Y. KUBOTA
KŌDAI MATH. SEM. REP.
26 (1975), 267—288

ON THE FOURTH COEFFICIENT OF MEROMORPHIC UNIVALENT FUNCTIONS

BY YOSHIHISA KUBOTA

1. Let Σ_0 denote the class of functions $g(z)$ univalent in $|z|>1$, regular apart from a simple pole at the point at infinity and having the expansion at that point

$$g(z)=z+\sum_{n=1}^{\infty} b_n z^{-n}. \tag{1}$$

It is well known that $|b_2|\leqq 2/3$ [2], [8]. Further for the coefficients b_n with even subscripts $n=2m$ the following result is known: If $b_1=\cdots=b_{m-1}=0$, then $|b_{2m}|\leqq 2/(2m+1)$ [1], [6].

In this paper we shall prove the following theorem.

THEOREM. *Let $g(z)$ be a function of class Σ_0 having the expansion at the point at infinity*

$$g(z)=z+\sum_{n=1}^{\infty} b_n z^{-n}$$

where b_1 is real.

If $b_1\geqq 0$, then

$$\Re b_4\leqq \frac{2}{5}+\frac{729}{163840}$$

with equality holding only for the function $g^*(z)$ which satisfies the differential equation

$$z^2\left(\frac{dw}{dz}\right)^2\left(w^3-\frac{27}{64}w+\frac{27}{256}\right)=z^5-\frac{27}{128}z^3+\frac{27}{128}z^2+\frac{729}{65536}z$$
$$-\frac{66265}{32768}+\frac{729}{65536}z^{-1}+\frac{27}{128}z^{-2}-\frac{27}{128}z^{-3}+z^{-5}.$$

The expansion of $g^*(z)$ at the point at infinity begins

$$z+\frac{27}{128}z^{-1}-\frac{27}{256}z^{-2}-\frac{243}{65536}z^{-3}+\left(\frac{2}{5}+\frac{729}{163840}\right)z^{-4}+\cdots.$$

If $b_1\leqq 0$, then

Received Oct. 9, 1973.

$$\Re b_4 \leqq \frac{2}{5} + \frac{729}{163840} \Lambda^2, \qquad \Lambda = \frac{(3-4\cos^2\varphi)^3}{\cos^2\varphi(9-8\cos^2\varphi)^2}$$

with equality holding only for the function $\tilde{g}(z)$ *which satisfies the differential equation*

$$z^2\left(\frac{dw}{dz}\right)^2\left(w^3+\frac{27}{64}\Lambda w-\frac{27}{256}\Lambda\right)=z^5+\frac{27}{128}\Lambda z^3-\frac{27}{128}\Lambda z^2$$
$$+\frac{729}{65536}\Lambda^2 z-\left(2+\frac{729}{32768}\Lambda^2\right)+\frac{729}{65536}\Lambda^2 z^{-1}$$
$$-\frac{27}{128}\Lambda z^{-2}+\frac{27}{128}\Lambda z^{-3}+z^{-5}$$

where φ *is the real number satisfying*

$$\Im\left\{e^{i2\varphi}\int_0^1 (e^{i\varphi}t^3-2\cos\varphi\cdot t^2+e^{-i\varphi}t)^{1/2}dt\right\}=0,$$
$$0<\varphi<\pi, \qquad -0.44<\cos\varphi<-0.4.$$

The expansion of $\tilde{g}(z)$ *at the point at infinity begins*

$$z-\frac{27}{128}\Lambda z^{-1}+\frac{27}{256}\Lambda z^{-2}-\frac{243}{65536}\Lambda^2 z^{-3}+\left(\frac{2}{5}+\frac{729}{163840}\Lambda^2\right)z^{-4}+\cdots.$$

Here we remark that $\Lambda>1$. Our proof is due to Jenkins' General Coefficient Theorem.

2. Firstly we give several lemmas which will be used later on.

LEMMA A. *Let* $Q(w)dw^2=\alpha(w^3+\beta_1 w^2+\beta_2 w+\beta_3)dw^2$ *be a quadratic differential on the* w*-sphere and let*

$$g^*(z)=z+\sum_{n=1}^{\infty} b_n^* z^{-n}$$

be a function of class Σ_0 *which maps* $|z|>1$ *onto a domain* D *admissible with respect to* $Q(w)dw^2$. *Let* $g(z)$ *be a function of class* Σ_0 *having the expansion at the point at infinity*

$$g(z)=z+\sum_{n=1}^{\infty} b_n z^{-n}$$

where $b_1=b_1^*$. *Then*

$$\Re\alpha\{b_4-b_4^*+\beta_1(b_3-b_3^*)+(\beta_2+2b_1^*)(b_2-b_2^*)\}\leqq 0.$$

Equality occurs only for $g(z)\equiv g^*(z)$.

Proof. Let $\phi(w)$ be the inverse of $g^*(z)$ defined in D. We apply the General Coefficient Theorem in its extended form [7] with $\Re$ the w-sphere,

$Q(z)dz^2$ being $\alpha(w^3+\beta_1 w^2+\beta_2 w+\beta_3)dw^2$, the admissible domain D and the admissible function $g(\phi(w))$. The function $g(\phi(w))$ has the expansion at the point at infinity

$$w+\sum_{n=2}^{\infty} a_n w^{-n}$$

where

$$a_2=b_2-b_2^*,$$

$$a_3=b_3-b_3^*,$$

$$a_4=b_4-b_4^*+2b_1^*(b_2-b_2^*).$$

Hence we have the desired inequality. The equality statement follows from the general equality conditions in the General Coefficient Theorem [5].

LEMMA B. *Let $g(z)$ be a function of class Σ_0 having the expansion at the point at infinity*

$$g(z)=z+\sum_{n=1}^{\infty} b_n z^{-n}$$

where b_1 is real. Then

$$\Re(b_4+b_1b_2)\leqq 2/5.$$

Proof. Let E be the complement of the range of $g(z)$ and let a be an arbitrary point in E. Then $h(z)=g(z)-a$ has no zeros in $|z|>1$. Let $G_\mu(w)$ be the μ-th Faber polynomial which is defined by

$$G_\mu[h(z^2)^{1/2}]=z^\mu+\sum_{\nu=1}^{\infty} b_{\mu\nu} z^{-\nu}.$$

Then Grunsky's inequality [3] has the form

$$\Big|\sum_{\mu,\nu=1}^{m} \nu b_{\mu\nu} x_\mu x_\nu\Big| \leqq \sum_{\nu=1}^{m} \nu |x_\nu|^2.$$

Putting $m=5$, $x_1=x_2=x_3=x_4=0$, $x_5=1$ we have

$$\Re\Big(b_4+b_1b_2-\frac{1}{4}ab_1^2+\frac{1}{8}a^3b_1-\frac{9}{320}a^5\Big)\leqq\frac{2}{5}. \tag{2}$$

Assume that for all $a\in E$

$$\Re\Big(\frac{1}{4}ab_1^2-\frac{1}{8}a^3b_1+\frac{9}{320}a^5\Big)>0.$$

Then we have

$$\alpha(9\alpha^4-90\alpha^2\beta^2+45\beta^4-40b_1\alpha^2+120b_1\beta^2+80b_1^2)>0, \qquad a=\alpha+i\beta.$$

Let

$$D_1=\{w=u+iv:\ u>0,\ 9u^4-90u^2v^2+45v^4-40b_1u^2+120b_1v^2+80b_1^2>0\},$$

$$D_2=\{w=u+iv:\ u<0,\ 9u^4-90u^2v^2+45v^4-40b_1u^2+120b_1v^2+80b_1^2<0\}\,.$$

Since E is connected, it follows that E lies entirely in either D_1 or D_2. This contradicts that

$$\int_0^{2\pi} g(re^{i\theta})d\theta=0\,, \qquad (r>1)\,.$$

Hence there is a point a in E such that

$$\Re\Big(\frac{1}{4}ab_1^2-\frac{1}{8}a^3b_1+\frac{9}{320}a^5\Big)\leqq 0\,.$$

Then the ineqnality (2) gives the desired inequality.

The following lemma is a simple consequence of the area theorem.

LEMMA C. *Let $g(z)$ be a function of class Σ_0 having the expansion* (1) *at the point at infinity. Then*

$$|b_1|^2+2|b_2|^2+4|b_4|^2\leqq 1\,.$$

3. Next we give certain functions which play the role of extremal functions. In this section we consider quadratic differentials of the form $(w+2r)(w-r)^2dw^2$, $(r\geqq 0)$ and construct functions associated with them.

LEMMA 1. *Let Y be a non-negative real number and let $Q^*(w:Y)dw^2$ be the quadratic differential $(w+2Y)(w-Y)^2dw^2$. If X is a real number satisfying the condition*

$$80X^4-60X^2+4\leqq 3\sqrt{3}\,Y^{5/2}\leqq 64X^5-40X^3\,, \qquad \sqrt{\frac{5}{8}}\leqq X\leqq\frac{\sqrt{3}}{2}\,, \tag{3}$$

then there is a function $g^*(z:X, Y)$ of class Σ_0 which satisfies the differential equation

$$z^2\Big(\frac{dw}{dz}\Big)^2(w^3-3Y^2w+2Y^3)=z^5-2\mu z^3+2\mu z^2+\mu^2 z-2(\mu^2+1)+\mu^2z^{-1} \tag{4}$$
$$+2\mu z^{-2}-2\mu z^{-3}+z^{-5}\,, \qquad \mu=16X^4-12X^2+1$$

and which maps $|z|>1$ onto a domain admissible with respect to $Q^*(w:Y)dw^2$. The expansion of $g^*(z:X, Y)$ at the point at infinity begins

$$z-(2\mu-3Y^2)z^{-1}-2(\mu-Y^3)z^{-2}-3(\mu-Y^2)^2z^{-3}$$
$$+\Big(\frac{2}{5}+\Phi^*(\mu, Y)\Big)z^{-4}+\cdots$$

where

$$\Phi^*(\mu, Y)=-6\mu^2+(8Y^3+6Y^2)\mu-\frac{42}{5}Y^5\,.$$

Proof. There are two end domains $\mathcal{E}_1^*$, $\mathcal{E}_2^*$ and a half of an end domain h^*

in the trajectory structure of $Q^*(w:Y)dw^2$ on the upper half w-plane. For a suitable choice of determination the function

$$\zeta=\int[Q^*(w:Y)]^{1/2}dw$$

maps $\mathcal{E}_1^*$, $\mathcal{E}_2^*$, h^* respectively onto an upper half-plane, a lower half-plane and a domain $\Re\zeta>0$, $\Im\zeta>0$.

On the other hand there are two end domains $\mathcal{E}_1$, $\mathcal{E}_2$ and a half of an end domain h in the trajectory structure of the quadratic differential

$$z^{-7}(z-1)^2(z-e^{i\beta})^2(z-e^{-i\beta})^2(z-e^{i\gamma})^2(z-e^{-i\gamma})^2dz^2\,, \qquad 0\leqq\beta\leqq\gamma\leqq\pi$$

on the domain $|z|>1$, $\Im z>0$. For a suitable choice of determination the function

$$\zeta=\int z^{-7/2}(z-1)(z-e^{i\beta})(z-e^{-i\beta})(z-e^{i\gamma})(z-e^{-i\gamma})dz \tag{5}$$

maps $\mathcal{E}_1$, $\mathcal{E}_2$, h respectively onto an upper half-plane, a lower half-plane and a domain $\Re\zeta>0$, $\Im\zeta>0$.

If Y, β and γ satisfy the condition

$$\begin{aligned}
&\frac{4}{5}-4(4\cos^2\beta+2\cos\beta-1)\geqq-\frac{12\sqrt{3}}{5}Y^{5/2}\,,\\
&\frac{4}{5}\cos\frac{5}{2}\beta-4(4\cos^2\beta+2\cos\beta-1)\cos\frac{1}{2}\beta\leqq-\frac{12\sqrt{3}}{5}Y^{5/2}\,,\\
&\frac{4}{5}\cos\frac{5}{2}\gamma-4(4\cos^2\gamma+2\cos\gamma-1)\cos\frac{1}{2}\gamma\geqq0\,,\\
&2\cos\beta+2\cos\gamma+1=0\,,
\end{aligned} \tag{6}$$

then we can combine the above two functions to obtain a function which maps the domain $|z|>1$, $\Im z>0$ into the upper half w-plane. We put $X=\cos\frac{1}{2}\beta$. Then the condition (6) is equivalent to the condition (3). By reflection this function extends to a function $g^*(z:X,Y)$ which maps $|z|>1$ onto a domain admissible with respect to $Q^*(w:Y)dw^2$. The function $g^*(z:X,Y)$ satisfies the differential equation (4). Inserting

$$w=z+b_0+b_1z^{-1}+b_2z^{-2}+b_3z^{-3}+b_4z^{-4}+\cdots$$

in (4) we have

$$\begin{aligned}
&b_0=0\,,\\
&b_1=-2\mu+3Y^2\,,\\
&b_2=-2\mu+2Y^3\,,
\end{aligned}$$

$$b_3=-3\mu^2+6\mu Y^2-3Y^4,$$

$$b_4=\frac{2}{5}+\Phi^*(\mu, Y).$$

This completes the proof of Lemma 1.

Let $\mathfrak{D}^*$ denote the closed domain in the XY-plane defined by $80X^4-60X^2+4\leqq 3\sqrt{3}\,Y^{5/2}\leqq 64X^5-40X^3$, $\sqrt{5/8}\leqq X\leqq\sqrt{3}/2$ and $Y\geqq 0$.

LEMMA 2. *On* $\mathfrak{D}^*$

$$\Phi^*(\mu, Y)\leqq\frac{729}{163840}, \qquad \mu=16X^4-12X^2+1.$$

Equality occurs only for $\mu=\frac{27}{256}$, $X=\sqrt{\frac{3}{8}+\frac{\sqrt{347}}{64}}$, $Y=\frac{3}{8}$.

Proof. We have

$$\Phi^*(\mu, Y)=-6\Big(\mu-\frac{4Y^3+3Y^2}{6}\Big)^2+\frac{8}{3}Y^6-\frac{22}{5}Y^5+\frac{3}{2}Y^4$$

$$\leqq\frac{8}{3}Y^6-\frac{22}{5}Y^5+\frac{3}{2}Y^4\equiv\Psi^*(Y),$$

$$\Psi^{*\prime}(Y)=2Y^3(8Y-3)(Y-1).$$

Further if $(X, Y)\in\mathfrak{D}^*$, then $0\leqq Y\leqq 1$. Hence on $\mathfrak{D}^*$

$$\Phi^*(\mu, Y)\leqq\Phi^*\Big(\frac{27}{256}, \frac{3}{8}\Big)=\frac{729}{163840}.$$

Obviously

$$\Big(\sqrt{\frac{3}{8}+\frac{\sqrt{347}}{64}}, \frac{3}{8}\Big)\in\mathfrak{D}^*.$$

Thus we have the desired result.

LEMMA 3. *Let*

$$R_1^*=\{(b_1, b_2):\ 0\leqq b_1\leqq 0.43,\ -0.49\leqq b_2\leqq 0\},$$

$$R_2^*=\{(b_1, b_2):\ 0.43\leqq b_1\leqq 0.6,\ -0.38\leqq b_2\leqq 0\}.$$

If (b_1, b_2) *is a point in* $R_1^*\cup R_2^*$, *then there is a point* (X, Y) *in* $\mathfrak{D}^*$ *such that*

$$b_1=-2\mu+3Y^2=-32X^4+24X^2-2+3Y^2,$$

$$b_2=-2\mu+2Y^3=-32X^4+24X^2-2+2Y^3.$$

Proof. $\mathfrak{D}^*$ is bounded by the curves

$$C_1:\ Y=0, \qquad \Big(\sqrt{\frac{5}{8}}\leqq X\leqq\sqrt{\frac{3}{8}+\frac{\sqrt{145}}{40}}\Big),$$

$$C_2:\ X=\frac{\sqrt{3}}{2}, \qquad \left(\left(\frac{16}{27}\right)^{1/5}\leqq Y\leqq 1\right),$$

$$C_3:\ 3\sqrt{3}\,Y^{5/2}=64X^5-40X^3, \qquad \left(\sqrt{\frac{5}{8}}\leqq X\leqq\frac{\sqrt{3}}{2}\right)$$

and

$$C_4:\ 3\sqrt{3}Y^{5/2}=80X^4-60X^2+4, \qquad \left(\sqrt{\frac{3}{8}+\frac{\sqrt{145}}{40}}\leqq X\leqq\frac{\sqrt{3}}{2}\right).$$

Let C_k^* denote the image curve of C_k by the mapping

$$\begin{aligned} b_1&=-32X^4+24X^2-2+3Y^2, \\ b_2&=-32X^4+24X^2-2+2Y^3. \end{aligned} \tag{7}$$

Then C_1^* is the segment defined by $b_2=b_1$, $(-0.4\leqq b_1\leqq 0.5)$ and C_2^* is the curve defined by $27(b_2+2)^2=4(b_1+2)^3$, $(3(16/27)^{2/5}-2\leqq b_1\leqq 1)$. If $(0.6, b_2)\in C_2^*$, then $b_2<-0.38$. Put

$$C_3^*:\ b_1=\sigma_3^*(X),\ b_2=\tau_3^*(X), \qquad \left(\sqrt{\frac{5}{8}}\leqq X\leqq\frac{\sqrt{3}}{2}\right).$$

Then we have

$$\frac{d\tau_3^*}{dX}=(64X^3-24X)\{-2+4\cdot 3^{-4/5}X(64X^5-40X^3)^{1/5}\}\leqq 0$$

and

$$\tau_3^*\left(\frac{\sqrt{3}}{2}\right)=0.$$

This implies that $\tau_3^*(X)\geqq 0$ for $\sqrt{5/8}\leqq X\leqq\sqrt{3}/2$. Put

$$C_4^*:\ b_1=\sigma_4^*(X),\ b_2=\tau_4^*(X), \qquad \left(\sqrt{\frac{3}{8}+\frac{\sqrt{145}}{40}}\leqq X\leqq\frac{\sqrt{3}}{2}\right).$$

Then we have

$$\frac{d\sigma_4^*}{dX}=(64X^3-24X)\{-2+4\cdot 3^{-1/5}(80X^4-60X^2+4)^{-1/5}\}>0$$

and

$$\begin{aligned} \frac{d\tau_4^*}{dX}&=(64X^3-24X)\{-2+4\cdot 3^{-4/5}(80X^4-60X^2+4)^{1/5}\} \\ &<0, \qquad \left(X<\sqrt{\frac{3}{8}+\frac{\sqrt{3130}}{160}}\right), \\ &>0, \qquad \left(X>\sqrt{\frac{3}{8}+\frac{\sqrt{3130}}{160}}\right). \end{aligned}$$

Further we have

$$\sigma_4^*(0.83)<-0.09, \qquad \tau_4^*(0.83)<-0.49$$

and

$$\sigma_4^*(0.866)>0.43, \qquad \tau_4^*(0.866)<-0.53.$$

Here we remark that

$$\sqrt{\frac{3}{8}+\frac{\sqrt{145}}{40}}<0.83<\sqrt{\frac{3}{8}+\frac{\sqrt{3130}}{160}}<0.866<\frac{\sqrt{3}}{2}\,.$$

Summing up the results we conclude that R_1^* and R_2^* are contained in the image of $\mathfrak{D}^*$ by the mapping (7).

4. In this section we consider quadratic differentials of the form $w(w-re^{i\theta})(w-re^{-i\theta})dw^2$, $(r\geqq 0,\ 0<\theta<\pi)$ and construct functions associated with them.

Let $Q(w:r,\theta)dw^2$ be a quadratic differential of the form $w(w-re^{i\theta})(w-re^{-i\theta})dw^2$, $(r\geqq 0,\ 0<\theta<\pi)$ and let Φ denote the union of all trajectories of $Q(w:r,\theta)dw^2$ which have a limiting end point at a zero of $Q(w:r,\theta)$. Let S_w denote the w-sphere.

LEMMA 4. *No cases can occur except the following three cases:*
(i) $S_w-\bar{\Phi}$ *consists of five end domains,*
(ii) $S_w-\bar{\Phi}$ *consists of five end domains and a strip domain,*
(iii) $S_w-\bar{\Phi}$ *consists of five end domains and two strip domains.*

This lemma follows from Theorems 3.2, 3.3, 3.5 and 3.6 in [4].

LEMMA 5. $S_w-\bar{\Phi}$ *consists of five end domains if and only if* θ *satisfies the condition*

$$\Im\left\{e^{i2\theta}\int_0^1(e^{i\theta}t^3-2\cos\theta\cdot t^2+e^{-i\theta}t)^{1/2}dt\right\}=0\,. \tag{8}$$

Proof. Let C denote the segment joining 0 to $re^{i\theta}$. In case (i) we have

$$\Im\left\{\int_C[Q(w:r,\theta)]^{1/2}dw\right\}=0 \tag{9}$$

and in cases (ii), (iii) we have

$$\Im\left\{\int_C[Q(w:r,\theta)]^{1/2}dw\right\}\neq 0\,.$$

The condition (9) implies the condition (8).

We show that there is a real number φ satisfying the condition (8) and estimate the value

$$\Re\left\{e^{i2\varphi}\int_0^1(e^{i\varphi}t^3-2\cos\varphi\cdot t^2+e^{-i\varphi}t)^{1/2}dt\right\}.$$

We use the following lemma which is well known.

LEMMA D. *If* $f(t)$ *is twice continuously differentiable for* $a\leqq t\leqq b$, *then*

$$-\frac{(b-a)^3}{12n^2}\max\{f''(t)\}\leqq\int_a^b f(t)dt-\frac{b-a}{2n}[f(a)+f(b)+2\{f(t_1)$$

$$+\cdots+f(t_{n-1})\}]\leqq-\frac{(b-a)^3}{12n^2}\min\{f''(t)\}$$

where

$$t_k=a+\frac{k(b-a)}{n}, \qquad k=1, 2, \cdots, n-1.$$

We take the determination of $(e^{i\theta}t^3-2\cos\theta\cdot t^2+e^{-i\theta}t)^{1/2}$ such that

$$e^{i2\theta}\int_0^1(e^{i\theta}t^3-2\cos\theta\cdot t^2+e^{-i\theta}t)^{1/2}dt=R(\theta)+iI(\theta),$$

$$R(\theta)=-\frac{1}{\sqrt{2}}\int_0^1 t^{1/2}(1-t)^{1/2}[\{(1-2t\cos 2\theta+t^2)^{1/2}+(1-t)\cos\theta\}^{1/2}\cos 2\theta$$

$$+\{(1-2t\cos 2\theta+t^2)^{1/2}-(1-t)\cos\theta\}^{1/2}\sin 2\theta]dt,$$

$$I(\theta)=-\frac{1}{\sqrt{2}}\int_0^1 t^{1/2}(1-t)^{1/2}[\{(1-2t\cos 2\theta+t^2)^{1/2}+(1-t)\cos\theta\}^{1/2}\sin 2\theta$$

$$-\{(1-2t\cos 2\theta+t^2)^{1/2}-(1-t)\cos\theta\}^{1/2}\cos 2\theta]dt.$$

Firstly we show that $I(\varphi_1)>0$, $I(\varphi_2)<0$ where $\cos\varphi_1=-0.44$, $\cos\varphi_2=-0.4$. Then, since $I(\theta)$ is continuous, it follows that there is a real number φ such that $I(\varphi)=0$, $-0.44<\cos\varphi<-0.4$, $0<\varphi<\pi$.

We put

$$F(t:\theta)=f_1(t)\{f_2(t:\theta)\sin 2\theta-f_3(t:\theta)\cos 2\theta\},$$

$$f_1(t)=t^{1/2}(1-t)^{1/2},$$

$$f_2(t:\theta)=\{(1-2t\cos 2\theta+t^2)^{1/2}+(1-t)\cos\theta\}^{1/2},$$

$$f_3(t:\theta)=\{(1-2t\cos 2\theta+t^2)^{1/2}-(1-t)\cos\theta\}^{1/2}.$$

By an easy calculation we have

$$0.3\leqq f_1(t)\leqq 0.5,$$

$$-\frac{4}{3}\leqq f_1'(t)\leqq\frac{4}{3},$$

$$-\frac{250}{27}\leqq f_1''(t)\leqq -2$$

for $0.1\leqq t\leqq 0.9$. Further we have

$$f_2'(t:\theta)=\frac{1}{2}\{f_2(t:\theta)\}^{-1}\{t-\cos 2\theta-(1-2t\cos 2\theta+t^2)^{1/2}\cos\theta\}$$

$$\times(1-2t\cos 2\theta+t^2)^{-1/2},$$

$$f_2''(t:\theta)=-\frac{1}{4}\{f_2(t:\theta)\}^{-3}\{t-\cos 2\theta-(1-2t\cos 2\theta+t^2)^{1/2}\cos\theta\}^2$$

$$\times\{3(1-2t\cos 2\theta+t^2)^{1/2}+2(1-t)\cos\theta\}(1-2t\cos 2\theta+t^2)^{-3/2}$$

$$-\cos\theta\cdot\{f_2(t:\theta)\}^{-1}\{t-\cos 2\theta-(1-2t\cos 2\theta+t^2)^{1/2}\cos\theta\}$$

$$\times(1-2t\cos 2\theta+t^2)^{-1}$$

$$+\frac{1}{2}\sin^2\theta\cdot\{f_2(t:\theta)\}^{-1}(1-2t\cos 2\theta+t^2)^{-1/2}.$$

Let φ_1 be a real number such that $\cos\varphi_1=-0.44$. We remark the following facts: for $0.1\leqq t\leqq 0.9$

$$1.1325\leqq 1-2t\cos 2\varphi_1+t^2\leqq 2.9131,$$

$$0.6682\leqq(1-2t\cos 2\varphi_1+t^2)^{1/2}+(1-t)\cos\varphi_1\leqq 1.6628,$$

$$1.181\leqq t-\cos 2\varphi_1-(1-2t\cos 2\varphi_1+t^2)^{1/2}\cos\varphi_1\leqq 2.2638,$$

$$2.4006\leqq 3(1-2t\cos 2\varphi_1+t^2)^{1/2}+2(1-t)\cos\varphi_1\leqq 5.0324.$$

Hence we have

$$0.8174\leqq f_2(t:\varphi_1)\leqq 1.2895,$$

$$0.2682\leqq f_2'(t:\varphi_1)\leqq 1.3013,$$

$$-9.4754\leqq f_2''(t:\varphi_1)\leqq 1.4614$$

for $0.1\leqq t\leqq 0.9$. Similarly we have

$$1.2083\leqq f_3(t:\varphi_1)\leqq 1.3232,$$

$$0.0541\leqq f_3'(t:\varphi_1)\leqq 0.2963,$$

$$-0.4224\leqq f_3''(t:\varphi_1)\leqq 0.2808$$

for $0.1\leqq t\leqq 0.9$. Hence we have

$$F''(t:\varphi_1)\geqq -5$$

for $0.1\leqq t\leqq 0.9$. On the other hand we have

$F(0.1:\varphi_1)<0.02837$, $F(0.15:\varphi_1)<0.02537$, $F(0.2:\varphi_1)<0.01932$,

$F(0.25:\varphi_1)<0.01138$, $F(0.3:\varphi_1)<0.00228$, $F(0.35:\varphi_1)<-0.00748$,

$F(0.4:\varphi_1)<-0.01749$, $F(0.45:\varphi_1)<-0.02744$, $F(0.5:\varphi_1)<-0.03702$,

$F(0.55:\varphi_1)<-0.04596$, $F(0.6:\varphi_1)<-0.05400$, $F(0.65:\varphi_1)<-0.06086$,

$F(0.7:\varphi_1)<-0.06623$, $F(0.75:\varphi_1)<-0.06972$, $F(0.8:\varphi_1)<-0.07084$,

$F(0.85:\varphi_1)<-0.06884$, $F(0.9:\varphi_1)<-0.06244$.

Therefore using Lemma D we have

$$\frac{1}{\sqrt{2}}\int_{0.1}^{0.9} F(t:\varphi_1)dt<\frac{1}{\sqrt{2}}\Big(-\frac{0.8}{32}\cdot 0.96913+\frac{0.8^3}{12\cdot 16^2}\cdot 5\Big) \tag{10}$$
$$<-0.01654\,.$$

Since

$$f_2(t:\varphi_1)\sin 2\varphi_1-f_3(t:\varphi_1)\cos 2\varphi_1\leqq f_2(0:\varphi_1)\sin 2\varphi_1-f_3(0.1:\varphi_1)\cos 2\varphi_1$$
$$<0.14916\,,$$
$$f_1(t)\leqq 0.3$$

for $0\leqq t\leqq 0.1$, we have

$$\frac{1}{\sqrt{2}}\int_{0}^{0.1} F(t:\varphi_1)dt<0.00317\,. \tag{11}$$

Since

$$f_2(t:\varphi_1)\sin 2\varphi_1-f_3(t:\varphi_1)\cos 2\varphi_1\leqq f_2(0.9:\varphi_1)\sin 2\varphi_1-f_3(1:\varphi_1)\cos 2\varphi_1$$
$$<0$$

for $0.9\leqq t\leqq 1$, we have

$$\frac{1}{\sqrt{2}}\int_{0.9}^{1} F(t:\varphi_1)dt<0\,. \tag{12}$$

By (10), (11) and (12) we have

$$I(\varphi_1)>0.01337>0\,.$$

Let φ_2 be a real number such that $\cos\varphi_2=-0.4$. Then we have

$$F''(t:\varphi_2)\leqq 7$$

for $0.1\leqq t\leqq 0.9$ and

$$F(0.1:\varphi_2)>0.05857,\quad F(0.15:\varphi_2)>0.06275,\quad F(0.2:\varphi_2)>0.06280,$$
$$F(0.25:\varphi_2)>0.06016,\quad F(0.3:\varphi_2)>0.05569,\quad F(0.35:\varphi_2)>0.04995,$$
$$F(0.4:\varphi_2)>0.04335,\quad F(0.45:\varphi_2)>0.03622,\quad F(0.5:\varphi_2)<0.02882,$$
$$F(0.55:\varphi_2)>0.02135,\quad F(0.6:\varphi_2)>0.01404,\quad F(0.65:\varphi_2)>0.00707,$$
$$F(0.7:\varphi_2)>0.00063,\quad F(0.75:\varphi_2)>-0.00506,\quad F(0.8:\varphi_2)<-0.00976,$$
$$F(0.85:\varphi_2)>-0.01313,\quad F(0.9:\varphi_2)>-0.01466\,.$$

Hence using Lemma D we have

$$\frac{1}{\sqrt{2}}\int_{0.1}^{0.9} F(t:\varphi_2)dt>\frac{1}{\sqrt{2}}\Big(\frac{0.8}{32}\cdot 0.87367-\frac{0.8^3}{12\cdot 16^2}\cdot 7\Big) \tag{13}$$
$$>0.0146\,.$$

Since

$$f_2(t:\varphi_2)\sin 2\varphi_2-f_3(t:\varphi_2)\cos 2\varphi_2\geqq f_2(0.1:\varphi_2)\sin 2\varphi_2-f_3(0:\varphi_2)\cos 2\varphi_2 >0$$

for $0\leqq t\leqq 0.1$, we have

$$\frac{1}{\sqrt{2}}\int_0^{0.1}F(t:\varphi_2)dt>0\,. \tag{14}$$

Since

$$f_2(t:\varphi_2)\sin 2\varphi_2-f_3(t:\varphi_2)\cos 2\varphi_2\geqq f_2(1:\varphi_2)\sin 2\varphi_2-f_3(0.9:\varphi_2)\cos 2\varphi_2 >-0.08502\,,$$

$$0\leqq f_1(t)\leqq 0.3$$

for $0.9\leqq t\leqq 1$, we have

$$\frac{1}{\sqrt{2}}\int_{0.9}^{1}F(t:\varphi_2)dt>-0.00181\,. \tag{15}$$

By (13), (14) and (15) we have

$$I(\varphi_2)<-0.01279<0\,.$$

Next we estimate the value $R(\varphi)$, where φ is a real number such that $I(\varphi)=0$, $-0.44<\cos\varphi<-0.4$, $0<\varphi<\pi$. We put

$$F_1(t)=f_1(t)\{f_2(t:\varphi_1)\cos 2\varphi_1+f(t:\varphi_1,\varphi_2)\sin 2\varphi_2\}\,,$$

$$F_2(t)=f_1(t)\{f_2(t:\varphi_2)\cos 2\varphi_2+f(t:\varphi_2,\varphi_1)\sin 2\varphi_1\}\,,$$

$$f(t:\varphi_j,\varphi_k)=\{(1-2t\cos 2\varphi_j+t^2)^{1/2}-(1-t)\cos\varphi_k\}^{1/2}\,.$$

Then obviously

$$-\frac{1}{\sqrt{2}}\int_0^1 F_1(t)dt<R(\varphi)<-\frac{1}{\sqrt{2}}\int_0^1 F_2(t)dt\,.$$

For the function $F_1(t)$ we have

$$F_1''(t)\geqq -0.6$$

for $0.1\leqq t\leqq 0.9$ and

$F_1(0.1)<-0.41277$, $F_1(0.15)<-0.50020$, $F_1(0.2)<-0.57030$,

$F_1(0.25)<-0.62811$, $F_1(0.3)<-0.67606$, $F_1(0.35)<-0.71540$,

$F_1(0.4)<-0.74679$, $F_1(0.45)<-0.77046$, $F_1(0.5)<-0.78643$,

$F_1(0.55)<-0.79440$, $F_1(0.6)<-0.79392$, $F_1(0.65)<-0.78422$,

$F_1(0.7)<-0.76418$, $F_1(0.75)<-0.73214$, $F_1(0.8)<-0.68552$.

$F_1(0.85)<-0.62009$, $F_1(0.9)<-0.52776$.

Hence using Lemma D we have

$$\frac{1}{\sqrt{2}}\int_{0.1}^{0.9} F_1(t)dt < -0.39018 . \tag{16}$$

Since

$$f_2(t:\varphi_1)\cos 2\varphi_1 + f(t:\varphi_1, \varphi_2)\sin 2\varphi_2$$
$$\leqq f_2(0:\varphi_1)\cos 2\varphi_1 + f(0:\varphi_1, \varphi_2)\sin 2\varphi_2 < -1.32611$$

for $0 \leqq t \leqq 0.1$, we have

$$\frac{1}{\sqrt{2}}\int_0^{0.1} F_1(t)dt < -\frac{1}{\sqrt{2}}\cdot 1.32611\int_0^{0.1} t^{1/2}(1-t)^{1/2}dt < -0.01898 . \tag{17}$$

Since

$$f_2(t:\varphi_1)\cos 2\varphi_1 + f(t:\varphi_1, \varphi_2)\sin 2\varphi_2$$
$$\leqq f_2(0.9:\varphi_1)\cos 2\varphi_1 + f(0.9:\varphi_1, \varphi_2)\sin 2\varphi_2 < -1.75923$$

for $0.9 \leqq t \leqq 1$, we have

$$\frac{1}{\sqrt{2}}\int_{0.9}^{1} F_1(t)dt < -\frac{1}{\sqrt{2}}\cdot 1.75923\int_{0.9}^{1} t^{1/2}(1-t)^{1/2}dt < -0.02518 . \tag{18}$$

By (16), (17) and (18) we have

$$-\frac{1}{\sqrt{2}}\int_0^1 F_1(t)dt > 0.43434 .$$

For the function $F_2(t)$ we have

$$F_2''(t) \leqq 24.2$$

for $0.1 \leqq t \leqq 0.9$ and

$$F_2(0.1) > -0.45906, \quad F_2(0.15) > -0.55619, \quad F_2(0.2) > -0.63394,$$
$$F_2(0.25) > -0.69799, \quad F_2(0.3) > -0.75099, \quad F_2(0.35) > -0.79436,$$
$$F_2(0.4) > -0.82885, \quad F_2(0.45) > -0.85475, \quad F_2(0.5) > -0.87203,$$
$$F_2(0.55) > -0.88047, \quad F_2(0.6) > -0.87951, \quad F_2(0.65) > -0.86833,$$
$$F_2(0.7) > -0.84573, \quad F_2(0.75) > -0.80987, \quad F_2(0.8) > -0.75793,$$
$$F_2(0.85) > -0.68528, \quad F_2(0.9) > -0.58295 .$$

Hence using Lemma D we have

$$\frac{1}{\sqrt{2}}\int_{0.1}^{0.9} F_2(t)dt > -0.43552 . \tag{19}$$

Since

$$f_2(t:\varphi_2)\cos 2\varphi_2+f(t:\varphi_2,\varphi_1)\sin 2\varphi_1$$
$$\geqq f_2(0.1:\varphi_2)\cos 2\varphi_2+f(0.1:\varphi_2,\varphi_1)\sin 2\varphi_1>-1.53019$$

for $0\leqq t\leqq 0.1$, we have

$$\frac{1}{\sqrt{2}}\int_0^{0.1}F_2(t)dt>-\frac{1}{\sqrt{2}}\cdot 1.53019\int_0^{0.1}t^{1/2}(1-t)^{1/2}dt>-0.02219. \tag{20}$$

Since

$$f_2(t:\varphi_2)\cos 2\varphi_2+f(t:\varphi_2,\varphi_1)\sin 2\varphi_1$$
$$\geqq f_2(1:\varphi_2)\cos 2\varphi_2+f(1:\varphi_2,\varphi_1)\sin 2\varphi_1>-1.99057$$

for $0.9\leqq t\leqq 1$, we have

$$\frac{1}{\sqrt{2}}\int_{0.9}^{1}F_2(t)dt>-\frac{1}{\sqrt{2}}\cdot 1.99057\int_{0.9}^{1}t^{1/2}(1-t)^{1/2}dt>-0.02886. \tag{21}$$

By (19), (20) and (21) we have

$$-\frac{1}{\sqrt{2}}\int_0^1 F_2(t)dt<0.48657.$$

Summing up the results we have the following lemma.

LEMMA 6. *There is a real number φ such that $I(\varphi)=0$, $-0.44<\cos\varphi<-0.4$, $0<\varphi<\pi$. For such a φ*

$$0.434<R(\varphi)<0.487.$$

LEMMA 7. *Let φ be a real number such that $I(\varphi)=0$, $-0.44<\cos\varphi<-0.4$, $0<\varphi<\pi$ and let Y be a non-negative real number. Let $\tilde{Q}(w:Y)dw^2$ be the quadratic differential*

$$\left\{w^3+\frac{1}{3}Y^2(3-4\cos^2\varphi)w+\frac{2}{27}Y^3\cos\varphi(9-8\cos^2\varphi)\right\}dw^2.$$

If X is a real number satisfying the condition

$$-256X^5+160X^3\leqq 5R(\varphi)Y^{5/2}\leqq -160X^4+120X^2-8, \tag{22}$$

then there is a function $\tilde{g}(z:X,Y)$ of class Σ_0 which satisfies the differential equation

$$z^2\left(\frac{dw}{dz}\right)^2\left\{w^3+\frac{1}{3}Y^2(3-4\cos^2\varphi)w+\frac{2}{27}Y^3\cos\varphi(9-8\cos^2\varphi)\right\} \tag{23}$$
$$=z^5-2\mu z^3+2\mu z^2+\mu^2 z-2(\mu^2+1)+\mu^2 z^{-1}+2\mu z^{-2}-2\mu z^{-3}+z^{-5},$$
$$\mu=16X^4-12X^2+1,$$

and which maps $|z|>1$ onto a domain admissible with respect to $\tilde{Q}(w:Y)dw^2$. The expansion of $\tilde{g}(z:X,Y)$ at the point at infinity begins

$$z-\left\{2\mu+\frac{1}{3}Y^2(3-4\cos^2\varphi)\right\}z^{-1}-2\left\{\mu-\frac{1}{27}Y^3\cos\varphi(9-8\cos^2\varphi)\right\}z^{-2}$$

$$-3\left\{\mu+\frac{1}{9}Y^2(3-4\cos^2\varphi)\right\}^2 z^{-3}+\left\{\frac{2}{5}+\tilde{\Phi}(\mu,\ Y)\right\}z^{-4}+\cdots$$

where

$$\tilde{\Phi}(\mu,\ Y)=-6\mu^2-\left(2Y^2-\frac{8}{3}Y^3\cos\varphi-\frac{8}{3}Y^2\cos^2\varphi+\frac{64}{27}Y^3\cos^3\varphi\right)\mu$$

$$+\frac{14}{405}Y^5(27\cos\varphi-60\cos^3\varphi+32\cos^5\varphi)\,.$$

Proof. Since $\tilde{Q}(w:Y)dw^2=Q\left(w+\frac{2}{3}Y\cos\varphi : Y,\ \varphi\right)dw^2$, there are two end domains $\tilde{\mathscr{E}}_1$, $\tilde{\mathscr{E}}_2$ and a half of an end domain $\tilde{h}$ in the trajectory structure of $\tilde{Q}(w:Y)dw^2$ on the upper half w-plane. For a suitable choice of determination the function

$$\zeta=\int[\tilde{Q}(w:Y)]^{1/2}dw$$

maps $\tilde{\mathscr{E}}_1$, $\tilde{\mathscr{E}}_2$, $\tilde{h}$ respectively onto an upper half-plane, a lower half-plane and a domain $\Re\zeta>0$, $\Im\zeta>0$. If Y, β and γ satisfy the condition

$$(24)\quad \begin{aligned} &2Y^{5/2}R(\varphi)\leqq\frac{4}{5}-4(4\cos^2\beta+2\cos\beta-1)\,,\\ &Y^{5/2}R(\varphi)\geqq\frac{4}{5}\cos\frac{5}{2}\beta-4(4\cos^2\beta+2\cos\beta-1)\cos\frac{1}{2}\beta\,,\\ &Y^{5/2}R(\varphi)\leqq\frac{4}{5}\cos\frac{5}{2}\gamma-4(4\cos^2\gamma+2\cos\gamma-1)\cos\frac{1}{2}\gamma\,,\\ &2\cos\beta+2\cos\gamma+1=0\,, \end{aligned}$$

then we can combine this function with (5) to obtain a function which maps the domain $|z|>1$, $\Im z>0$ into the upper half w-plane. We put $X=\cos\frac{1}{2}\beta$. Then the condition (24) is equivalent to the condition (22). By reflection this function extends to a function $\tilde{g}(z:X,\ Y)$ which maps $|z|>1$ onto a domain admissible with respect to $\tilde{Q}(w:Y)dw^2$. The function $\tilde{g}(z:X,\ Y)$ satisfies the differential equation (23). Inserting

$$w=z+b_0+b_1z^{-1}+b_2z^{-2}+b_3z^{-3}+b_4z^{-4}+\cdots$$

in (23) we have

$$\begin{aligned} &b_0=0\,,\\ &b_1=-2\mu-\frac{1}{3}Y^2(3-4\cos^2\varphi)\,,\\ &b_2=-2\mu+\frac{2}{27}Y^3\cos\varphi(9-8\cos^2\varphi)\,, \end{aligned}$$

$$b_3=-3\mu^2-\frac{2}{3}\mu Y^2(3-4\cos^2\varphi)-\frac{1}{27}Y^4(3-4\cos^2\varphi)^2,$$

$$b_4=\frac{2}{5}+\tilde{\Phi}(\mu, Y).$$

This completes the proof of Lemma 7.

Let $\tilde{\mathfrak{D}}$ denote the closed domain in the XY-plane defined by $Y^{5/2}\leqq-65.708X^4+49.28X^2-3.286$, $Y^{5/2}\geqq-117.972X^5+73.733X^3$ and $Y\geqq0$. Since $0.434<R(\varphi)<0.487$, if $(X, Y)\in\tilde{\mathfrak{D}}$ then X and Y satisfy the condition (22).

LEMMA 8. *On* $\tilde{\mathfrak{D}}$

$$\tilde{\Phi}(\mu, Y)\leqq\frac{729}{163840}\Lambda^2, \qquad \Lambda=\frac{(3-4\cos^2\varphi)^3}{\cos^2\varphi(9-8\cos^2\varphi)^2}.$$

Equality occurs only for

$$\mu=-\frac{27}{256}\Lambda,\quad X=\sqrt{\frac{3}{8}+\frac{\sqrt{320-27\Lambda}}{64}},\quad Y=\frac{-9(3-4\cos^2\varphi)}{8\cos\varphi(9-8\cos^2\varphi)}.$$

Proof. By an easy calculation we have

$$\tilde{\Phi}(\mu, Y)=-6\Big[\mu+\Big\{\frac{1}{18}Y^2(3-4\cos^2\varphi)-\frac{2}{81}Y^3\cos\varphi(9-8\cos^2\varphi)\Big\}\Big]^2$$
$$+\tilde{\Psi}(Y)\leqq\tilde{\Psi}(Y),$$

$$\tilde{\Psi}(Y)=\frac{8}{2187}Y^6\cos^2\varphi(9-8\cos^2\varphi)^2+\frac{22}{1215}Y^5\cos\varphi(9-8\cos^2\varphi)$$
$$\times(3-4\,\mathrm{cop}^2\varphi)+\frac{1}{54}Y^4(3-4\cos^2\varphi)^2,$$

$$\tilde{\Psi}'(Y)=\frac{2}{729}Y^3\{8\cos\varphi(9-8\cos^2\varphi)Y+9(3-4\cos^2\varphi)\}$$
$$\times\{\cos\varphi(9-8\cos^2\varphi)Y+3(3-4\cos^2\varphi)\}$$

and

$$0\leqq Y<2<-\frac{3(3-4\cos^2\varphi)}{\cos\varphi(9-8\cos^2\varphi)}$$

for all points (X, Y) in $\tilde{\mathfrak{D}}$. Hence on $\tilde{\mathfrak{D}}$

$$\tilde{\Phi}(\mu, Y)\leqq\tilde{\Phi}\Big(-\frac{27}{256},\ -\frac{9(3-4\cos^2\varphi)}{8\cos\varphi(9-8\cos^2\varphi)}\Big)=\frac{729}{163840}\Lambda^2.$$

Since $1.02561<\Lambda<1.37842$, if

$$\Big(\sqrt{\frac{3}{8}+\frac{\sqrt{320-27\Lambda}}{64}},\ Y\Big)\in\partial\tilde{\mathfrak{D}}$$

then $Y=0$ or $1<Y$. On the other hand

$$0<-\frac{9(3-4\cos^2\varphi)}{8\cos\varphi(9-8\cos^2\varphi)}<1.$$

Hence we have

$$\left(\sqrt{\frac{3}{8}+\frac{\sqrt{320-27A}}{64}},\ -\frac{9(3-4\cos^2\varphi)}{8\cos\varphi(9-8\cos^2\varphi)}\right)\in\tilde{\mathfrak{D}}.$$

LEMMA 9. *Let*

$$\tilde{R}_1=\{(b_1, b_2):\ -0.46\leqq b_1\leqq 0,\ 0\leqq b_2\leqq 0.425\},$$
$$\tilde{R}_2=\{(b_1, b_2):\ -0.6\leqq b_1\leqq -0.46,\ 0\leqq b_2\leqq 0.33\}.$$

If (b_1, b_2) *is a point in* $\tilde{R}_1\cup\tilde{R}_2$, *then there is a point* (X, Y) *in* $\tilde{\mathfrak{D}}$ *such that*

$$b_1=-2\mu-\frac{1}{3}Y^2(3-4\cos^2\varphi)=-32X^4+24X^2-2-\frac{1}{3}Y^2(3-4\cos^2\varphi),$$

$$b_2=-2\mu+\frac{2}{27}Y^3\cos\varphi(9-8\cos^2\varphi)$$
$$=-32X^4+24X^2-2+\frac{2}{27}Y^3\cos\varphi(9-8\cos^2\varphi).$$

Proof. $\tilde{\mathfrak{D}}$ is bounded by the curves

$$C_1:\ Y=0,\qquad (\lambda_2\leqq X\leqq\lambda_1),$$
$$C_2:\ Y^{5/2}=-65.708X^4+49.28X^2-3.286,\qquad (\lambda_0\leqq X\leqq\lambda_1),$$
$$C_3:\ Y^{5/2}=-117.972X^5+73.733X^3,\qquad (\lambda_0\leqq X\leqq\lambda_2).$$

Here we remark that $0.72<\lambda_0<0.73$, $0.82<\lambda_1<0.823$, $0.79<\lambda_2<0.791$. Let $\tilde{C}_k$ denote the image curve of C_k by the mapping

$$\begin{aligned} b_1&=-32X^4+24X^2-2-\frac{1}{3}Y^2(3-4\cos^2\varphi),\\ b_2&=-32X^4+24X^2-2+\frac{2}{27}Y^3\cos\varphi(9-8\cos^2\varphi).\end{aligned}\tag{25}$$

Then $\tilde{C}_1$ is the segment defined by $b_2=b_1$, $(-32\lambda_1^4+24\lambda_1^2-2\leqq b_1\leqq -32\lambda_2^4+24\lambda_2^2-2)$. Put

$$\tilde{C}_2:\ b_1=\tilde{\sigma}_2(X),\qquad b_2=\tilde{\tau}_2(X),\qquad (\lambda_0\leqq X\leqq\lambda_1).$$

Then we have

$$\frac{d\tilde{\sigma}_2}{dX}=-128X^3+48X+\frac{4}{15}(3-4\cos^2\varphi)(262.832X^3-98.56X)$$
$$\times(-65.708X^4+49.28X^2-3.286)^{-1/5},$$
$$\frac{d\tilde{\tau}_2}{dX}=-128X^3+48X-\frac{4}{45}\cos\varphi(9-8\cos^2\varphi)(262.832X^3-98.56X)$$
$$\times(-65.708X^4+49.28X^2-3.286)^{1/5}.$$

Since $-0.44<\cos\varphi<-0.4$ and $(-65.708X^4+49.28X^2-3.286)^{1/5}<1.358$ for $0.72\leqq X\leqq\lambda_1$, we have

$$\frac{d\tilde{\tau}_2}{dX}<-23.918X^3+8.971X<0$$

for $0.72\leqq X\leqq\lambda_1$. Further $\tilde{\tau}_2(0.8)<0$, whence $\tilde{\tau}_2(X)<0$ for $0.8\leqq X\leqq\lambda_1$. Since

$$\begin{aligned}(-65.708X^4+49.28X^2-3.286)^{1/5}&>1.307\,, \qquad (0.72\leqq X\leqq 0.745)\,,\\ &>1.188\,, \qquad (0.745\leqq X\leqq 0.78)\,,\\ &>1.059\,, \qquad (0.78\leqq X\leqq 0.8)\,,\end{aligned}$$

we have

$$\begin{aligned}\frac{d\tilde{\sigma}_2}{dX}&<-1.052X^3+0.396X<0\,, \qquad (0.72\leqq X\leqq 0.745)\,,\\ &<11.301X^3-4.236X<2.06\,, \qquad (0.745\leqq X\leqq 0.78)\,,\\ &<28.386X^3-10.643X<6.02\,, \qquad (0.78\leqq X\leqq 0.8)\,.\end{aligned}$$

Hence we have

$$\begin{aligned}\tilde{\sigma}_2(X)&\leqq\tilde{\sigma}_2(0.72)<-0.6721\,, \qquad (0.72\leqq X\leqq 0.745)\,,\\ &=\int_{0.745}^{X}\frac{d\tilde{\sigma}_2}{dX}dX+\tilde{\sigma}_2(0.745)\\ &\qquad <2.06\cdot 0.035-0.7061=-0.634\,, \qquad (0.745\leqq X\leqq 0.78)\,,\\ &=\int_{0.78}^{X}\frac{d\tilde{\sigma}_2}{dX}dX+\tilde{\sigma}_2(0.78)\\ &\qquad <6.02\cdot 0.02-0.7242=-0.6038\,, \qquad (0.78\leqq X\leqq 0.8)\,.\end{aligned}$$

These results imply that if $(b_1, b_2)\in\tilde{C}_2$ and $b_2>0$ then $b_1<-0.6$. Put

$$\tilde{C}_3:\ b_1=\tilde{\sigma}_3(X)\,, \qquad b_2=\tilde{\tau}_3(X)\,, \qquad (\lambda_0\leqq X\leqq\lambda_2)\,.$$

Then we have

$$\begin{aligned}\frac{d\tilde{\sigma}_3}{dX}&=-128X^3+48X+\frac{4}{15}(3-4\cos^2\varphi)(589.86X^4-221.199X^2)\\ &\qquad\times(-117.972X^5+73.733X^3)^{-1/5}\,,\\ \frac{d\tilde{\tau}_3}{dX}&=-128X^3+48X-\frac{4}{45}\cos\varphi(9-8\cos^2\varphi)(589.86X^4-221.199X^2)\\ &\qquad\times(-117.972X^5+73.733X^3)^{1/5}\,.\end{aligned}$$

Since $(-117.972X^5+73.733X^3)^{1/5}<1.363$ for $0.72\leqq X\leqq\lambda_2$, we have

$$\frac{d\tilde{\sigma}_3}{dX}>256.589X^4-128X^3-96.222X^2+48X>0$$

for $0.72\leqq X\leqq\lambda_2$. Further $\tilde{\sigma}_3(0.78)>0$, whence $\tilde{\sigma}_3(X)>0$ for $0.78\leqq X\leqq\lambda_2$. Since

$$(-117.972X^5+73.733X^3)^{1/5}>1.114\,,\qquad (0.72\leqq X\leqq 0.77)\,,$$
$$>0.985\,,\qquad (0.77\leqq X\leqq 0.78)\,,$$

we have

$$\frac{d\tilde{\tau}_3}{dX}>179.907X^4-128X^3-67.466X^2+48X>0\,,\qquad (0.73\leqq X\leqq 0.77)\,,$$
$$>159.262X^4-128X^3-59.724X^2+48X>-0.903\,,\qquad (0.77\leqq X\leqq 0.78)\,.$$

Hence we have

$$\tilde{\tau}_3(X)\geqq\tilde{\tau}_3(0.73)>0.33\,,\qquad (0.73\leqq X\leqq 0.75)\,,$$
$$\geqq\tilde{\tau}_3(0.75)>0.425\,,\qquad (0.75\leqq X\leqq 0.77)\,,$$
$$=\int_{0.77}^{X}\frac{d\tilde{\tau}_3}{dX}dX+\tilde{\tau}_3(0.77)$$
$$>-0.903\cdot 0.01+0.5118=0.5027\,,\qquad (0.77\leqq X\leqq 0.78)\,.$$

Further we remark the following facts:

$$\tilde{\sigma}_3(0.73)<-0.64\,,$$
$$-0.58<\tilde{\sigma}_3(0.75)<-0.46\,,$$
$$-0.24<\tilde{\sigma}_3(0.77)\,.$$

These results imply that if $(b_1, b_2)\in\tilde{C}_3$ and $-0.6\leqq b_1\leqq -0.46$ then $b_2>0.33$ and that $(b_1, b_2)\in\tilde{C}_3$ and $-0.46\leqq b_1\leqq 0$ then $b_2>0.425$. Therefore we conclude that $\tilde{R}_1$ and $\tilde{R}_2$ are contained in the image of $\tilde{\mathfrak{D}}$ by the mapping (25),

5. Now we prove our theorem. Firstly we consider the case $b_1\geqq 0$. We divide this case into several subcases.

Case 1. $\Re b_2\geqq 0$. By Lemma B we have

$$\Re b_4\leqq\Re(b_4+b_1b_2)\leqq\frac{2}{5}\,.$$

Case 2. $0\leqq b_1\leqq 0.43$, $-0.49\leqq\Re b_2\leqq 0$. In this case by Lemma 3 there is a point (X_0, Y_0) in $\mathfrak{D}^*$ such that

$$b_1=-32X_0^4+24X_0^2-2+3Y_0^2\,,$$
$$\Re b_2=-32X_0^4+24X_0^2-2+2Y_0^3\,.$$

We apply Lemma A with $Q(w)dw^2=(w^3-3Y_0^2w+2Y_0^3)dw^2$, $g^*(z)=g^*(z: X_0, Y_0)$. Then we have

$$\Re\{b_4+(-64X_0^4+48X_0^2-4+3Y_0^2)i\Im b_2\}\leqq\frac{2}{5}+\Phi^*(\mu_0, Y_0),$$

$$\mu_0=16X_0^4-12X_0^2+1.$$

Hence by using Lemma 2 we obtain

$$\Re b_4\leqq\frac{2}{5}+\frac{729}{163840}.$$

Case 3. $0.43\leqq b_1\leqq 0.6$, $-0.38\leqq\Re b_2\leqq 0$. As in Case 2 we obtain

$$\Re b_4\leqq\frac{2}{5}+\frac{729}{163840}.$$

Case 4. $0\leqq b_1\leqq 0.43$, $\Re b_2\leqq -0.49$. By Lemma C we have

$$4|b_4|^2\leqq 1-2\cdot 0.49^2<\frac{16}{25}.$$

This implies that $\Re b_4<2/5$.

Case 5. $0.43\leqq b_1\leqq 0.6$, $\Re b_2\leqq -0.38$. By Lemma C we have

$$4|b_4|^2\leqq 1-0.43^2-2\cdot 0.38^2<\frac{16}{25}.$$

This implies that $\Re b_4<2/5$.

Case 6. $0.6\leqq b_1$, $\Re b_2\leqq 0$. By Lemma C we have

$$4|b_4|^2\leqq 1-0.6^2=\frac{16}{25}.$$

This implies that $\Re b_4\leqq 2/5$.

Thus we obtain that if $b_1\geqq 0$ then

$$\Re b_4\leqq\frac{2}{5}+\frac{729}{163840}.$$

The equality statement follows from Lemma 1, Lemma 2 and Lemma A.

Next we consider the case $b_1\leqq 0$. We also divide this case into several subcases.

Case 1. $-0.46\leqq b_1\leqq 0$, $0\leqq\Re b_2\leqq 0.425$ or $-0.6\leqq b_1\leqq -0.46$, $0\leqq\Re b_2\leqq 0.33$. In this case by Lemma 9 there is a point (X_0, Y_0) in $\tilde{\mathfrak{D}}$ such that

$$b_1=-32X_0^4+24X_0^2-2-\frac{1}{3}Y_0^2(3-4\cos^2\varphi),$$

$$\Re b_2=-32X_0^4+24X_0^2-2+\frac{2}{27}Y_0^3\cos\varphi(9-8\cos^2\varphi)$$

where φ is a real number such that $I(\varphi)=0$, $-0.44<\cos\varphi<-0.4$, $0<\varphi<\pi$. We apply Lemma A with

$$Q(w)dw^2=\left\{w^3+\frac{1}{3}Y_0^2(3-4\cos^2\varphi)w+\frac{2}{27}Y_0^3\cos\varphi(9-8\cos^2\varphi)\right\}dw^2,$$

$$g^*(z)=\tilde{g}(z: X_0, Y_0).$$

Then we have

$$\Re\left[b_4+\left\{-64X_0^4+48X_0^2-4-\frac{1}{3}Y_0^2(3-4\cos^2\varphi)\right\}i\Im b_2\right]\leqq\frac{2}{5}+\tilde{\Phi}(\mu_0, Y_0),$$

$$\mu_0=16X_0^4-12X_0^2+1.$$

Hence by Lemma 8 we obtain

$$\Re b_4\leqq\frac{2}{5}+\frac{729}{163840}\Lambda^2.$$

Case 2. $-0.46\leqq b_1\leqq 0$, $0.425\leqq\Re b_2$. By Lemma C we have

$$4|b_4|^2\leqq 1-2\cdot 0.425^2<\frac{16}{25}.$$

Case 3. $-0.6\leqq b_1\leqq -0.46$, $0.33\leqq\Re b_2$. By Lemma C we have

$$4|b_4|^2\leqq 1-0.46^2-2\cdot 0.33^2<\frac{16}{25}.$$

Case 4. $b_1\leqq -0.6$, $0\leqq\Re b_2$. By Lemma C we have

$$4|b_4|^2\leqq 1-0.6^2=\frac{16}{25}.$$

Case 5. $\Re b_2\leqq 0$. By Lemma B we have

$$\Re b_4\leqq\Re(b_4+b_1b_2)\leqq\frac{2}{5}.$$

Thus we obtain that if $b_1\leqq 0$ then

$$\Re b_4\leqq\frac{2}{5}+\frac{729}{163840}\Lambda^2.$$

By Lemma 7 and Lemma 8 it follows that equality really occurs. Hence it follows that there is only one real number φ such that $I(\varphi)=0$, $-0.44<\cos\varphi<-0.4$, $0<\varphi<\pi$. The equality statement follows from Lemma 8 and Lemma A.

References

[1] Duren, P. L., Coefficients of meromorphic schlicht functions. Proc. Amer. Math. Soc., **28** (1971), 169-172.

[2] Golusin, G. M., Some evaluations of the coefficients of univalent functions. Mat. Sbornik, **3** (1938), 321-330.

[3] Grunsky, H., Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen. Math. Zeits., **4** (1939), 29-61.

[4] JENKINS, J. A., Univalent functions and conformal mapping. Berlin-Göttingen-Heidelberg, Springer-Verlag, 1958.

[5] JENKINS, J. A., An extension of the general coefficient theorem. Trans. Amer. Math. Soc., **95** (1960), 387-407.

[6] JENKINS, J. A., On certain coefficients of univalent functions. II. Trans. Amer. Math. Soc., **96** (1960), 534-545.

[7] JENKINS, J. A., An addendum to the general coefficient theorem. Trans. Amer. Math. Soc., **107** (1963), 125-128.

[8] SCHIFFER, M., Sur un problème d'extrémum de la représentation conforme. Bull. Soc. Math., **65** (1937), 48-55.

TOKYO GAKUGEI UNIVERSITY.