{{}_-I+{}_+I} U(t)K(z,t)dt ,$$

where

$$K(z,t)=\frac{y}{(x-t)^2+y^2} \quad \text{and} \quad \underset{z}{\mathrm{grad}}\, K(z,t)=\frac{1}{(x-t^2)+y^2} .$$

We estimate $\underset{z}{\mathrm{grad}}\, K(z,t)$ for $t\in{}_+I$ and $z\in G$. Put $t=1+\varepsilon'$ and $z=1+re^{i\theta}$. Then by $t\in {}_{+}I$, $1+\varepsilon'\leqq\frac{1}{k}=\frac{1}{1-\varepsilon^2}$ and $r>\varepsilon$ for $z\in G$. Hence by $\varepsilon<\varepsilon^*<\frac{1}{4}$ and $\varepsilon<r$ we have

$$\varepsilon'\leqq\varepsilon^2<\frac{\varepsilon}{4}<\frac{r}{4}. \tag{8}$$

By (8) $(x-t)^2+y^2\geqq r^2-2r\varepsilon'\cos\theta+\varepsilon'^2\geqq r^2-2r\varepsilon'\geqq\frac{r^2}{2}$: $t\in{}_{+}I$, $z\in G$ and

$$|\operatorname{grad}_z K(z, y)|\leqq\frac{2}{r^2}. \tag{9}$$

We have also

$$\sqrt{1-z^2}\leqq r^{\frac{1}{2}}(r+2)^{\frac{1}{2}} \quad\text{and}\quad \sqrt{1+kz}\leqq(r+2)^{\frac{1}{2}}; \qquad t\in{}_{+}I,\quad z\in G. \tag{10}$$

By $\varepsilon^2<\frac{\varepsilon}{4}<\frac{r}{4}$, $\varepsilon^4<\frac{r^2}{16}$

$$\sqrt{1-kz}\leqq r^{\frac{1}{2}}\sqrt[4]{19}, \qquad t\in{}_{+}I,\quad z\in G. \tag{11}$$

For $t\in{}_{-}I$ we have the same estimation for $z\in G$. Hence by (9), (10), (11) we have

$$\begin{aligned}|\operatorname{grad}_\zeta U(\zeta)|&\leqq\frac{M}{2\pi}\int_{-I+{}_{+}I}\operatorname{grad} K(z, t)\left|\frac{dz}{d\xi}\right|\left|\frac{d\xi}{dt}\right|dt\\ &\leqq\frac{M}{2\pi}\int_{-I+{}_{+}I}\frac{2}{r^2}\sqrt{1-z^2}\sqrt{1-k^2z^2}\,Kdt\\ &\leqq\frac{2M(2+r)\sqrt[4]{19}}{\pi r}\varepsilon^2K.\end{aligned}$$

Whence $\operatorname{grad}_\zeta U(\zeta)\leqq\frac{6\sqrt[4]{19}M\varepsilon^2K}{\pi}$: $r\geqq1$ and $\operatorname{grad}_\zeta U(\zeta)\leqq\frac{6\sqrt[4]{19}M\varepsilon K}{\pi}$: $r\leqq1$. Thus

$$|\operatorname{grad} U(\zeta)|\leqq6\sqrt[4]{19}M\varepsilon K.$$

Now by (6) $K\varepsilon\to0$ as $\varepsilon\to0$. Hence for any $\gamma>0$ there exists $\varepsilon_0<\varepsilon^*$ such that

$$|\operatorname{grad} U(\zeta)|<\gamma \quad\text{in}\quad R\left(\frac{1}{3}, \frac{K'}{K}, \zeta\right) \qquad\text{for}\quad \varepsilon<\varepsilon_0.$$

LEMMA 9. *Let R be a rectangle $\{-\theta\leqq\operatorname{Re}\zeta\leqq\theta,\ 0\leqq\operatorname{Im}\zeta\leqq2\mathfrak{M}\theta\}$. Let ${}^{U}_{\delta}I^i_n({}^{L}_{\delta}I^i_n)$ be a slit: $0<\delta<\frac{1}{2}$ as follow*

$$\begin{aligned}{}^{U}_{\delta}I^i_n&=\left\{\operatorname{Re}\zeta=\frac{2i\theta}{n}-\theta,\ \left(\frac{3+\delta}{2}\right)\mathfrak{M}\theta\leqq\operatorname{Im}\zeta\leqq2\mathfrak{M}\theta\right\},\\ &\qquad\qquad\qquad\qquad\qquad\qquad i=1, 2, \cdots, n-1.\\ {}^{L}_{\delta}I^i_n&=\left\{\operatorname{Re}\zeta=\frac{2i\theta}{n}-\theta,\ 0\leqq\operatorname{Im}\zeta\leqq\left(\frac{1}{2}-\delta\right)\mathfrak{M}\theta\right\}.\end{aligned}$$

We denote this rectangle with the slits by $R(\theta, \mathfrak{M}\theta, \delta, n)$. Let R' be a rectangle *in* $R(\theta, \mathfrak{M}\theta, \delta, n)$ *such that* $-\frac{\theta}{3}\leqq \mathrm{Re}\leqq\frac{\theta}{3}, \frac{3\mathfrak{M}\theta}{4}\leqq \mathrm{Im}\,\zeta\leqq\frac{5\mathfrak{M}\theta}{4}$. *Then for any given* $\varepsilon>0$, *there exist numbers* $\mathfrak{M}, n, \delta$ *such that*

$$|\operatorname*{grad}_{\zeta} U(\zeta)|<\varepsilon \quad in \quad R',$$

for any harmonic function $U(\zeta)$ *in* $R(\theta, \mathfrak{M}\theta, \delta, n)$ *such that* $|U(\zeta)|\leqq 1$, $U(\zeta)=0$ *on* $\sum_i({}^{u}_{\delta}I^i_n+{}^{l}_{\delta}I^i_n)$ *and* $D(U(\zeta))\leqq 1$.

Proof. At first we determine $\mathfrak{M}$. By 4) of lemma 8, for any $\varepsilon>0$ there exists a number $\mathfrak{M}$ (this is equivalent to the existence of $k=1-\varepsilon^2$) such that $|\mathrm{grad}\, U(\zeta)|<\varepsilon$ in $R\left(\frac{\theta}{3}, \frac{\mathfrak{M}\theta}{2}\right)=\left\{-\frac{\theta}{3}\leqq \mathrm{Re}\,\zeta\leqq\frac{\theta}{3}, \frac{\mathfrak{M}\theta}{2}\leqq \mathrm{Im}\,\zeta\leqq\frac{3\mathfrak{M}\theta}{2}\right\}$ for any harmonic function $U(\zeta)$ in $R\left(\theta, \frac{\mathfrak{M}\theta}{2}\right)=\left\{-\theta\leqq \mathrm{Re}\,\zeta\leqq\theta, \frac{\mathfrak{M}\theta}{2}\leqq \mathrm{Im}\,\zeta\leqq\frac{3\mathfrak{M}\theta}{2}\right\}$ vanishing on the horizontal sides and $|U(\zeta)|\leqq 1$ on vertical sides. Fix $\mathfrak{M}$ and denote it by $\mathfrak{M}_0$. Secondly we determine δ. Let $G(\zeta, p)$ be a Green's function of $R\left(\theta, \frac{\mathfrak{M}\theta}{2}\right)$. Then there exists a const. M such that $\left|\mathrm{grad}\frac{\partial}{\partial n}G(\zeta, p)\right|\leqq M$ for $\zeta\in\partial R\left(\theta, \frac{\mathfrak{M}\theta}{2}\right)$ and $p\in R\left(\frac{\theta}{3}, \frac{\mathfrak{M}\theta}{4}\right)=\left\{-\frac{\theta}{3}\leqq \mathrm{Re}\,\zeta\leqq\frac{\theta}{3}, \frac{3\mathfrak{M}\theta}{4}\leqq \mathrm{Im}\,\zeta\leqq\frac{5\mathfrak{M}\theta}{4}\right\}$. A rectangle with vertical slits is mapped by $z=e^{i\zeta}$ onto a circular trapezoid with circular slits. Hence Lemma 1′ is applicable to a rectangle. Let $R_\delta=\left\{-\theta\leqq \mathrm{Re}\,\zeta\leqq\theta, 0\leqq \mathrm{Im}\,\zeta\leqq\frac{\mathfrak{M}\theta}{2}\right\}$ with vertical slits $\{{}^{l}_{\delta}I^i_n\}$. Let δ_0 be the number and fix it, where

$$\delta_0\leqq\frac{\varepsilon^2\pi^2}{16M^2\theta}. \tag{12}$$

Let $U_n(\zeta)$ be a harmonic function in R_δ such that $D(U_n(\zeta))\leqq 1$ vanishing on $\{{}^{l}_{\delta}I^i_n\}$. Then by Lemma 7 $\lim_n D(U_n(\zeta))\geqq\frac{1}{\delta_0}\int_{\mathrm{Im}\,\zeta=\frac{\mathfrak{M}\theta}{2}}U(\zeta)^2 d\theta$. Hence there exists a number n_0 such that

$$\int_{\mathrm{Im}\,\zeta=\frac{\mathfrak{M}\theta}{2}}U_n^2(\zeta)d\theta\leqq 2\delta_0=\frac{\varepsilon^2\pi^2}{8M^2\theta} \qquad \text{for} \quad n\geqq n_0. \tag{13}$$

Fix such n_0. Then such numbers $\mathfrak{M}_0, \delta_0, n_0$ are required numbers. Similar fact occurs in $\left\{-\theta<\mathrm{Re}\,\zeta<\theta, \frac{3\mathfrak{M}\theta}{2}\leqq \mathrm{Im}\,\zeta\leqq 2\mathfrak{M}\theta\right\}$. Let $U(\zeta)$ be a harmonic function in $R(\theta, \mathfrak{M}_0\theta, \delta_0, n_0)$ satisfying the condition of Lemma 9. Put $U_1(\zeta)=\frac{1}{2\pi}\int_A U(t)\frac{\partial}{\partial n}G(t, \zeta)ds$ and $U_2(\zeta)=\int_B U(t)\frac{\partial}{\partial n}G(t, \zeta)ds$, where $G(t, \zeta)$ is a Green's function of $R\left(\theta, \frac{\mathfrak{M}\theta}{2}\right)$, A and B are vertical and horizontal sides. Then $|U_1(\zeta)|\leqq 1$ on A and $=0$ on B. Hence $|\mathrm{grad}\, U_1(\zeta)|<\frac{\varepsilon}{2}$ in $R\left(\frac{\theta}{3}, \frac{\mathfrak{M}\theta}{2}\right)$. By Schwarz's inequality $U_2(\zeta)$ satisfies by (13) $\int_B |U_2(\zeta)|d\theta<\frac{\pi\varepsilon}{2M}$ and $|\mathrm{grad}\, U_2(\zeta)|<\frac{\varepsilon}{2}$ in $R\left(\frac{\theta}{3}, \frac{\mathfrak{M}\theta}{4}\right)=\left\{-\frac{\theta}{3}\leqq \mathrm{Re}\,\zeta\leqq\frac{3\theta}{4}, \frac{3\mathfrak{M}\theta}{4}\leqq \mathrm{Im}\,\zeta\leqq\frac{5\mathfrak{M}\theta}{4}\right\}$. Thus $|\mathrm{grad}\, U(\zeta)|<\varepsilon$ in $R\left(\frac{\theta}{3}, \frac{\mathfrak{M}\theta}{4}\right)$ for any harmonic function $U(\zeta)$ in $R(\theta, \mathfrak{M}_0\theta, \delta_0, n_0)$ satisfying the condition of Lemma 9.

Strong surface with exception δ and deviation ε. Let R be the same leaf of $R(\theta, \mathfrak{M}_0\theta, \delta_0, n_0)$ of Lemma 9. Identify $\{{}^{U}_{\delta}I^i_n+{}^{L}_{\delta}I^i_n\}$ of $\hat{R}$ and $R(\theta, \mathfrak{M}_0\theta, \delta_0, n_0)$. Then we have a surface $\tilde{R}$. As case of Lemma 5 $\left|\frac{d}{dt}f(t)\right|\leqq 2\sqrt{2}\,\varepsilon$: proj. $t\in R\left(\frac{\theta}{3}, \frac{\mathfrak{M}_0\theta}{4}\right)$ in Lemma 9 for any analytic function $f(t): t\in\tilde{R}$ with $|f(t)|\leqq\frac{1}{2}$ and $D(f(t))\leqq\frac{1}{4}$. Let l be an integer and put $\theta=\frac{\pi}{l}$ and let $\mathfrak{M}_0, \delta_0, n_0$ be numbers in Lemma 9 corresponding to θ. Let ${}_{-}I^i$ and ${}_{+}I^i$ $(i=1, 2, 3, \cdots, ln_0)$ in $\{a\leqq|w|\leqq ae^{\mathfrak{M}^0}\}: \mathfrak{M}^0=\frac{\pi\mathfrak{M}_0}{l}$ such that

$${}_{-}I^i=\left\{\arg w=\frac{2i\theta}{n_0}: a\leqq|w|\leqq ae^{\mathfrak{M}_0\left(\frac{1}{2}-\delta_0\right)}\right\},$$

$${}_{+}I^i=\left\{\arg w=\frac{2i\theta}{n_0},\ ae^{\left(\frac{3}{4}+\delta_0\right)\mathfrak{M}^0}\leqq|w|\leqq ae^{2\mathfrak{M}^0}\right\}.$$

Let $R^W=\{a\leqq|w|\leqq ae^{2\mathfrak{M}^0}\}-\sum_i({}_{-}I^i+{}_{+}I^i)$ and let $\hat{R}$ be the same leaf as R^W. Identify $\{{}_{-}I^i+{}_{+}I^i\}$. Then we have a surface $\tilde{R}^W$. Let $A(\theta_1, \theta_2)\frown\tilde{R}^W$ be the part of $\tilde{R}^W$ over $\theta_1<\arg w<\theta_2$. Then $A\left(\frac{2j_0\theta}{n_0}, \frac{2j_0\theta+2n_0\theta}{n_0}\right)\frown R^W$ is mapped conformally onto $R(\theta, \mathfrak{M}_0, \delta_0, n_0)$ in Lemma 9 by $w=ae^{-iz+\frac{2j_0\theta}{n_0}}$ where j_0 is an integer. Hence we have at once

a) Let j_1 and j_2 be integers such that $j_2-j_1\geqq n_0$. Then since $A\left(\frac{2j_1\theta}{n_0}, \frac{2j_2\theta}{n_0}\right)\supset A\left(\frac{2j_3\theta}{n_0}, \frac{2j_3\theta+2n_0\theta}{n_0}\right)$ for $j_1\leqq j_3\leqq j_2-n_0$, $\left|\frac{d}{dt}f(t)\right|\leqq 2\sqrt{2}\,\varepsilon$ in $A\left(\frac{2j_1\theta}{n_0}+\frac{2\theta}{3}, \frac{2j_2\theta}{n_0}-\frac{2\theta}{3}\right)\frown\tilde{R}^W$ over $ae^{\frac{3}{4}\mathfrak{M}^0}<|w|<ea^{\frac{5}{4}\mathfrak{M}^0}$ for and $f(t)$ which is analytic in $A\left(\frac{2j_1\theta}{n_0}, \frac{2j_2\theta}{n_0}\right)\frown\tilde{R}^W$ with $|f(t)|\leqq\frac{1}{2}$ and $D(f(t))\leqq\frac{1}{4}$.

b) Let $\delta=\frac{2\theta}{3}+\frac{2\theta}{n_0}$. Then $\left|\frac{d}{dt}f(t)\right|\leqq 2\sqrt{2}\,\varepsilon$ in $A(\theta_1+\delta, \theta_2-\delta)\frown\tilde{R}^W$ over $ae^{\frac{3}{4}\mathfrak{M}^0}<|w|<ae^{\frac{5}{4}\mathfrak{M}^0}$ for any $f(t)$ in $A(\theta_1, \theta_2)\frown\tilde{R}^W$ for $\theta_2-\theta_1\geqq 4\delta$ with $|f(t)|<\frac{1}{2}$, $D(f(t))\leqq\frac{1}{4}$.

In fact, $\theta_2-\theta_1\geqq 4\delta\geqq 2\theta+\frac{8\theta}{n_0}$. We can find $\theta_1\leqq\theta_1'<\theta_2'\leqq\theta_2$ such that $0\leqq\theta_2-\theta_1$

$\leqq\frac{2\theta}{n_0}$, $0\leqq\theta_1'-\theta_1\leqq\frac{2\theta}{n_0}$ and $\theta_1'=\frac{2j_1\theta}{n_0}$, $\theta_2'=\frac{2j_2\theta}{n_0}$, where j_1 and j_2 are integers. Now $j_2-j_1\geqq n_0$, hence by a) $\left|\frac{d}{dt}f(t)\right|\leqq 2\sqrt{2\varepsilon}$ in $A\left(\frac{2j_1\theta}{n_0}+\frac{2\theta}{3},\ \frac{2j_2\theta}{n_0}-\frac{2\theta}{3}\right)\cap\tilde{R}^W$ over $ae^{\frac{3}{4}\mathfrak{M}^0}<|w|<ae^{\frac{5}{4}\mathfrak{M}^0}$. Now $A\left(\frac{2j_1\theta}{n_0}+\frac{2\theta}{3},\ \frac{2j_2\theta}{n_0}-\frac{2\theta}{3}\right)\supset A(\theta_1+\delta,\ \theta_2-\delta)$ by $\frac{2j_1\theta}{n_0}+\frac{2\theta}{3}-\theta_1<\frac{2\theta}{3}+\frac{2\theta}{n_0}<\delta$ and $\theta_2-\left(\frac{2j_2\theta}{n_0}-\frac{2\theta}{3}\right)\leqq\frac{2\theta}{3}+\frac{2\theta}{n_0}<\delta$ and we have b).

In general, let R be a ring surface consisting of two leaves obtained by identifying radial slits over $a<|w|ae^{\mathfrak{M}}$. If $\left|\frac{d}{dt}f(t)\right|<\varepsilon$ over $\{ae^{\frac{3}{4}\mathfrak{M}}<|w|<ae^{\frac{5}{4}\mathfrak{M}}\}$ $\cap A(\theta_1+\delta,\ \theta_2-\delta)\cap\tilde{R}$ for any analytic function $f(t)$ in $A(\theta_1,\theta_2)\cap\tilde{R}$ with $|f(t)|<\frac{1}{2}$ and $D(f(t))\leqq\frac{1}{4}$: $\theta_2-\theta_1>4\delta$, we call $\tilde{R}$ a strong surface with exception δ and deviation ε. In fact the surface $\tilde{R}^W$ discussed above is a strong surface with exception $\frac{2\theta}{3}+\frac{2\theta}{n_0}$ and deviation $2\sqrt{2\varepsilon}$.

α,β-thin set. Let $\tilde{G}$ be a strong surface with exception δ and with deviation ε over $a<|z|<ae^{\mathfrak{M}}$. Let F be a closed set in $\tilde{G}$. We say F is α,β-thin set in G, if F is so thinly distributed that there exists a Jordan curve Γ in $G-F$ and $\hat{G}-F$ such that 1) proj Γ separates $|z|=a$ from $|z|=ae^{\mathfrak{M}}$, 2) length of $\Gamma\leqq\alpha a$. 3) $\left|\frac{d}{dt}f(t)\right|<\beta\varepsilon$ on $\Gamma\cap A^{\tilde{G}}(\theta_1+\delta,\ \theta_2-\delta)$ for any analytic function $f(t)$ in $(\tilde{G}-F)$ $\cap A^{\tilde{G}}(\theta_1,\theta_2):\theta_2-\theta_1\geqq 4\delta$ with $|f(t)|\leqq\frac{1}{2}$ and $D(f(t))\leqq\frac{1}{4}$, where $A^{\tilde{G}}(\theta_1,\theta_2)$ means the part of $\tilde{G}$ over $\theta_1\leqq\arg z\leqq\theta_2$.

EXAMPLE 3. Let $U=|z|<1$ and $1>a_1>a_2\cdots\downarrow 0$ and

$$\sum_n \log\frac{a_{2n+1}}{a_{2n+2}}=\infty\,. \tag{14}$$

Let $J_n=\{\arg z=\pi,\ a_{2n+2}\leqq|z|\leqq a_{2n+1}\}$ be a slit and $R(a_{2n+2},a_{2n+1})=\{a_{2n+2}\leqq|z|\leqq a_{2n+1}\}$. Let $1>b_1'>b_1>b_2'>b_2\cdots\downarrow 0$ G_n be a ring $b_n\leqq|z|\leqq b_n'$ with slits $\sum_{i=1}^{j(n)}I^i_{j(n)}$ such that we can construct a strong surface $\tilde{G}_n$ with exception $\delta=\frac{1}{n}$ with deviation c_n, where $\lim_{n=\infty}\frac{\log c_n}{\log b_n}=\infty$, $\sum_n^\infty R(b_n,b_n')\cap\sum_n^\infty R(a_{2n+2},a_{2n+1})=0$ and

$$\lim_{n=\infty}\frac{\log b_n}{\log b_n'}=1\,. \tag{15}$$

Let $\mathfrak{F}$ be a unit circle wtih slits $\sum^\infty J_n+\sum_{n=1}^\infty\sum_{i=1}^{j(n)}I^i_{j(n)}$ and $\hat{\mathfrak{F}}$ be the same leaf as $\mathfrak{F}$. Identify $J_n+I_n^i$ of $\mathfrak{F}$ and $\hat{\mathfrak{F}}$. Then we have a Riemann surface $\tilde{\mathfrak{F}}$. Evidently $\tilde{\mathfrak{F}}$ has one boundary component $\mathfrak{p}$. The part of $\tilde{\mathfrak{F}}$ over $R(a_{2n+1},a_{2n+2})$ is a ring with module $=\frac{1}{2}\log\frac{a_{2n+1}}{a_{2n+2}}$ and separates $\partial\mathfrak{F}$ from $\mathfrak{p}$, hence $\tilde{\mathfrak{F}}$ is an end of another Riemann surface $\in O_g$ and $\mathfrak{p}$ is of harmonic dimension $=1$ and there exists only one Martin point p over $\mathfrak{p}$.

a) *Let* F_1 *be a set of radial slits in* $\tilde{\mathfrak{F}}-\sum_n \tilde{G}_n$.

b) $F_2\frown\tilde{G}_n$ *is an* α,β*-thin set for* $n=1,2,\cdots$.

c) *Let* $A(\theta_1,\theta_2)=\{\theta_1<\arg z<\theta_2\}$. *Let* $A^{\mathfrak{F}}(\theta_1,\theta_2,b'_n)$(*or* $\hat{\mathfrak{F}}$) *be the part of* $\mathfrak{F}$ *over* $A(\theta_1,\theta_2)$ *bounded by* $|z|=1$, $|z|=b'_n$, $\arg z=\theta_1$ *and* $\arg z=\theta_2$. *Let* $U_n(z)$ *be a harmonic function in* $A^{\mathfrak{F}}(\theta_1,\theta_2,b'_n)-F_1-F_2$ *such that* $U_n(z)=0$ *on* $|z|=1$, $=1$ *on* $|z|=b'_n$ *and has M.D.I. Then*

$$D(U_n(z))\geqq\frac{\gamma(\theta_2-\theta_1)}{-\log b'_n}:\ \ \gamma>0\,. \tag{16}$$

d) $v(p)\frown A^{\mathfrak{F}}(\theta_1,\theta_2,b)-F_1-F_2$ *is connected for* $\theta_2>\theta_1$. *If* F_1 *and* F_2 *satisfy the above conditions, then the part of* $(v(p)-F_1-F_2)$ *over* $A(\theta_1,\theta_2)\in O_{ABF}$ *for any* $\theta_2>\theta_1$.

Proof. Assume there exists a non const. analytic fuuction $f(t)$ in the part $v(p)-F_1-F_2$ over $A(\theta_1,\theta_2)$. Then $f(t)$ is a finite number of sheets covering. $\sup|f(t)|<\infty$ implies $D(f(t))<\infty$. Hence we can suppose $|f(t)|<\frac{1}{2}$, $D(f(t))\leqq\frac{1}{4}$. Let n_0 be the number such that $4\delta_{n_0}<\frac{\theta_2-\theta_1}{4}$. Let $\theta'_1=\frac{3\theta_1+\theta_2}{4}$, $\theta'_2=\frac{\theta_1+3\theta_2}{4}$. Then $\theta_1<\theta'_1<\theta'_2<\theta_2$. Let $(v(p)-F_1-F_2)\frown A(\theta'_1,\theta'_2)$ be the part of $v(p)-F_1-F_2$ over $A(\theta'_1,\theta'_2)$. The existence of $f(t)$ in $(v(p)-F_1-F_2)\frown A(\theta_1,\theta_2)$ implies there exists a Jordan curve Γ_n such that $\left|\frac{d}{dt}f(t)\right|<\beta c_n$ on Γ_n in $(v(p)-F_1-F_2)\frown A(\theta'_1,\theta'_2)$ and length of $\Gamma_n<\alpha b_n$. Let $\mathfrak{F}_n$ be the part of $\mathfrak{F}\frown(v(p)-F_1-F_2)$ over $A(\theta'_1,\theta'_2)$ bounded by $\partial v(p)$, F_1, F_2 $\arg z=\theta'_1$, $\arg z=\theta'_2$ and Γ_n. Let $U_n(z)$ be a harmonic function in $\mathfrak{F}_n$ such that $U_n(z)=0$ on $\partial v(p)$, $=1$ on Γ_n. Then as case of example 2 we have

$$D(U_n(z))\leqq 0\Big(\frac{1}{-\log b_n-\log c_n-\log\alpha\beta}\Big).$$

On the other hand by condition c) $D(U_n(z))\geqq 0\Big(\frac{\theta'_2-\theta''_1}{-\log b_n}\Big)$. This is a contradiction by $\lim\frac{\log c_n}{\log b_n}=\infty$. Hence we have the conclusion.

EXAMPLE 4. Let $\frac{1}{3}<a_1<a_2,\cdots\uparrow 1$ with $\sum_n\log\frac{1-a_{2n+1}}{1-a_{2n+2}}=\infty$. Let G_n be a ring $\{b_n\leqq|z|\leqq b'_n\}$ with slits $\sum_{i=1}^{j(n)} I^i_{j(n)}$ such that

1) $\frac{1}{3}<b_n<b'_n<b_{n+1}<b'_{n+1}\cdots\uparrow 1$ and $\sum_n\{b_n\leqq|z|\leqq b'_n\}\frown\sum\{a_{2n+1}\leqq|z|\leqq a_{2n+2}\}=0$.

2) $\tilde{G}_n$ is a strong surface with exception $\delta_n=\frac{1}{n}$ and deviation $\varepsilon_n:\lim_{n=\infty}\varepsilon_n=0$.

3) $\underset{|z|=\sqrt{b_n b_n'}}{\omega_n(z)} \leqq \frac{1}{n}$, where $\omega_n(z)$ is a harmonic function in G_n such that $0<\omega_n(z)\leqq 1$ on G_n and $=0$ on $\sum_i^{j(n)} I^i_{j(n)}$ (this condition is easily satisfied by Lemma 8. 2) for sufficiently many slits). Let $\mathfrak{F}$ be a unit circle with slits $I^i_{j(n)}$ ($n=1, 2, 3, \cdots$, $i=1, 2, \cdots, j(n)$). Let $\hat{\mathfrak{F}}$ be the same leaf as $\mathfrak{F}$. Identify $I^i_{j(n)}$ of $\mathfrak{F}$ and $\hat{\mathfrak{F}}$. Then we have a Riemann surface $\tilde{\mathfrak{F}}$ over $|z|<1$ with one boundary component on $|z|=1$. The part of over $G_n(b_n, b_n')$ is an strong surface. At first we investigate the structure of the boundary. Then

1) *$\tilde{\mathfrak{F}}$ has no singular point relative to N-Martin and Martin topology.*
2) *There exsits only one point on $e^{i\theta}$ relative to N-Martin topology.*

Proof of 1) Let $\tilde{\mathfrak{F}}'$ be the part of $\tilde{\mathfrak{F}}$ over $1>|z|>\frac{1}{3}$. Then $\tilde{\mathfrak{F}}'$ has relative boundary $\partial\tilde{\mathfrak{F}}'$ on $|z|=\frac{1}{3}$. We suppose N-Martin topology is defined on $\tilde{\mathfrak{F}}'$. Let $\Delta_{n,i}=\left\{1-\frac{1}{n}\leqq|z|<1, \frac{2\pi i}{n}\leqq \arg z\leqq\frac{2\pi(i+1)}{n} : i=0, 1, \cdots, n-1\right\}$. Let G be a domain in $\tilde{\mathfrak{F}}'$ and let $\omega(G, t)$: $t\in\tilde{\mathfrak{F}}'$ be capacitary potential, i. e. $\omega(G, t)$ is the harmonic function in $\tilde{\mathfrak{F}}'-G$ such that $\omega(G, t)=0$ on $\partial\tilde{\mathfrak{F}}'$, $=1$ on G and has M. D. I. Let $U(z)$ be a harmonic function in $\left\{\frac{1}{3}<|z|<1\right\}-\Delta_{n,i}$ such that $U(z)=1$ on $\Delta_{n,i}$, $=0$ on $|z|= 1$ and $U(z)$ has M. D. I. Then

$$D(U(z))\leqq\frac{-\pi\log(\text{diameter of } \Delta_{n,i})}{\pi\log\frac{2}{3}}\downarrow 0 \quad \text{as} \quad n\to\infty\,.$$

Let $U'(z)$ be a harmonic function in $\left\{\frac{1}{3}<|z|<1\right\}-\sum_{n,i} I_{n,i}$ such that $U'(z)=1$ on $\Delta_{n,i}$, $=0$ on $|z|=\frac{1}{3}$ and has M.D.I. Then $D(U'(z))\leqq D(U(z))$ and $\frac{\partial}{\partial n}U'(z)=0$ on $\sum I_{n,i}$. Put $U'(t)=U'(z)$ ($z=\text{proj}\, t$) in $\tilde{\mathfrak{F}}'-\Delta^{\tilde{\mathfrak{F}}}_{n,i}$, where $\Delta^{\tilde{\mathfrak{F}}}_{n,i}$ is the part of $\tilde{\mathfrak{F}}'$ over $\Delta_{n,i}$. Then $U'(t)$ is harmonic in $\tilde{\mathfrak{F}}'-\Delta^{\tilde{\mathfrak{F}}}_{n,i}$. Hence $D(\omega(\Delta^{\tilde{\mathfrak{F}}}_{n,i}, t))\leqq 2D(U'(z))\downarrow 0$ as $n\to\infty$. This implies $\omega(\Delta^{\tilde{\mathfrak{F}}}_{n,i}, t)\to 0$ as $n\to\infty$. Assume there exists a singular point p relative to N-Martin topology. Then $\omega(p, t)=\lim_{m=\infty}\omega(v_m(p), t)>0$. Where $v_m(p)$ is a neighbourhood of p relative to N-Martin topology. Let t_0 be a point and let n_0 be a number such than

$$\omega(\Delta^{\tilde{\mathfrak{F}}}_{n_0,i}, t_0)<\frac{1}{3}\omega(p, t_0) \qquad \text{for} \quad i=1, 2, \cdots, n_0\,. \tag{17}$$

Now $\sum_i^{n_0}\omega(\Delta^{\tilde{\mathfrak{F}}}_{n_0,i}\cap v_m(p), t)\geqq\omega(v_m(p), t)\geqq\omega(p, t)$, where we suppose $\text{proj}\, v_m(p)\subset\left(|z|>1-\frac{1}{n_0}\right)$. Since $\omega(\Delta^{\tilde{\mathfrak{F}}}_{n_0,i}\cap v_m(p), t)\downarrow$ as $m\to\infty$, there exists at least one $\Delta^{\tilde{\mathfrak{F}}}_{n_0,i_0}$ such that $\omega(\Delta^{\tilde{\mathfrak{F}}}_{n_0,i_0}\cap v_m(p), t_0)\geqq\frac{1}{n_0}\omega(p, t_0)$ for any m. Let $m\to\infty$. Then

$\lim_{m=\infty}\omega(\Delta_{n_0,i_0}\cap v_m(p), t)=\alpha\omega(p, t)$ by $\overline{\cap v_m(p)\cap\Delta_{n_0,i_0}}\subset p$. On the other hand, $\lim_{m}\omega(\Delta_{n_0,i_0}\cap v_m(p), t)>0$ implies $\sup(\lim_{m}\omega(\Delta_{n_0,i_0}\cap v_m(p), t))=1$ and $\alpha=1$, whence $\frac{1}{3}\omega(p, t_0)\geqq\lim_{m}\omega(\Delta_{n_0,i_0}\cap v_m(p), t_0)=\omega(p, t_0)$ by (17). This is a contradiction. Hence there exsits no singular point relative to N-Martin topology. Assume there exists a singular point p relative to Martin topology. Then $\lim_{m} w(v_m(p), t)$ $=w(p, t)>0$, where $v_m(p)$ is a neighbourhood of p relative to Martin topology and $w(G, t)$ is H. M. of G i. e. is the least positive superharmonic function in $\tilde{\mathfrak{F}}'$ larger than 1 on G. Now $w(\Delta^{\tilde{\mathfrak{F}}}_{n_0,i_0}, t)\leqq\omega(\Delta^{\tilde{\mathfrak{F}}}_{n_0,i_0}, t)$. Hence we can prove similarly as case of $\omega(p, z)$ that there exists no singular point relative to Martin topology.

Proof of 2) To prove 2) we use following three facts.

a) Let t and $\hat{t}$ be points in $\mathfrak{F}$ and $\tilde{\mathfrak{F}}$ such that $\operatorname{proj} t_1=\operatorname{proj} t_2=z$. Let $U(t)$ be a harmonic function in $\tilde{\mathfrak{F}}'$ such that $|U(t)|<M$. Then $|U(t)-U(\hat{t})|\to 0$ as $|z|\to 1$.

In fact, consider $U(t)-U(\hat{t})$ over $b_n<|z|<b'_n$. Then $|U(t)-U(\hat{t})|\leqq 2M\omega_n(z)$. Hence by the maximum principle $|U(t)-U(\hat{t})|<2M\times\max(\varepsilon_n, \varepsilon_{n+1})$ over $\sqrt{b_n b'_n}\leqq|z|\leqq\sqrt{b_{n+1}b'_{n+1}}$ and we have a).

b) Let $U(t)$ be a harmonic function in $\tilde{\mathfrak{F}}'$ such that $U(t)$ has M. D. I. among all harmonic functions with the same value as $U(z)$ on $\partial\tilde{\mathfrak{F}}'$ over $\tilde{\mathfrak{F}}'$. Let G be a domain in $\tilde{\mathfrak{F}}'$. Then $\sup_{t\in G}|U(t)|\leqq\sup_{t\in\partial G}|U(t)|$.

Because let $\{\tilde{\mathfrak{F}}'_n\}$ be an exhaustion of $\tilde{\mathfrak{F}}'$ such that $\partial\tilde{\mathfrak{F}}'_n\supset\partial\tilde{\mathfrak{F}}'$ for any n and $\tilde{\mathfrak{F}}'_n\uparrow\tilde{\mathfrak{F}}'$. Let $U_n(t)$ be a harmonic function in $\tilde{\mathfrak{F}}'_n$ such that $U_n(t)=U(t)$ on $\partial\tilde{\mathfrak{F}}'$ and $\frac{\partial}{\partial n}U_n(t)=0$ on $\partial\tilde{\mathfrak{F}}'_n-\partial\tilde{\mathfrak{F}}'$. Then $U_n(t)\to U(t)$. Clearly for $U_n(t)$, by the maximum principle $\sup_{t\in G}|U_n(t)|\leqq\sup_{t\in\partial G}|U(t)|$. Hence we have b).

c) Let $U(t)$ be a harmonic function in $\tilde{\mathfrak{F}}'$ with $D(U(t))<\infty$. Then there exists a curve $\tilde{\Gamma}_n: n=1, 2, \cdots$ consisting of two components: $\tilde{\Gamma}_n=\Gamma_n+\hat{\Gamma}_n$ in such that $\Gamma_n\subset\mathfrak{F}'$, $\hat{\Gamma}_n\subset\tilde{\mathfrak{F}}'$, $\operatorname{proj}\Gamma_n=\operatorname{proj}\hat{\Gamma}_n$, $\operatorname{proj}\tilde{\Gamma}_n\cap\sum_{n,i}I^i_n=0$, $\operatorname{proj}\tilde{\Gamma}_n$ separates $z=e^{i\theta}$ from $|z|=\frac{1}{3}$, $\operatorname{proj}\tilde{\Gamma}_n\to e^{i\theta}$ as $n\to\infty$ and $\int_{\tilde{\Gamma}_n}|\operatorname{grad} U(t)|dt\to 0$ as $n\to\infty$.

Proof. We suppose $e^{i\theta}=1$. Map U (unit circle with slits I^i_n) by $\xi=\log z$ in the ξ-plane. Then $z=1\to\xi=0$, $I^i_{j(n)}\to a$ horizontal slits ${}_{\xi}I^i_{j(n)}$. Let $\Delta_1=\left\{0\leqq\operatorname{Im}\xi\leqq l, \frac{\pi}{2}\leqq\arg\xi\leqq\frac{3\pi}{4}\right\}$, $\Delta_2=\left\{-l\leqq\operatorname{Re}\xi\leqq 0, \frac{3\pi}{4}<\arg\xi\leqq\frac{5\pi}{4}\right\}$, $\Delta_3=\left\{0\geqq\operatorname{Re}\xi\geqq -l, \frac{5\pi}{4}<\arg\xi\leqq\frac{3\pi}{2}\right\}$, where $l=1$. Let $\eta=g(\xi)$ be a one to one mapping from $\Delta_1+\Delta_2+\Delta_3$ to $\left\{0\leqq|\eta|\leqq l, \frac{\pi}{2}\leqq\arg\eta\leqq\frac{3\pi}{2}\right\}$ such that

$$(r=\operatorname{Im}\xi,\ \theta=\arg\xi)\ \text{in}\ \Delta_1, \qquad (r=-\operatorname{Re}\xi,\ \theta=\arg\xi)\ \text{in}\ \Delta_2$$

$$\text{and}\quad (r=-\operatorname{Im}\xi,\ \theta=\arg\xi)\ \text{in}\ \Delta_3, \qquad \text{where}\ re^{i\theta}=\eta=g(\xi).$$

We see by computation $\eta=g(\xi)$ is a quasiconformal mapping with maximal dilation quotient $=K\leqq\frac{3+\sqrt{5}}{2}$. Let $R^{\varDelta}(a_{2n+1}, a_{2n+2})=\varDelta_1\cap(-\log a_{2n+2}\leqq \mathrm{Im}\,\xi\leqq -\log a_{2n+1})\}+\varDelta_2\cap\{\log a_{2n+1}\leqq \mathrm{Re}\,\xi\leqq\log a_{2n+2}\}+\varDelta_3\cap\{\log a_{2n+2}\geqq \mathrm{Im}\,\xi\geqq\log a_{2n+1}\}$. Then $g(\xi)$ maps $R^{\varDelta}(a_{2n+1}, a_{2n+2})$ onto a semiring $R^{\eta}(a_{2n+1}, a_{2n+2})=\{-\log a_{2n+1}\leqq |\eta|\leqq -\log a_{2n+2}, \frac{\pi}{2}\leqq\arg\eta\leqq\frac{3\pi}{2}\}$. We remark ${}^{\xi}I^i_{j(n)}$ contained in $(\varDelta_1+\varDelta_2)\cap R^{\varDelta}(a_{2n+1}, a_{2n+2})\rightarrow$ a circular slit ${}_{\eta}I^i_{j(n)}$ in $R^{\eta}(a_{2n+1}, a_{2n+2})$ and there is no slit in $\varDelta_2\cap R^{\varDelta}(a_{2n+1}, a_{2n+2})$. Hence $R^{\eta}(a_{2n+1}, a_{2n+2})$ has only circular slits. Let $\hat{R}^{\varDelta}(a_{2n+1}, a_{2n+2})$ be the same leaf as $R^{\varDelta}(a_{2n+1}, a_{2n+2})$. Identify ${}_{\xi}I^i_n$ of $R^{\varDelta}(a_{2n+1}, a_{2n+2})$ and $\hat{R}^{\varDelta}(a_{2n+1}, a_{2n+2})$. Then we have a surface $\tilde{R}^{\varDelta}(a_{2n+1}, a_{2n+2})$. We construct a surface $\tilde{R}^{\eta}(a_{2n+1}, a_{2n+2})$ from $R^{\eta}(a_{2n+1}, a_{2n+2})$ similarly. Then $\eta=g(\xi)$ continued to a quasiconformal mapping from $\tilde{R}^{\varDelta}(a_{2n+1}, a_{2n+2})$ to $\tilde{R}^{\eta}(a_{2n+1}, a_{2n+2})$ with the same maximal dilatation quotient except on $\sum I^i_n$. Consider the function $U(t): t\in\tilde{\mathfrak{F}}'$. $z=\mathrm{proj}\,t$. Then $U(\eta)=U(\exp(g^{-1}(\eta)))$ is not harmonic in ${}_{\eta}\tilde{R}(a_{2n+1}, a_{2n+2})$ but a Dirichlet bounded function and

$$\sum_n D_{\tilde{R}(a_{2n+1},\, a_{2n+2})}(U(\eta))\leqq\sum KD_{\tilde{R}(a_{2n+1},\, a_{2n+2})}(U(\xi))\leqq KD_{\tilde{\mathfrak{F}}'}(U(t))<\infty\,.$$

Put $\eta=re^{i\theta}$ and $L(r)=\int_{C_r}\left|\frac{\partial u}{\partial\theta}(re^{i\theta})\right|d\theta$, where $C_r=\{|\eta|=r\}$ is contained in $\sum{}_{\eta}\tilde{R}(a_{2n+1}, a_{2n+2})$ and composed of two components, $C(r)$ does not intersect ${}_{\xi}I^i_n$ except a set of r of measure zero. By Schwarz's inequality

$$\int_{\Sigma\lambda_n}\frac{L^2(r)}{r}dr\leqq\sum_n D_{{}_{\eta}\tilde{R}(a_{2n+1},\, a_{2n+2})}(U(\eta))\qquad<\infty\,,$$

where λ_n is an interval $=\{-\log a_{2n+2}, -\log a_{2n+1}\}$. By $\sum_n\log\left(\frac{\log a_{2n+1}}{\log a_{2n+2}}\right)\sim\sum_n\log\frac{1-a_{2n+1}}{1-a_{2n+2}}=\infty$, we see there exists a sequence $\{r_n\}$ such that C_{r_n} does not touch $\{I^i_n\}$ and $L(r_n)\rightarrow 0$ as $n\rightarrow\infty$. Let $\Gamma_n=g^{-1}(C_{r_n})$. Then clearly $\mathrm{proj}\,\Gamma_n\rightarrow z=1$. Hence Γ_n is a required curve.

Let $U(t)$ be a harmonic function in $\tilde{\mathfrak{F}}'$ such that $U(t)$ has M.D.I. among all harmonic functions with the same value as $U(t)$ on $\partial\tilde{\mathfrak{F}}'$. Then by a), b) and c) $U(t)$ has a limit as $\mathrm{proj}\,t\rightarrow z=1$. Hence there exists only one N-Martin point over $e^{i\theta}$. Let p be the N-Martin point over $e^{i\theta}$. Then since there exists only one point p over $e^{i\theta}$, $\mathrm{dist}\,(e^{i\theta}, \mathrm{proj}\,\partial v_n(p))>0$ for any $v_n(p)=\left\{t: \text{Martin distance } (t, p)<\frac{1}{n}\right\}$. Let F_1 be a set of radial slits in the part of $\tilde{\mathfrak{F}}_1$ over $\sum R(a_{2n+1}, a_{2n+2})$. Now the part of $\tilde{\mathfrak{F}}'$ over $b_n<|z|<b'_n$ is an strong surface with δ_n, ε_n. Let F_2 be a closed set in $\tilde{\mathfrak{F}}'$ such that F_2 is an α,β-thin set in the part of $\tilde{\mathfrak{F}}'$ over $R(b_n, b'_n)$ $(n=1, 2, \cdots)$ and 2) $D(U(z))>\gamma(\theta_2-\theta_1)$: $\gamma>0$ where $U(z)$ is a harmonic function in $\tilde{\mathfrak{F}}'\cap\left(\frac{1}{3}<|z|<1,\ \theta_1<\arg z<\theta_2\right)-F_1-F_2$ such that $U(z)=0$

on $|z|=\frac{1}{3}$, $U(z)=1$ on $|z|=1$ and $U(z)$ has M. D. I. Then we have as example 3 following

PROPOSITION.

$$(v_n(p)-F_1-F_2)\in O_{ABF}\,.$$

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[5] See (2).
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[7] See Lemma 8.

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