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QUANTIC MANIFOLDS WITH PARA-COKÄHLERIAN STRUCTURES

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Following J. M. Souriau [1] a quantic manifold $(Q, \bar{\omega})$ is a Hausdorff manifold having a Pfaffian structure defined by $d\wedge\bar{\omega}=\bar{\Omega}$ where $\bar{\Omega}$ is a *pre-symplectic* form with $\dim(\ker\bar{\Omega})\equiv 1$. The present paper is concerned with a class of quantic manifolds such that $Q=K\times h$ where K is a *para-Kählerian* manifold and h is a time-like vector. Such manifolds are called by the author *quantic manifolds with para-coKählerian structure* and are denoted by Q_k. Some properties of the *self-orthogonal* Grassman manifolds over Q_k are studied and a simple result regarding minimal immersions in Q_k is stated. Next is investigated the behaviour of a tangential concurrent vector field (in the sense of K. Yano and B. Y. Chen [2]) of immersed para-Kählerian manifolds in Q_k. In the last section the notion of "minimal harmonic inclusion" for an isotropic (or total null) submanifold is defined, and is applied to Planck submanifolds of Q_k.

1. Preliminaries.

Let (M, Ω) be a *potential symplectic* manifold M (of dimension $2n$), i.e. such that

$$\Omega=d\wedge\omega, \qquad \omega\in\Lambda^1(M). \tag{1}$$

If M is a Hausdorff manifold, then M is *quantificable* [1] and the quantic manifold derived from M is defined as the direct product $Q=M\times T$. By a definition of J. M. Souriau [1] a Hausdorff manifold $\bar{M}$, is a general quantic manifold if the following conditions are fulfilled:

(i) The existence on $\bar{M}$ of a differentiable field of 1-forms $\bar{p}\to\bar{\omega}$ $(\bar{p}\in\bar{M})$, which gives to $\bar{M}$ a Pfaffian structure defined by $d\wedge\bar{\omega}=\bar{\Omega}$; $\dim(\mathrm{Ker}\,\bar{\Omega})\equiv 1$;

(ii) $\dim(\ker(\bar{\omega})\cap\ker(\bar{\Omega}))=0$.

In consequence of the above definitions, one may state that

(i)′ $\bar{M}$ is pre-symplectic;

(ii)′ $\bar{M}$ is a foliated manifold;

(iii) $\bar{M}$ is a fiber space whose basis is a symplectic manifold (M, Ω) and $\dim M=\dim\bar{M}-\dim\ker(\bar{\Omega})$.

Now suppose that M is a *para-Kählerian* manifold [3] (denoted by K) and let $T_p(K)$ be the tangent space to K at $p\in K$. As is known [4] with a real basis

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of $T_n(K)$ is injectively associated a *Witt basis* (W basis). One has the following decomposition of Witt

(2) $$T_p(K)=S_p\oplus S'_p$$

where S_p and S'_p are two *self-orthogonal* vectorial subspaces [5] of the same dimension n. The pair (S_p, S'_p) defines an *involutive automorphism* $\mathcal{U}$ satisfying $\mathcal{U}^2=+1$ [3]. If $h_\alpha\in S_p$ and $h_{\alpha'}\in S'_p$ ($\alpha=1,\cdots,n$; $\alpha'=\alpha+n$) are isotropic (real) vectors of the W basis, one has $\mathcal{U}h_\alpha=h_{\alpha'}$, $\mathcal{U}h_{\alpha'}=h_\alpha$.

Remark. $T_p(K)$ may be also considered as the *orthogonal sum* of the n *hyperbolic* 2-planes $P_\alpha\equiv(h_\alpha, h_{\alpha'})$ [6], that is

$$T_p(K)=P_1\perp P_2\perp\cdots\perp P_n\,.$$

2. Quantic manifolds Q_k.

Assume that the pseudo-Riemannian metric of the manifold $Q=K\times T$ is of index $n+1$. Denote by $h=h_{2n+1}$ the time-like vector tangent to T. Then a unitary frame (or normed) $\{\bar{p}, h_A;\ A=1,2,\cdots,2n,\ 2n+1\}$ at $\bar{p}\in Q$ is defined by

(3) $$\begin{aligned}&\langle h_\alpha, h_{\beta'}\rangle=\delta_{\alpha\beta}\,,\qquad \langle h, h\rangle=1\,,\\&\langle h, h_\alpha\rangle=0=\langle h, h_{\alpha'}\rangle\,.\end{aligned}$$

The line element $d\bar{p}$ of Q is

(4) $$d\bar{p}=\bar{\theta}^A\otimes h_A$$

where $\{\bar{\theta}^A\}$ is the dual basis of $\{h_A\}$.

From (3) and (4) the metric of Q in terms of $\bar{\theta}^A$ is expressed by the quadratic *para-coHermitian* [7] form

(5) $$ds^2=2\textstyle\sum_2\bar{\theta}^\alpha\bar{\theta}^{\alpha'}+(\bar{\theta})^2\,.$$

The *para-Hermitian* component of ds^2 that is $2\sum_\alpha\bar{\theta}^\alpha\bar{\theta}^{\alpha'}$ is exchangeable with the 2-form of rank $2n$

(6) $$\bar{\Omega}=\textstyle\sum_\alpha\bar{\theta}^\alpha\wedge\bar{\theta}^{\alpha'}\,.$$

The manifold Q is structured by the connection

(7) $$\bar{\nabla}h_A=\bar{\theta}^B_A\otimes h_B$$

where $\bar{\theta}^B_A=\bar{l}^B_{AC}\bar{\theta}^C$ are the connection forms on the principal frame bundle $\mathcal{B}(Q)=\cup\{\bar{p}, h_A\}$ and from (3) one finds easily

(8) $$\bar{\theta}^\alpha_\beta+\bar{\theta}^{\beta'}_{\alpha'}=0\,.$$

(8′) $$\bar{\theta}^{2n+1}_\alpha+\bar{\theta}^{\alpha'}_{2n+1}=0\,,\qquad \bar{\theta}^{2n+1}_{2n+1}=0\,.$$

K and h being a para-Kählerian manifold and a time-like vector respectively, we shall call the quantic manifold defined by

$$Q=K\times h \tag{9}$$

a quantic manifold with *para-coKählerian structure* (denoted by Q_k).

By reasoning similar to that for coKählarian manifolds [8] and from (8), we deduce

$$d\wedge\bar{\theta}=0\,, \tag{10}$$

$$\nabla h=0\Rightarrow \bar{\theta}_{\alpha}^{2n+1}=0=\bar{\theta}_{2n+1}^{\alpha'} \tag{11}$$

and if $\mathcal{M}$ is the connection matrix on $\mathcal{B}(Q_k)$ one has

$$\mathcal{M}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \bar{\theta}_{\beta}^{\alpha} & 0 \\ 0 & 0 & \bar{\theta}_{\beta'}^{\alpha'} \end{pmatrix}. \tag{12}$$

In the following we shall call h and $\bar{\theta}$ *the canonical field* and the *canonical covector* of Q_k, respectively (h may be also called the *anisotropic* vector [6] corresponding to the splitting $T_{\bar{p}}(Q_k)=S_{\bar{p}}\oplus S_{\bar{p}}\oplus h$ of the tangent space $T_{\bar{p}}(Q_k)$ at $\bar{p}\in Q_k$). Let $\bar{\omega}$ be the 1-form which defines the quantic structure of Q_k and ω its induced value on K.

Since ω is *semi-basic* with respect to the Pfaffian structure of Q_k, we may write

$$\bar{\omega}=\omega+\bar{\theta}\,. \tag{13}$$

The connection $\bar{\nabla}$ being torsionless (since a para-coKählerian structure is integrable) by virtue of (12) the structure equations of Q_k are

$$\begin{aligned} &d\wedge\bar{\theta}^{\alpha}=\bar{\theta}^{\beta}\wedge\bar{\theta}_{\beta}^{\alpha}\,,\\ &d\wedge\bar{\theta}^{\alpha'}=\bar{\theta}^{\beta'}\wedge\bar{\theta}_{\beta'}^{\alpha'}\,,\\ &d\wedge\bar{\theta}=0 \end{aligned} \tag{14}$$

and

$$\begin{aligned} &d\wedge\bar{\theta}_{\beta}^{\alpha}=\bar{\Omega}_{\beta}^{\alpha}+\bar{\theta}_{\beta}^{\gamma}\wedge\bar{\theta}_{\gamma}^{\alpha}\\ &d\wedge\bar{\theta}_{\beta'}^{\alpha'}=\bar{\Omega}_{\beta'}^{\alpha'}+\bar{\theta}_{\beta'}^{\gamma'}\wedge\bar{\theta}_{\gamma'}^{\alpha'} \end{aligned} \tag{14'}$$

where $\bar{\Omega}_{\beta}^{\alpha}$, $\bar{\Omega}_{\beta'}^{\alpha'}$ are the curvature 2-forms.

3. Self-orthonormal Grassman manifolds $G^n(T_p^*(Q_k))$ over Q_k.

Consider the *simple unitary* form σ (resp. σ') of the self-orthogonal n-plane spanned by h_α (resp. $h_{\alpha'}$). Accordingly one has

(15) $$\bar{\sigma}=\bar{\theta}^1\wedge \cdots \wedge\bar{\theta}^n .$$

(15′) $$\bar{\sigma}'=\bar{\theta}^{1'}\wedge \cdots \wedge\bar{\theta}^{n'}$$

and by (14) we get

(16) $$d\wedge\bar{\sigma}=-\tau\wedge\bar{\sigma} ,$$

(16′) $$d\wedge\bar{\sigma}'=\tau\wedge\bar{\sigma}'$$

where $\tau=\sum_\alpha\bar{\theta}^\alpha_\alpha$; $\sum_\alpha\bar{\theta}^\alpha_\alpha+\sum_{\alpha'}\bar{\theta}^{\alpha'}_{\alpha'}=0$.

It follows from (16) and (16′) that the two self-orthogonal subspaces $S_{\bar{p}}$ and $S_{\bar{p}}$ define on Q_k a G-structure of type $G=GL(n;R)\times GL(n;R)$ [9], and consequently an (n,n) *foliation* on Q_k. Hence we may say that $\bar{\sigma}$ (resp. $\bar{\sigma}'$) defines a *Grassman manifold* $G^n(T^*_{\bar{p}}(Q_k))$ (resp. $G'^n(T^*_{\bar{p}}(Q_k))$ of dimension n over the dual space $T^*_{\bar{p}}(Q_k)$. We shall call $\bar{\sigma}$ and $\bar{\sigma}'$ the *self-orthogonal Grassmann manifolds* over Q_k and τ the *trace* 1-form associated with the W-basis $\{h_\alpha, h_{\alpha'}\}$.

Remark. If $\tau=0$, the connection $\bar{\nabla}$ is *proper spin-euclidean* [10] ($\tau=0$, defines the one modular linear group on $S_{\bar{p}}$).

Since

(17) $$\bar{\sigma}\wedge\bar{\sigma}'\wedge\bar{\theta}={}_*(1)$$

is the volume element of Q_k, we readily find

(18) $${}_*\bar{\theta}=\bar{\sigma}\wedge\bar{\sigma}'$$

and by means of (16) and (16′) we get

(19) $$d\wedge({}_*\bar{\theta})=0\Rightarrow\delta\bar{\theta}=0\Rightarrow\Delta\bar{\theta}\equiv(d\delta+\delta d)\bar{\theta}=0 .$$

But by virtue of the property ${}_{**}(\ )=-(\ )$ of the star operator, we have also

(20) $$\Delta({}_*\bar{\theta})=0 .$$

Thus we may say that the simple unit $2n$-form ${}_*\bar{\theta}$ satisfies the general Maxwell equations in vacuum.

Moreover one finds

(21) $$d\wedge{}_*\bar{\sigma}=\tau\wedge{}_*\bar{\sigma} ,$$

(21′) $$d\wedge{}_*\bar{\sigma}'=-\tau\wedge{}_*\bar{\sigma}'$$

and this shows that both n-forms $\bar{\sigma}$ and $\bar{\sigma}'$ wchich are visibly *orthogonal* $(\bar{\sigma},\bar{\sigma}')=0$) are *co-completely integrable.*

Putting

(21″) $$\tau=l_\alpha\bar{\theta}^\alpha+l_{\alpha'}\bar{\theta}^{\alpha'}$$

one finds from (21) and (21′)

$$\delta\bar{\sigma}=(-1)^{n+1}\textstyle\sum_{\alpha}(-1)^{\alpha-1}l_{\alpha}\bar{\theta}^{1}\wedge\cdots\wedge\hat{\bar{\theta}}^{\alpha}\wedge\cdots\bar{\theta}^{n}, \tag{22}$$

$$\delta\bar{\sigma}'=(-1)^{n}\textstyle\sum_{\alpha'}(-1)^{\alpha'-1}l_{\alpha'}\bar{\theta}^{1}\wedge\cdots\wedge\hat{\bar{\theta}}^{\alpha'}\wedge\cdots\bar{\theta}^{n'} \tag{22′}$$

(the roof indicates the missing term).

Making now use of G. de Rham formula [11] for $\bar{\sigma}$ and $\bar{\sigma}'$, that is

$$d\wedge(\bar{\sigma}\wedge_{*}(d\wedge\bar{\sigma}')-\bar{\sigma}'\wedge_{*}(d\wedge\bar{\sigma})+\delta\bar{\sigma}\wedge_{*}\bar{\sigma}'-\delta\bar{\sigma}'\wedge_{*}\bar{\sigma})$$
$$=\Delta\bar{\sigma}\wedge_{*}\bar{\sigma}'-\Delta\bar{\sigma}'\wedge_{*}\bar{\sigma}=0\Rightarrow\Delta\bar{\sigma}\wedge\bar{\sigma}-\Delta\bar{\sigma}'\wedge\bar{\sigma}=0$$

one finds with the help of (16), (16′), (22) and (22′)

$$\Delta\bar{\sigma}=0\Leftrightarrow\Delta\bar{\sigma}'=0\,.$$

We may state the preceding results as follows:

THEOREM. *Let Q_k be a quantic mvnıfold with para-coKählerian structure and let h be the canonıcal field of Q_k and $\bar{\sigma}, \bar{\sigma}'$ the simple unitary forms of the self-orthogonal sub-spaces $S_{\bar{p}}, S'_{\bar{p}}$, respectıvely. Then*

(i) *$\bar{\sigma}$ (resp. $\bar{\sigma}'$) defines a Grassman manifold of dimensıon n;*

(ii) *h is an infinitesimal automorphism of the G-structure defined by the volume element of Q_k (in other words h ıs divergence-free) and the adjoint, ${}_{*}\bar{\theta}$ of the canonical covector $\bar{\theta}$ satısfies Maxwell general equations ın vacuum;*

(iii) *$\bar{\sigma}$ and $\bar{\sigma}'$ are co-completely ıntegrable and $\Delta\bar{\sigma}=0\Leftrightarrow\Delta\bar{\sigma}'=0$.*

4. Minimal immersion in Q_k.

Consider first the immersion $x:\mathring{K}\rightarrow Q_k$ where $\mathring{K}$ is a para-Kählerian manifold of dimension $2q$. If $i=1,\cdots,q$; $i'=i+n$ are the tangential indices associated with x and dp, θ^{i}, $\theta^{i'}$ θ^{α}_{β} and $\theta^{\alpha'}_{\beta'}$ the restrictions on $\mathring{K}$ of $d\bar{p}$, $\bar{\theta}^{\alpha}$, $\bar{\theta}^{\alpha'}$, $\bar{\theta}^{\alpha}_{\beta}$ and $\bar{\theta}^{\alpha'}_{\beta'}$ respectively, we may write

$$dp=\theta^{i}\otimes h_{i}+\theta^{i'}\otimes h_{i'}\,. \tag{23}$$

Let $T^{\perp}_{p}(\mathring{K})=\{h_r, h_{r'}\}$ be the normal space to $\mathring{K}$ at p ($r=q+\cdots n$; $r'=r+n$ are the normal indices corresponding to the isotropic normal vectors associated with x). From (23) we find that the adjoint of the line element dp is

$$\begin{aligned}{}_{*}dp=&\textstyle\sum(-1)^{i-1}h_{i}\theta^{1}\wedge\cdots\wedge\theta^{q}\wedge\theta^{1'}\wedge\cdots\wedge\hat{\theta}^{i'}\wedge\cdots\wedge\theta^{q'}\\&+\textstyle\sum(-1)^{i'-1}h_{i'}\theta^{1}\wedge\cdots\wedge\hat{\theta}^{i}\wedge\cdots\wedge\theta^{q}\wedge\theta^{1'}\wedge\cdots\wedge\theta^{q'}\end{aligned} \tag{24}$$

$$d\wedge_{*}dp=H_{*}(1)\,;\qquad {}_{*}(1)\ \text{volume element of}\ \mathring{K} \tag{25}$$

where $H\in T^{\perp}_{p}(\mathring{K})$ represents as is known the *mean curvature* vector associated with x. From (7) and (14) one finds by straight forward calculation

$$d\wedge{}_*dp=0\Rightarrow H=0.$$

Remark. This result is analogous to the well known property of Kählerian subspaces of a Kählerian space.

Next consider the immersion $x:\overset{\circ}{Q}\to\overset{\circ}{Q}_k$ where Q is a *para-coKählerian* manifold of dimension $2q+1$. In this case the line element dp of $\overset{\circ}{Q}$ is

$$dp=\theta^i\otimes h_i+\theta^{i'}\otimes h_{i'}+\theta\otimes h \tag{26}$$

and one finds

$$\begin{aligned}{}_*dp=(\textstyle\sum(-1)^{i-1}h_i\theta^1\wedge\cdots\wedge\theta^q\wedge\theta^{1'}\wedge\cdots\wedge\hat{\theta}^{i'}\wedge\cdots\wedge\theta^{q'}\\ +\textstyle\sum(-1)^{i'-1}h_{i'}\theta^1\wedge\cdots\wedge\hat{\theta}^i\wedge\cdots\wedge\theta^q\wedge\theta^{1'}\wedge\cdots\wedge\theta^{q'})\wedge\theta\\ +h\theta^1\wedge\cdots\wedge\theta^q\wedge\theta^{1'}\wedge\cdots\wedge\theta^{q'}.\end{aligned}$$

Taking account of (10) and (11) one readly gets

$$d\wedge{}_*dp=0\Rightarrow H=0,$$

and so we have the

THEOREM. *Any immersion of a para-Kählerian or a para-coKählerian manifold in Q_k is minimal.*

5. Concurrent tangential vector fields over a para-Kahlerian submanifold of Q_k.

Let $x:\overset{\circ}{K}\to Q_k$ be the immersion considered at section 4, and let

$$X=t^ih_i+t^{i'}h_{i'} \tag{27}$$

be a tangential vector field over $\overset{\circ}{K}$. Following K. Xano and B. Y. Chen [2], X is *concurrent* if we have

$$dp+\nabla X=0. \tag{28}$$

By (7), (23) and (27) we get from (28)

$$dt^i+\theta^i+t^j\theta^i_j=0,\qquad i,j=1,\cdots,q,\quad i'=i+n;\quad j'=j+n. \tag{29}$$

$$dt^{i'}+\theta^{i'}+t^{j'}\theta^{i'}_{j'}=0, \tag{30}$$

$$t^i\theta^r_i=0;\qquad r=q+1,\cdots,n,\quad r'=r+n, \tag{31}$$

$$t^{i'}\theta^{r'}_{i'}=0 \tag{32}$$

and by exterior differentiation one finds that the necessary and sufficient conditions for the above system to be closed are

(33) $$\det(\Omega^i_j)=0\,, \qquad \det(\Omega^{i'}_{j'})=0\,,$$

(34) $$t^i\Omega^r_i=0\,, \qquad t^{i'}\Omega^{r'}_{i'}=0\,.$$

Further, since the second fundamental forms associated with x are

(35) $$\varphi_r=-\langle dp, \nabla h_r\rangle=\theta^{r'}_{i'}\theta^{i'}\,,$$

(36) $$\varphi_{r'}=-\langle dp, \nabla h_{r'}\rangle=\theta^r_i\theta^i$$

the *Lispchitz-Killing curvatures* $K(p, h_r)$, $K(p, h_{r'})$ associated with x are defined by

(37) $$K(p, h_r)=\det(\varphi_r)\,,$$

(38) $$K(p, h_{r'})=\det(\varphi_{r'})\,.$$

Thus one gets from (31) and (32)

$$K(p, h_r)=0=K(p, h_{r'})\,.$$

THEOREM. *Let $x: \overset{\circ}{K}\to Q_k$ be the immersion of a para-Kählerian manifold in a quantic manifold with para-coKählerian structure. If $\overset{\circ}{K}$ admits a concurrent tangential field then all Lipschitz-Killing curvatures associated with x vanish*

Now consider the invariant $(2q-1)$-form

(39) $$\Theta=\sum_i(-1)^{i-1}\langle X, h_{i'}\rangle\theta^1\wedge\cdots\wedge\hat{\theta}^i\wedge\cdots\wedge\theta^q\wedge\theta^{1'}\wedge\cdots\wedge\theta^{q'}$$
$$+\sum_{i'}(-1)^{i'-1}\langle X, h_i\rangle\theta^1\wedge\cdots\wedge\theta^q\wedge\theta^{1'}\wedge\cdots\wedge\hat{\theta}^{i'}\wedge\cdots\wedge\theta^{q'}$$

which is an integral relation of invariance for X, that is

(40) $$X\lrcorner\Theta=0\,.$$

By (14) and (27) we have

(41) $$d\wedge\Theta=-2q_*(1)$$

and since

(42) $$X\lrcorner_*(1)=\Theta$$

we obtain

(43) $$L_{X*}(1)=-2q_*(1)$$

L_X: Lie differentiation with respect to the vector field X

(44) $$L_X\Theta=-2q\Theta\,.$$

Thus (43) and (44) show that X is a *homothetic infinitesimal transformation over*

Q_k *and a conformal infinitesimal transformation* of the G-structure defined on $\overset{\circ}{K}$ by Θ, respectively.

Further, the dual form α of X being

$$\alpha=\sum_i t^{i'}\theta^i+\sum_i t^i\theta^{i'}\,; \qquad i'=i+n \tag{45}$$

one finds by means of (29), (30)

$$\alpha=d\sum_i t^i t^{i'}=\frac{1}{2}d\langle X, X\rangle\,. \tag{46}$$

Calling α the *concurent tangential covector* associated with x, (46) shows that α is a coboundary.

Now let $\overset{\circ}{\Omega}$ be the restriction of $\bar{\Omega}$ on $\overset{\circ}{K}$ and let $\tilde{\alpha}$ be the dual form of X with respect to $\overset{\circ}{\Omega}$. That is the isomorphism

$$j:\ \wedge^2(\overset{\circ}{K})\longrightarrow\wedge^1(\overset{\circ}{K})\,,\qquad \overset{\circ}{\Omega}\longrightarrow X\lrcorner\,\overset{\circ}{\Omega}=\tilde{\alpha}\,. \tag{4.7}$$

Since

$$\overset{\circ}{\Omega}=\sum_i\theta^i\wedge\theta^{i'} \tag{48}$$

one finds

$$\tilde{\alpha}=\sum_i t^i\theta^{i'}-\sum_i t^{i'}\theta^i \tag{49}$$

and by (29) and (30)

$$d\wedge\tilde{\alpha}=-q\overset{\circ}{\Omega}\,. \tag{50}$$

Consequently we deduce

$$L_X\overset{\circ}{\Omega}=-q\overset{\circ}{\Omega} \tag{51}$$

and this shows that X is a *conformal infinitesimal transformation* of the symplectic structure $S_p(q, R)$ defined by $\overset{\circ}{\Omega}$ on $\overset{\circ}{K}$ ($\overset{\circ}{K}$ is not compact). On the other hand if we denote by $X_\alpha=-\overset{\circ}{\Omega}{}^{-1}(\alpha)$ the *Hamiltonian field* corresponding to α (by virtue of (46) on may say that $\frac{1}{2}\langle X, X\rangle$ is the *energy integral* of X_α) it is readily seen that

$$X_\alpha=t^{i'}h_{i'}-t^i h_i \tag{52}$$

and one finds

$$L_{X_\alpha *}(1)=0\,. \tag{53}$$

Hence X_α is an *infinitesimal automorphism* of the G-structure defined by the volume element of $\overset{\circ}{K}$.

Fro the above we have the

THEOREM. *Let $\overset{\circ}{K}$ be an 2q-dimensional para-Kählerian submanifold of a quantic manifold with para-coKählerian structure. If $\overset{\circ}{K}$ admits a concurrent tangential vector field X, and α is the dual form of X, (or the concurrent tangential covector) then*

(i) *X is a homothetic infinitesimal transformation over $\overset{\circ}{K}$.*

(ii) *X is a conformal infinitesimal transformation of the G-structure defined on $\overset{\circ}{K}$ by $_*\alpha$, and of the induced symplectic structure $\overset{\circ}{\Omega}$ on $\overset{\circ}{K}$.*

(iii) *α is a coboundary and its associated Hamiltonian field with respect to $\overset{\circ}{\Omega}$ is an infinitesimal automorphism of the G-structure defined by the volume element of $\overset{\circ}{K}$.*

6. Planck manifolds.

Being given a quantic manifold Q any (horizontal) submanifold of Q, defined by $\bar{\omega}=0$ $(\omega=0)$ is called a *Planck manifold* (denoted by $\mathcal{P}$). Since the reciprocal image of $d\wedge\bar{\omega}$ is also zero, it follows that any Planck manifold has an isotropic metric structure [1] (or is *total null*). In consequence of the splitting $T_{\bar{p}}(Q_k)=S_{\bar{p}}\oplus S'_{\bar{p}}\oplus h$, the *index* [6] of $T_{\bar{p}}(Q_k)$ is n (that is the maximal isotropic subspace of $T_{\bar{p}}(Q_k)$ is of dimension n). Let then $\mathcal{P}$ be a Planck manifold of dimension $q\leqq n$ and $T_p(\mathcal{P})$ and $T_p^{\perp}(\mathcal{P})$ the tangent space and the normal space at $p\in\mathcal{P}$ respectively. If $q=n$ one has $T_p(\mathcal{P})\equiv T_p^{\perp}(\mathcal{P})$ and in this case we shall call $\mathcal{P}$ a *self orthogonal* Planck manifold (or of maximal dimension). If $q<n$ one has $T_p(\mathcal{P})\subset T_p^{\perp}(\mathcal{P})$ and $\mathcal{P}$ is called an *isotropic* Planck manifold.

For later convenience, in stating some results, the following definition will be made.

Definition. Let $x\in M\to\bar{M}$ be the inclusion of an isotropic manifold M in a pseudo-Riemannian manifold $\bar{M}$ and let dp be the line element of M. We say that x is a *minimal harmonic inclusion* if $d\wedge_* dp=0\Rightarrow \Delta p=0$, holds.

Suppose now that $T_p(\mathcal{P})\subseteq S_p$, and denote by h_i $(i, j=1, 2, \cdots q)$ and h_r $(r=q+1\cdots n)$ the normal tangential isotropic vector and the normal transveral isotropic vector, respectively, associated with the inclusion $x:\mathcal{P}\to Q_k$.

Since

$$dp=\theta^i\otimes h_i \tag{54}$$

the adjoint $_*dp$ is expressed by

$$_*dp=\sum_i(-1)^i h_{i'}\theta^1\wedge\cdots\wedge\hat{\theta}^i\wedge\cdots\wedge\theta^q. \tag{55}$$

Thus if $q=n$ we deduce

$$d\wedge_* dp=-\tau\wedge_* dp \tag{56}$$

where $\tau=\sum_i\theta_i^i$ is the trace 1-form associated with x.

In case $q<n$ we shall introduce the following quadratic forms associated with x

$$\varphi_r=-\langle \mathcal{U}dp, \nabla h_r\rangle \tag{57}$$

where $\mathcal{U}$ is the parahermitian operator defined at section 1. By straight forward calculation one finds

$$d\wedge_* dp=-\tau\wedge_* dp-\{\textstyle\sum_r(\text{trace}\,\varphi_r)h_{r'}\}_*(1) \tag{58}$$

where $r'=r+n$ and $_*(1)$ is the volume element of $\mathcal{P}$.

Calling φ_r the *para-Hermitian quadratic forms* associated with the inclusion $x:\mathcal{P}\to Q_k$, we formulate the

THEOREM. *Let $x:\mathcal{P}\to Q_k$ be the inclusion of a Plack manifold $\mathcal{P}$ in a quantic manifold Q_k with para-coKählerian structure and let τ and φ_r be the trace 1-form and the para-Hermitian quadratic forms associated with x, respectively. Then*

(i) *If $\mathcal{P}$ is self-orthogonal, the necessary and sufficient condition that $\mathcal{P}$ be minimal harmonic is that τ vanishes;*

(ii) *If $\mathcal{P}$ is isotropic, the necessary and sufficient conditions that $\mathcal{P}$ be minimal harmonic is that both τ and trace (φ_r) vanish.*

Remark. From (16) and we deduce if $\mathcal{P}$ is self-orthogonal, then the above results may be expressed as follows:

The necessary and sufficient condition that $\mathcal{P}$ be minimal harmonic is that the associated Grassman manifold σ be harmonic.

That is $\Delta p=0\Leftrightarrow\Delta\sigma=0$; σ is the restriction of $\bar{\sigma}$ on $\mathcal{P}$. This property is in some regards related to the theory of harmonic simple forms constructed by Tachibana [12].

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