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ON THE INFLUENCE OF A CONFORMAL KILLING TENSOR ON THE REDUCIBILITY OF COMPACT RIEMANNIAN SPACES

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§ 0. Let M be an n-dimensional Riemannian space whose metric tensor is given by g_{ab}.[1] A contravariant vector field v^a is called an infinitesimal conformal transformation or a conformal Killing vector if there exists a scalar function ρ such that

$$\nabla_a v_b + \nabla_b v_a = 2\rho g_{ab},$$

where $v_a = g_{ab}v^b$ and ∇_a means the covariant derivation with respect to the Riemannian connection. Especially, a conformal Killing vector v^a is called an infinitesimal isometry or a Killing vector if $\rho=0$. In a compact reducible Riemannian space, the following theorem is well known.

THEOREM (Tachibana [1][2]). *In a compact reducible Riemannian space, an infinitesimal conformal transformation is an infinitesimal isometry.*

On the other hand, as a generalization of a conformal Killing vector, Kashiwada [3] has defined a conformal Killing tensor, that is, a skew-symmetric tensor $u_{a_1\cdots a_r}$ is called a conformal Killing tensor of degree r if there exists a skew-symmetric tensor $\rho_{a_1\cdots a_{r-1}}$ such that

$$\nabla_c u_{a_1\cdots a_r} + \nabla_{a_1} u_{ca_2\cdots a_r} = 2\rho_{a_2\cdots a_r} g_{ca_1} - \sum_{i=2}^{r} (-1)^i (\rho_{a_1\cdots \hat{a}_i \cdots a_r} g_{ca_i} + \rho_{ca_2\cdots \hat{a}_i \cdots a_r} g_{a_1 a_i}), \tag{0.1}$$

where $\hat{a}_i$ means that a_i is omitted. This $\rho_{a_1\cdots a_{r-1}}$ is called the associated tensor of $u_{a_1\cdots a_r}$. Especially, $u_{a_1\cdots a_r}$ is called a Killing tensor if $\rho_{a_1\cdots a_{r-1}}=0$.

The purpose of the paper is to discuss the relation between the existence of a conformal Killing tensor and the reducibility of compact Riemannian spaces as a generalization of the above theorem.

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1) Indices $a, b, c, \cdots$ run over $1, \cdots, n$.

2) See the bibliography at the end of the paper.

§1. Preliminaries. Let M^n be an $n(\geqq 2)$ dimensional connected[3] reducible Riemannian space. Then there exists a system of coordinate neighborhoods $\{U_\alpha\}$ such that in each U_α the line element is given by the form

$$ds^2 = g_{\lambda\mu}(x^\nu)dx^\lambda dx^\mu + g_{ij}(x^k)dx^i dx^j.^{4)}$$

If we define $\varphi_a{}^b$ by

$$(\varphi_a{}^b) = \begin{pmatrix} \delta_\mu{}^\lambda & 0 \\ 0 & -\delta_i{}^j \end{pmatrix}$$

in each U_α, then they define a tensor field of type (1,1) over M^n. The metric tensor g_{ab} and the tensor $\varphi_a{}^b$ satisfy

$$\varphi_a{}^b \varphi_b{}^c = \delta_a{}^c, \tag{1.1}$$

$$g_{ab}\varphi_c{}^b = g_{cb}\varphi_a{}^b, \tag{1.2}$$

$$\nabla_a \varphi_c{}^b = 0. \tag{1.3}$$

If we put $g_{ab}\varphi_c{}^b = \varphi_{ac}$ and $g^{ab}\varphi_b{}^c = \varphi^{ac}$, then they are symmetric tensors and it holds that $\varphi_a{}^a = p - q$.

Since tensor equations are independent to choice of coordinate systems, these equations hold good in any allowable coordinates and equations appeared hereafter will be considered there too.

Throughout the paper we shall assume that M^n is an n-dimensional reducible Riemannian space.

Let $\xi_{(a)}{}^{(b)} = \xi_{a_1 \cdots a_p}{}^{b_1 \cdots b_q}$ be a tensor of type (q, p). If it commutes with $\varphi_a{}^b$ at a pair of indices remaining other indices fixed, then it is called pure with respect to the two indices. For example, it is pure with respect to a_k and b_h, if

$$\xi_{a_1 \cdots c \cdots a_p}{}^{(b)} \varphi_{a_k}{}^c = \varphi_c{}^{b_h} \xi_{(a)}{}^{b_1 \cdots c \cdots b_q}$$

and pure with respect to a_k and a_h, if

$$\xi_{a_1 \cdots c \cdots a_h \cdots a_p}{}^{(b)} \varphi_{a_k}{}^c = \varphi_{a_h}{}^c \xi_{a_1 \cdots a_k \cdots c \cdots a_p}{}^{(b)}.$$

$\xi_{(a)}{}^{(b)}$ is called pure if it is pure with respect to any pair of indices. The equation (1.2) means that metric tensor g_{ab} is pure. The purity of tensors is invariant under rising (resp. lowering) of their indices by g^{ab} (resp. g_{ab}). Let $R_{abc}{}^d$ be the Riemannian curvature, then $R_{abc}{}^d$ and Ricci tensor $R_{ab} = R_{cab}{}^c$ are pure by virtue of Tachibana's lemma [2].

3) We shall always assume that M is connected.

4) Indices λ, μ, ν (resp. i, j, k) run over $1, \cdots, p$ (resp. $p+1, \cdots, p+q=n$) and p is a constant positive integer over M^n.

§ 2. **An integral formula.** In this section we shall assume our M^n to be orientable.

Let $u_{a_1\cdots a_r}$ be any tensor field and we define

$$\overset{*}{u}_{(a)}=\overset{*}{u}_{a_1\cdots a_r}=\varphi_{a_1}{}^{b}u_{ba_2\cdots a_r},$$

$$A_{b(a)}(u)=\nabla_b u_{(a)}-\varphi_b{}^c\nabla_c\overset{*}{u}(a).$$

Denoting the square of $A_{b(a)}(u)$ by $A^2(u)$, we can obtain

$$\nabla^b(A_{b(a)}(u)u^{(a)})=(\nabla^b A_{b(a)}(u))u^{(a)}+\frac{1}{2}A^2(u),$$

from which and Green's theorem it follows

THEOREM 1 (Tachibana [1]). *In a compact orientable space M^n, the integral formula*

$$\int_M\left[(\nabla^b\nabla_b u_{(a)}-\varphi^{bc}\nabla_b\nabla_c\overset{*}{u}_{(a)})u^{(a)}+\frac{1}{2}A^2(u)\right]d\sigma=0$$

is valid for any tensor $u_{(a)}$, where $d\sigma$ means the volume element of M.

§ 3. **Non-existence of a conformal Killing tensor.** We consider a conformal Killing tensor $u_{a_1\cdots a_r}$ of degree r, then from (0. 1) the associated tensor $\rho_{a_1\cdots a_{r-1}}$ satisfies

$$\nabla^b u_{ba_2\cdots a_r}=(n-r+1)\rho_{a_2\cdots a_r}. \tag{3. 1}$$

Operating ∇_b to (0. 1), we have

$$\begin{aligned}&\nabla_b\nabla_c u_{a_1\cdots a_r}+\nabla_b\nabla_{a_1}u_{ca_2\cdots a_r}\\&=2\tau_{ba_2\cdots a_r}g_{ca_1}-\sum_{i=2}^{r}(-1)^i(\tau_{ba_1\cdots\hat{a}_i\cdots a_r}g_{ca_i}+\tau_{bca_2\cdots\hat{a}_i\cdots a_r}g_{a_1a_i}),\end{aligned} \tag{3. 2}$$

where $\tau_{ba_2\cdots a_r}=\nabla_b\rho_{a_2\cdots a_r}$. By interchanging indices b, c, a_1, as $b\to c\to a_1\to b$ and $b\to a_1\to c\to b$ in the equation (3. 2) and substracting the latter equation from the sum of (3. 2) and the former, we have

$$\begin{aligned}&2\nabla_b\nabla_c u_{a_1\cdots a_r}+\sum_{i=1}^{r}R_{bca_i}{}^{e}u_{a_1\cdots e\cdots a_r}-(R_{ba_1c}{}^{e}+R_{ca_1b}{}^{e})u_{ea_2\cdots a_r}\\&-\sum_{i=2}^{r}(R_{ba_1a_i}{}^{e}u_{ca_2\cdots e\cdots a_r}+R_{ca_1a_i}{}^{e}u_{ba_2\cdots e\cdots a_r})\\&=2\tau_{ba_2\cdots a_r}g_{ca_1}+2\tau_{ca_2\cdots a_r}g_{ba_1}-2\tau_{a_1\cdots a_r}g_{bc}\\&-\sum_{i=2}^{r}(-1)^i(\tau_{bca_2\cdots\hat{a}_i\cdots a_r}+\tau_{cba_2\cdots\hat{a}_i\cdots a_r})g_{a_1a_i}\end{aligned} \tag{3. 3}$$

$$-\sum_{i=2}^{r}(-1)^{i}(\tau_{ba_1\cdots\hat{a}_i\cdots a_r}-\tau_{a_1ba_2\cdots\hat{a}_i\cdots a_r})g_{ca_i}$$

$$-\sum_{i=2}^{r}(-1)^{i}(\tau_{ca_1\cdots\hat{a}_i\cdots a_r}-\tau_{a_1ca_2\cdots\hat{a}_i\cdots a_r})g_{ba_i}.$$

Transvecting (3. 3) with g^{bc}, we have

$$\begin{aligned}&\nabla^b\nabla_b u_{a_1\cdots a_r}+R_{a_1}{}^{e}u_{ea_2\cdots a_r}-\sum_{i=2}^{r}R^{b}{}_{a_1a_i}{}^{e}u_{ba_2\cdots e\cdots a_r}\\ &=-(n-r-1)\tau_{a_1\cdots a_r}-\sum_{i=2}^{r}(-1)^{i}\tau_{a_ia_1\cdots\hat{a}_i\cdots a_r}.\end{aligned}\tag{3. 4}$$

By virtue of equations (0. 1), (3. 3) and the purity of curvature, we have

$$\begin{aligned}&\varphi^{bc}\nabla_b\nabla_c\overset{*}{u}_{a_1\cdots a_r}=\varphi^{bc}(\nabla_b\nabla_c u_{fa_2\cdots a_r})\varphi_{a_1}{}^{f}\\ &=-R_{a_1}{}^{e}u_{ea_2\cdots a_r}+\sum_{i=2}^{r}R^{b}{}_{a_1a_i}{}^{e}u_{ba_2\cdots e\cdots a_r}+2\tau_{a_1\cdots a_r}\\ &\quad-(p-q)\tau_{fa_2\cdots a_r}\varphi_{a_1}{}^{f}-\sum_{i=2}^{r}(-1)^{i}\tau_{bca_2\cdots\hat{a}_i\cdots a_r}\varphi_{a_ia_1}\varphi^{bc}\\ &\quad-\sum_{i=2}^{r}(-1)^{i}(\tau_{bca_1\cdots\hat{a}_i\cdots a_r}-\tau_{cba_2\cdots\hat{a}_i\cdots a_r})\varphi_{a_i}{}^{b}\varphi_{a_1}{}^{c}.\end{aligned}\tag{3. 5}$$

Substracting (3. 5) from (3. 4), we have

$$\begin{aligned}&\nabla^b\nabla_b u_{a_1\cdots a_r}-\varphi^{bc}\nabla_b\nabla_c\overset{*}{u}_{a_1\cdots a_r}\\ &=-(n-r+1)\tau_{a_1\cdots a_r}-\sum_{i=2}^{r}(-1)^{i}\tau_{a_ia_1\cdots\hat{a}_i\cdots a_r}\\ &\quad+(p-q)\tau_{fa_2\cdots a_r}\varphi_{a_1}{}^{f}+\sum_{i=2}^{r}\tau_{bca_2\cdots\hat{a}_i\cdots a_r}\varphi_{a_ia_1}\varphi^{bc}\\ &\quad+\sum_{i=2}^{r}(-1)^{i}(\tau_{bca_2\cdots\hat{a}_i\cdots a_r}-\tau_{cba_2\cdots\hat{a}_i\cdots a_r})\varphi_{a_i}{}^{b}\varphi_{a_1}{}^{c}.\end{aligned}\tag{3. 6}$$

In the equation getting by transvection (3. 6) with $u^{a_1\cdots a_r}$, its right hand side is the sum of the following five terms $c_1,\cdots,c_5$.

$$\begin{aligned}c_1&=-(n-r+1)\tau_{a_1\cdots a_r}u^{a_1\cdots a_r}=-(n-r+1)\nabla_{a_1}\rho_{a_2\cdots a_r}u^{a_1\cdots a_r}\\ &=-(n-r+1)\nabla_{a_1}(\rho_{a_2\cdots a_r}u^{a_1\cdots a_r})+(n-r+1)\rho_{a_2\cdots a_r}\nabla_{a_1}u^{a_1\cdots a_r},\end{aligned}$$

where the first term of the right hand side vanishes by applying Green's theorem when it is integrated. Hereafter we substitute $\overset{*}{=}$ for $=$ when we abbrieviate the terms which vanish by integrations. Taking account of (0. 1) and (3. 1), we have

$$c_1 \overset{*}{=} (n-r+1)^2 \rho_{a_2\cdots a_r}\rho^{a_2\cdots a_r},$$

$$c_2 = -\sum_{i=2}^{r}(-1)^i \tau_{a_i a_1\cdots \hat{a}_i\cdots a_r} u^{a_1\cdots a_r}$$

$$\overset{*}{=} -(n-r+1)(r-1)\rho_{a_2\cdots a_r}\rho^{a_2\cdots a_r},$$

$$c_3 = (p-q)\tau_{f a_2\cdots a_r}\varphi_{a_1}{}^f u^{a_1\cdots a_r}$$

$$\overset{*}{=} -(p-q)\rho_{a_2\cdots a_r}\rho^{a_2\cdots a_r} + (p-q)(r-1)\rho_{ba_3\cdots a_r}\rho^{ca_3\cdots a_r}\varphi_c{}^b,$$

$$c_4 = \sum_{i=2}^{r}(-1)^i \tau_{bca_2\cdots \hat{a}_i\cdots a_r}\varphi_{a_i a_1}\varphi^{bc}u^{a_1\cdots a_r} = 0,$$

$$c_5 = \sum_{i=2}^{r}(-1)^i (\tau_{bca_2\cdots \hat{a}_i\cdots a_r} - \tau_{cba_2\cdots \hat{a}_i\cdots a_r})\varphi_{a_i}{}^b \varphi_{a_1}{}^c u^{a_1\cdots a_r}$$

$$\overset{*}{=} -2(r-1)[\rho_{a_2\cdots a_r}\rho^{a_2\cdots a_r} - (p-q)\rho_{ba_3\cdots a_r}\rho^{ca_3\cdots a_r}\varphi_c{}^b$$

$$+ (r-2)\rho_{bea_4\cdots a_r}\rho^{cfa_4\cdots a_r}\varphi_c{}^b\varphi_f{}^e].$$

Thus we get

$$(\nabla^b\nabla_b u_{(a)} - \varphi^{bc}\nabla_b\nabla_c \overset{*}{u}_{(a)})u^{(a)}$$

$$\overset{*}{=} [(n-r+1)^2 - (n-r+3)(r-1) - (p-q)^2]\rho_{a_2\cdots a_r}\rho^{a_2\cdots a_r} \tag{3. 7}$$

$$+3(p-q)(r-1)\rho_{ba_3\cdots a_r}\rho^{ca_3\cdots a_r}\varphi_c{}^b$$

$$-2(r-1)(r-2)\rho_{bea_4\cdots a_r}\rho^{cfa_4\cdots a_r}\varphi_c{}^b\varphi_f{}^e.$$

Substituting (3. 7) into the integral formula of Theorem 1, we have

$$\int_M \left[B + \frac{1}{2}A^2(u)\right]d\sigma = 0, \tag{**}$$

where B = the right hand side of (3. 7).

From the hypothesis, M^n is locally isometric to the direct product of a p-dimensional Riemannian space and a q-dimensional one and we can suitably choose a basis at any point such that

$$g_{ab} = \delta_{ab},$$

$$(\varphi_{ab}) = \begin{pmatrix} \delta_{\lambda\mu} & 0 \\ 0 & -\delta_{ij} \end{pmatrix}.$$

If we put $p-q=s$, then with respect to the basis, we have

$$B=[n^2-3(r-1)n-s^2+3s(r-1)]\rho_{\lambda\mu a_4\cdots a_r}\rho^{\lambda\mu a_4\cdots a_r}$$
$$+[n^2-3(r-1)n-s^2-3s(r-1)]\rho_{ija_4\cdots a_r}\rho^{ija_4\cdots a_r}$$
$$+2[n^2-3(r-1)n-s^2+4(r-1)(r-2)]\rho_{\lambda i a_4\cdots a_r}\rho^{\lambda i a_4\cdots a_r}.$$

Hence if the following (3. 8)$\sim$(3. 10) is satisfied, $B\geqq 0$ holds good.

$$n^2-3(r-1)n-s^2+3s(r-1)>0, \tag{3. 8}$$

$$n^2-3(r-1)n-s^2-3s(r-1)>0, \tag{3. 9}$$

$$n^2-3(r-1)n-s^2+4(r-1)(r-2)>0. \tag{3. 10}$$

Taking account of $n-s=2q>0$ and $n+s=2p>0$, (3. 8) and (3. 9) are equivalent to

$$s+n-3(r-1)>0, \tag{3. 11}$$

$$n-s-3(r-1)>0. \tag{3. 12}$$

Now we assume that r satisfies $3(r-1)<n$. Then from (3. 11) and (3. 12), we have

$$3(r-1)-n<s<n-3(r-1). \tag{3. 13}$$

It is easily seen that (3. 10) is a consequence of (3. 13), and (3. 13) is also written in the form

$$\frac{3}{2}(r-1)<p<n-\frac{3}{2}(r-1), \tag{3. 14}$$

taking account of $p+q=n$ and $p-q=s$.

The following (3. 15) is equivalent to (3. 14).

$$3(r-1)<2p \qquad \left(p\leqq\frac{n}{2}\right). \tag{3. 15}$$

Hence if (3. 15) is satisfied, then (3. 8)$\sim$(3. 10) are satisfied and we obtain $B\geqq 0$. Thus (3. 15) and (**) imply $B=0$ and hence $\rho_{a_2\cdots a_r}=0$ holds good.

Consequently we get the following

THEOREM 2. *Let M^n be a Riemannian space which is compact and locally isometric to the direct product of a $p(\leqq n/2)$-dimensional Riemannian space and a $(n-p)$-dimensional one. Then M^n can not admit a conformal Killing tensor of degree r satisfying $3(r-1)<2p$ which is not a Killing tensor.*

THEOREM 3. *If a compact Riemannian space admits a non-Killing conformal Killing tensor of degree r satisfying $3(r-1)<n$, then it is irreducible or locally isometric to $M^i\times M^{n-i}$ $(0<i\leqq 3(r-1)/2)$, where M^j means a j-dimensional Riemannian space.*

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