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PSEUDO-UMBILICAL SURFACES IN EUCLIDEAN SPACES

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Recently, the author introduced the notion of αth curvatures of first and second kinds for surfaces in higher dimensional euclidean space [2, 3]. The main purpose of this paper is to study these curvatures more detail. In §1, we derive some integral formulas for the αth curvatures of first and second kinds. In §2, we get some applications of these formulas to pseudo-umbilical surfaces.

§1. Integral formulas for αth curvatures.

Let M^2 be an oriented closed Riemannian surface with an isometric immersion x: $M^2 \to E^{2+N}$. Let $F(M^2)$ and $F(E^{2+N})$ be the bundles of orthonormal frames of M^2 and E^{2+N} respectively. Let B be the set of elements $b=(p, e_1, e_2, \cdots, e_{2+N})$ such that $(p, e_1, e_2) \in F(M^2)$ and $(x(p), e_1, \cdots, e_{2+N}) \in F(E^{2+N})$ whose orientation is coherent with that of E^{2+N}, identifying e_i with $dx(e_i)$, $i=1, 2$. Then $B \to M^2$ may be considered as a principal bundle with fibre $O(2)\times SO(N)$ and $\tilde{x}$: $B \to F(E^{2+N})$ is naturally defined by $\tilde{x}(b)=(x(p), e_1, \cdots, e_{2+N})$.

The structure equations of E^{2+N} are given by

$$(1)\qquad \begin{gathered} dx=\sum_A \omega'_A e_A, \qquad de_A=\sum_B \omega'_{AB} e_B, \\ d\omega'_B=\sum_B \omega'_B \wedge \omega'_{BA}, \qquad d\omega'_{AB}=\sum_C \omega'_{AC}\wedge\omega'_{CB}, \qquad \omega'_{AB}+\omega'_{BA}=0, \\ A, B, C, \cdots=1, 2, \cdots, 2+N, \end{gathered}$$

where ω'_A, ω'_{AB} are differential 1-forms on $F(E^{2+N})$. Let ω_A, ω_{AB} be the induced 1-forms on B from ω'_A, ω'_{AB} by the mapping $\tilde{x}$. Then we have

$$(2)\qquad \begin{gathered} \omega_r=0, \qquad \omega_{ir}=\sum_j A_{rij}\omega_j, \qquad A_{rij}=A_{rji}, \\ i, j, \cdots=1, 2; \qquad r, t, \cdots=3, \cdots, 2+N. \end{gathered}$$

From (1), we get

$$(3)\qquad d\omega_i=\sum_j \omega_j\wedge\omega_{ji}, \qquad d\omega_{AB}=\sum_C \omega_{AC}\wedge\omega_{CB},$$

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Let $(p, e_1, e_2, \bar{e}_3(p), \cdots, \bar{e}_{2+N}(p)), p\in U$, be a local cross-section of $B\to F(M^2)$ and for any unit normal vector e at $x(p), p\in U$, put $e=e_{2+N}=\sum\xi_r\bar{e}_r(p)$. Denoting the restriction of A_{rij} onto the image of this local cross-section by $\bar{A}_{rij}$, we may put

$$A_{2+Nij}=\sum_r \xi_r\bar{A}_{rij}.$$

Hence the Lipschitz-Killing curvature $G(p, e)$ is given by

$$G(p, e)=\det(A_{2+Nij})=(\sum_r\xi_r\bar{A}_{r11})(\sum_s\xi_s\bar{A}_{s22})-(\sum_t\xi_t\bar{A}_{t12})^2. \tag{4}$$

The right hand sides is a quadratic form of $\xi_3, \cdots, \xi_{2+N}$. Hence, choosing a suitable cross-section, we can write $G(p, e)$ as

$$G(p, e)=\sum_{\alpha=1}^{N}\lambda_\alpha(p)\xi_{\alpha+2}\xi_{\alpha+2}, \qquad \lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_N. \tag{5}$$

We call this local cross-section of $B\to F(M^2)$, the *Frenet cross-section in the sense of Ōtsuki*, and the frame $(p, e_1, e_2, \bar{e}_3, \cdots, \bar{e}_{2+N})$ the *Ōtsuki frame*. We call the curvature λ_α, the αth curvature of the second kind [2, 5]. By means of the method of definitions, λ_α are defined continuously on the whole manifold M^2, and the Ōtsuki frame is defined uniquely on the subset in which $\lambda_1>\lambda_2>\cdots>\lambda_N$. With respect to the Ōtsuki frame the curvatures:

$$\mu_\alpha(p)=\frac{1}{2}\operatorname{trace}(\bar{A}_{2+\alpha ij}), \qquad \alpha=1, \cdots, N, \tag{6}$$

are called the *αth curvatures of the first kind* [2].

With respect to the Ōtsuki frame, we have

$$\omega_{1r}\wedge\omega_{2r}=\lambda_{r-2}dV, \qquad dV=\omega_1\wedge\omega_2, \qquad r=3, \cdots, 2+N, \tag{7}$$

$$\omega_{1r}\wedge\omega_{2t}+\omega_{1t}\wedge\omega_{2r}=0, \qquad r\neq t; \qquad r, t=3, \cdots, 2+N. \tag{8}$$

In the following, by a *spherical immersion* $\bar{x}$: $M^n\to E^{n+N}$ of a manifold M^n into a euclidean $(n+N)$-space E^{n+N} we mean that *M^n is immersed into a hypersphere of E^{n+N} centered at the origin* of E^{n+N} by the immersion $\bar{x}$. Let $X(p)$ denote the position vector of $x(p)$ in E^{2+N} with respect to the origin of E^{2+N}, $[\overset{N+1\text{ terms}}{\ ,\cdots\cdots,\ }]$ the combined operation of exterior product and vector product in E^{2+N}, and $(\overset{2\text{ terms}}{\ ,\ })$ the combined operation of exterior product and scalar product in E^{2+N}. Then, with respect to the Ōtsuki frame $(p, e_1, e_2, \bar{e}_3, \cdots, \bar{e}_{2+N})$, we have

$$d(X, [d\bar{e}_r, \bar{e}_3, \cdots, \bar{e}_{2+N}])$$
$$=(dX, [d\bar{e}_r, \bar{e}_3, \cdots, \bar{e}_{2+N}])-\sum_s(X, [d\bar{e}_r, \bar{e}_3, \cdots, \bar{e}_{s-1}, d\bar{e}_s, \bar{e}_{s+1}, \cdots, \bar{e}_{2+N}])$$

$$=2(-1)^N\left\{\lambda_{r-2}(X\cdot\bar e_r)dV+\mu_{r-2}dV+\frac{1}{2}\sum_{s\neq r}(X\cdot\bar e_s)(\omega_{1r}\wedge\omega_{2s}+\omega_{1s}\wedge\omega_{2r})\right.$$

$$\left.+\frac{1}{2}\sum_{s\neq r}\omega_{rs}\wedge((X\cdot e_1)\omega_{2s}-(X\cdot e_2)\omega_{1s})\right\}$$

$$=2(-1)^N\left\{(X\cdot\bar e_r)\lambda_{r-2}dV+\mu_{r-2}dV+\sum_{s\neq r}\frac{1}{2}\omega_{rs}\wedge((X\cdot e_1)\omega_{2s}-(X\cdot e_2)\omega_{1s})\right\}.$$

Suppose that x: $M^2\to E^{2+N}$ is spherical. Then the last term of the above equations vanishes. Hence, we get

$$d(X,[d\bar e_r,\bar e_3,\cdots,\bar e_{2+N}])=2(-1)^N\{(X\cdot\bar e_r)\lambda_{r-2}+\mu_{r-2}\}dV. \tag{9}$$

Furthermore, suppose that there exist a unit normal vector field e defined globally on M^2 and a fixed integer α; $1\leqq\alpha\leqq N$, such that the Lipschitz-Killing curvature $G(p,e)=\lambda_\alpha(p)$ for all $p\in M^2$, then, by the definition of the Ōtsuki frame, we know that for each point p, there exists an Ōtsuki frame $(q, e_1, e_2, \bar e_3, \cdots, \bar e_{2+N})$ with $\bar e_{2+\alpha}=e$ on a neighborhood of p. Therefore, by the fact that

$$d(X,[d\bar e_r,\bar e_3,\cdots,\bar e_{2+N}])=d(X,[d\tilde e_r,\tilde e_3,\cdots,\tilde e_{2+N}])$$

for any two Ōtsuki frames $(p, e_1, e_2, \bar e_3, \cdots, \bar e_{2+N})$ and $(p, e_1', e_2', \tilde e_3, \cdots, \tilde e_{2+N})$ with $\bar e_r=\tilde e_r$, we have the following theorem:

THEOREM 1. *Let x: $M^2\to E^{2+N}$ be a spherical immersion of an oriented closed surface M^2 into E^{2+N}. If there exist a unit normal vector field e over M^2 and a fixed integer α, $1\leqq\alpha\leqq N$, such that the Lipschitz-Killing curvature $G(p,e)=\lambda_\alpha(p)$ for all $p\in M^2$, then we have*

$$-\int_M\mu_\alpha dV=\int_{M^2}(X\cdot e)\lambda_\alpha dV, \tag{10}$$

where μ_α is the αth curvature of first kind corresponding to e.

Furthermore, we have

$$\begin{aligned}&d(X,[dX,\bar e_3,\cdots,\bar e_{2+N}])\\ &=(dX,[dX,\bar e_3,\cdots,\bar e_{2+N}])-\sum_r(X,[dX,\bar e_3,\cdots,\bar e_{r-1},d\bar e_r,\bar e_{r+1},\cdots,\bar e_{2+N}])\\ &=2(-1)^{N-1}(1+\sum_r(X\cdot\bar e_r)\mu_{r-2})dV.\end{aligned} \tag{11}$$

Hence, if we define the mean curvature vector H by

$$H=\sum_r\mu_{r-2}\bar e_r, \tag{12}$$

then, by integrating both sides of (11) over M^2 and applying the Stokes theorem, we get

LEMMA 2. *Let x: $M^2 \to E^{2+N}$ be an immersion of an oriented closed surface M^2 into E^{2+N}. Then we have*

$$\int_{M^2} dV = -\int_{M^2} (X \cdot H) dV.$$

In theorem 1, if e is parallel to the mean curvature vector and $\alpha=1$, then we have the following corollary:

COROLLARY 1. *Let x: $M^2 \to E^{2+N}$ be a spherical immersion of an oriented closed surface M^2 into E^{2+N}. If the Lipschitz-Killing curvature in the direction of mean curvature vector is nowhere negative and the mean curvature normal is nowhere zero, then we have*

$$-\int_{M^2} h dV = \int_{M^2} (X \cdot e) \lambda_1 dV,$$

where $H=he$.

Proof. If the mean curvature vector $H \neq 0$ everywhere and the Lipschitz-Killing curvature in the direction of H is nowhere negative, then we can easily verify that $G(p, e)=\lambda_1(p)$ for all $p \in M^2$. Therefore, by theorem 1, we get the above integral formula.

§ 2. Pseudo-umbilical surfaces in Euclidean spaces.

In [4], the author proved that

LEMMA 3. *Let x: $M^n \to E^{n+N}$ be an immersion of an oriented closed n-dimensional manifold M^n into E^{n+N}. Then the immersion x is spherical if and only if $X \cdot H = -1$.*

An immersion x: $M^n \to E^{n+N}$ is called *minimal* if the mean curvature vector $H=0$ everywhere. If the mean curvature vector H is nowhere vanished, then we can let e be a unit normal vector field in the direction of H. In this case, if the second fundamental form in the direction e, $\mathrm{II}_e = -dX \cdot de$, is propotional to the first fundamental form, $\mathrm{I} = dX \cdot dX$, everywhere, then the immersion x: $M^n \to E^{n+N}$ is called *pseudo-umbilical* [6].

The main purpose of this section is to prove the following:

THEOREM 4. *Let x: $M^2 \to E^{2+N}$ be a spherical immersion of an oriented closed surface M^2 into E^{2+N} with the mean curvature vector H nowhere zero. Then the immersion x is pseudo-umbilical if and only if the Lipschitz-Killing curvature in the direction $H=he$ is maximal, i.e. $G(p, e)=\lambda_1(p)$ for all $p \in M^2$.*

Proof. If the immersion x is pseudo-umbilical, then we can choose a local cross-section $(p, e_1, e_2, e_3', \cdots, e_{2+N}')$ such that $e_3'=e$ parallel to the mean curvature vector H. With respect to this cross-section, we can easily find that

$$A_{311}'=A_{322}'\neq 0, \qquad A_{411}'=-A_{422}', \cdots, A_{2+N11}'=-A_{2+N22}', \tag{13}$$

where A_{rij}' is the restriction of A_{rij} onto the image of this local cross-section. Moreover, by a suitable choosing of e_1, e_2, we can assume that

$$A_{312}'=A_{321}'=0. \tag{14}$$

Thus, the Lipschitz-Killing curvature $G(p, e)=(A_{311}')^2>0$ in the direction of mean curvature vector $H=he$ and the Lipschitz-Killing curvatures $G(p, e_r')\leqq 0$ for all $r=4, \cdots, 2+N$. From these results, we can easily verify that $G(p, e)=\lambda_1(p)$ for all $p\in M^2$.

Conversely, if the Lipschitz-Killing curvature $G(p, e)$ in the direction of mean curvature vector $H=he$ is equal to the first curvature of second kind $\lambda_1(p)$ everywhere, then, by Corollary 1, we have

$$-\int_{M^2} h\, dV=\int_{M^2}(X\cdot e)\lambda_1 dV. \tag{15}$$

Moreover, by the assumption that x is spherical and lemma 3, we have

$$h(X\cdot e)=-1. \tag{16}$$

Combining (15) and (16), we get

$$\int_{M^2}\left(\frac{1}{h}\right)(h^2-\lambda_1)dV=0. \tag{17}$$

Thus, by the fact $h>0$, and $h^2-\lambda_1\geqq 0$, we get $h^2-\lambda_1=0$. From this we can easily prove that the immersion x is pseudo-umbllical. This completes the proof of the theorem.

REMARK 1. If x: $M^n\to E^{n+N}$ is spherical, then we can prove that the mean curvature vector $H\neq 0$ everywhere. So that the condition of $H\neq 0$ everywhere in theorem 4 is not essential.

REMARK 2. In [1, 7], Yano and Chen proved that the only spherical pseudo-umbilical submanifolds M^n of dimension n in E^{n+3} with constant mean curvature, h=constant, are minimal submanifolds of some hyperspheres of E^{n+3} (not necessary centered at the origin of E^{n+3}). Thus, by theorem 4 and the Yano-Chen result, we know that if x: $M^2\to E^5$ is a spherical immersion of a closed oriented surface in E^5, then the Lipschitz-Killing curvature in the direction of H satisfies $G(p, e)=\lambda_1$=constant, $H=he$, when and only when M^2 is immersed as a minimal surface in some hypersphere of E^5.

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