KŌDAI MATH. SEM. REP.
22 (1970), 424—435

ON THE ABSOLUTE SUMMABILITY OF THE SERIES ASSOCIATED WITH A FOURIER SERIES AND ITS ALLIED SERIES

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1. Let S_n be the partial sum of an infinite series $\sum_{n=0}^{\infty} a_n$ and let

$$t_n=\sum_{\nu=0}^{n}\binom{n}{\nu}(\Delta^{n-\nu}\mu_\nu)S_\nu. \tag{1.1}$$

Then the sequence $\{t_n\}$ is known as the Hausdroff means of sequence $\{S_n\}$, where $\{\mu_\nu\}$ is a sequence of real or complex numbers and the sequence $\{\Delta^p\mu_\nu\}$ denotes the differences of order p.

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable by Hausdroff mean to the sum S, if

$$\lim_{n\to\infty} t_n=S,$$

whenever $S_n\to S$. The necessary and sufficient condition for the Hausdorff summability to be conservative is that the sequence $\{\mu_n\}$ should be a sequence of moment constant, i.e.;

$$\mu_n=\int_0^1 x^n d\chi(x),$$

where $\chi(x)$ is a real function of bounded variation in $0\leqq x\leqq 1$. We may suppose without loss of generality that $\chi(0)=0$. If also $\chi(1)=1$ and $\chi(+0)=0$, so that $\chi(x)$ is continuous at the origin, then μ_n is a regular moment constant and (H, μ_n) is a regular method of summation [3].

If

$$\sum_{n=1}^{\infty}|(t_n-t_{n-1})|<\infty, \tag{1.2}$$

then the series $\sum_{n=0}^{\infty} a_n$ is said to be absolutely summable (H, μ_n) or summable $|\mathrm{H}, \mu_n|$. It is also known that the Cesàro, Hölder and Euler methods of summation are the particular cases of the above method.

2. Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue in $(-\pi, \pi)$. Let its Fourier series be

Received December 23, 1969.

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)=\sum_{n=0}^{\infty}A_n(t)$$

and its allied series is

$$\sum_{n=1}^{\infty}(b_n\cos nt-a_n\sin nt)=\sum_{n=1}^{\infty}B_n(t).$$

We write

$$\phi(t)=\frac{1}{2}\{f(\theta+t)+f(\theta-t)\},$$

$$\psi(t)=\frac{1}{2}\{f(\theta+t)-f(\theta-t)\},$$

$$\Phi_\beta(t)=\frac{1}{\Gamma(\beta)}\int_0^t(t-u)^{\beta-1}\phi(u)du,\qquad \beta>0;$$

$$\Psi_\beta(t)=\frac{1}{\Gamma(\beta)}\int_0^t(t-u)^{\beta-1}\psi(u)du,\qquad \beta>0;$$

$$\Phi_0(t)=\phi(t)$$

and

$$\Psi_0(t)=\psi(t).$$

Further, let the function $g(x)$ be Lebesgue integrable in $(0,1)$, then for $\varepsilon>0$

$$g_\varepsilon^+(x)=\frac{1}{\Gamma(\varepsilon)}\int_0^x(x-u)^{\varepsilon-1}g(u)du$$

and

$$g_\varepsilon^-(x)=\frac{1}{\Gamma(\varepsilon)}\int_x^1(x-u)^{\varepsilon-1}g(u)du\,.$$

Again, let

$$U_n(t)=\sum_{\nu=1}^{n}e^{i\nu t},$$

$$\begin{aligned}H(n,x,t)&=E(n,x,t)+iF(n,x,t)\\&=\sum_{\nu=1}^{n}\nu^{\alpha-\beta}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}e^{i\nu t}\qquad(\alpha-\beta>0);\end{aligned}$$

$$\begin{aligned}J(n,x,t)&=E_1(n,x,t)+iF_1(n,x,t)\\&=\sum_{\nu=1}^{n}\nu^{\alpha-\beta+1}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}e^{i\nu t}\qquad(\alpha-\beta>0);\end{aligned}$$

$$G(n,t)=\int_0^1 d\chi(x)\sum_{\nu=1}^{n}\binom{n}{\nu}x^{\nu}(1-x)^{n-\nu}\nu^{\alpha-\beta+1}\cos\nu t$$

and

$$G_1(n,t)=\int_0^1 d\chi(t)\sum_{\nu=1}^{n}\binom{n}{\nu}x^{\nu}(1-x)^{n-\nu}\nu^{\alpha-\beta+1}\sin\nu t.$$

3. Concerning the absolute Hausdorff summability of a series associated with a Fourier series and its allied series, recently Tripathy [9] has proved the following theorems:

THEOREM A. *If*

(i) $$\int_0^{\pi} t^{-\alpha}|\alpha\phi(t)|<\infty \qquad (1>\alpha>0);$$

(ii) (H,μ_n) *is conservative*

and

(iii) $$\begin{cases} \textit{either (a) } \chi(x)=g^{-}_{1+\alpha+\delta}(x)+C & (\delta>0); \\ \textit{or (b) } \chi(x)=g^{+}_{1+\alpha+\delta}(x)+C & (\delta>0); \end{cases}$$

for some $g(x)\in L(0,1)$,

then the series $\sum_{n=1}^{\infty} n^{\alpha}A_n(t)$ *is summable* $|\mathrm{H},\mu_n|$ *at* $t=0$, *where* C *is an absolute constant.*

THEOREM B. *If*

(i) $$\psi(+0)=0,$$

(ii) $$\int_0^{\pi} t^{-\alpha}|d\psi(t)|<\infty \qquad (0<\alpha<1);$$

(iii) (H,μ_n) *is conservative*

and

(iv) $$\begin{cases} \textit{either (a) } \chi(x)=g^{-}_{1+\alpha+\delta}(x)+C & (\delta>0); \\ \textit{or (b) } \chi(x)=g^{+}_{1+\alpha+\delta}(x)+C & (\delta>0); \end{cases}$$

for some $g(x)\in L(0,1)$,

then the series $\sum_{n=1}^{\infty} n^{\alpha}B_n(t)$ *is summable* $|\mathrm{H},\mu_n|$, *at* $t=0$, *where* C *is an absolute constant.*

4. The object of this paper is to generalize the theorems A and B. In what follows, we shall prove the following theorems.

THEOREM I. *If*

(i) $$\Phi_\beta(+0)=0,$$

(ii) $$\int_0^\pi t^{-\alpha}|d\Phi_\beta(t)|<\infty,$$

(iii) $$(\mathrm{H}, \mu_n) \text{ is conservative}$$

and

(iv) $$\begin{cases} \textit{either (a) } \chi(x)=g^-_{1+\alpha+\delta}(x)+C & (\delta>0); \\ \textit{or (b) } \chi(x)=g^+_{1+\alpha+\delta}(x)+C & (\delta>0); \end{cases}$$

for some $g(x)\in L(0,1)$,

then the series $\sum_{n=1}^\infty n^{\alpha-\beta}A_n(t)$ *is summable* $|\mathrm{H}, \mu_n|$, *at* $t=\theta$, *where C is an absolute constant and* $1>\alpha>\beta\geqq 0$ *or also* $1>\alpha\geqq\beta>0$.

THEOREM II. *If*

(i) $$\Psi_\beta(+0)=0,$$

(ii) $$\int_0^\pi t^{-\alpha}|d\Psi_\beta(t)|<\infty,$$

(iii) $$(\mathrm{H}, \mu_n) \text{ is conservative}$$

and

(iv) $$\begin{cases} \textit{either (a) } \chi(x)=g^-_{1+\alpha+\delta}(x)+C & (\delta>0); \\ \textit{or (b) } \chi(x)=g^+_{1+\alpha+\delta}(x)+C & (\delta>0); \end{cases}$$

for some $g(x)\in L(0,1)$,

then the series $\sum_{n=1}^\infty n^{\alpha-\beta}B_n(t)$ *is summable* $|\mathrm{H}, \mu_n|$, *at* $t=\theta$, *where C is an absolute constant and* $1>\alpha>\beta\geqq 0$ *or also* $1>\alpha\geqq\beta>0$.

It is clear that the theorems A and B follow as special cases for $\beta=0$ of our theorems.

It may also be remarked that if

$$\chi(x)=1-(1-x)^\gamma, \qquad \gamma>0;$$

the method (H, μ_n) reduces to the well known Cesàro method of order γ. Further if we choose $\alpha+\delta$ such that $\gamma>\alpha+\delta$, $\alpha>0$, $\delta>0$, then it can be proved that $\chi(x)-1$ is the $(1+\alpha+\delta)$th backward integral of

$$-\frac{\Gamma(1+\gamma)}{\Gamma(\gamma-\alpha-\delta)}(1-x)^{\gamma-\alpha-\delta-1}$$

and for $\alpha>0$, $\delta>0$, $\alpha+\delta<1$, $\gamma>\alpha+\delta$, $\chi(x)$ is also the $(1+\alpha+\delta)$th forward integral of

$$\frac{\gamma}{\Gamma(1-\delta-\alpha)}\left\{x^{-(\alpha+\delta)}+(1-\gamma)\int_0^x(1-v)^{\gamma-2}(x-v)^{-(\alpha+\delta)}dv\right\},$$

so the method of (C,γ), for $\alpha>0$, $\delta>0$, $\gamma>\alpha+\delta$, satisfies the hypothesis of theorems I and II [9].

Thus the following theorems become the corollaries of our theorems I and II.

THEOREM C [6]. *If*

(i) $$\Phi_\beta(+0)=0,$$

(ii) $$\int_0^\pi t^{-\alpha}|d\Phi_\beta(t)|<\infty,$$

then the series $\sum_{n=1}^\infty n^{\alpha-\beta}A_n(t)$, *at the point* $t=\theta$ *is summable* $|\mathrm{C},\gamma|$, *where* $1>\gamma>\alpha\geqq\beta\geqq 0$.

THEOREM D [7]. *If*

(i) $$\Psi_\beta(+0)=0,$$

(ii) $$\int_0^\pi t^{-\alpha}|d\Psi_\beta(t)|<\infty,$$

then the series $\sum_{n=1}^\infty n^{\alpha-\beta}B_n(t)$, *at the point* $t=\theta$ *is summable* $|\mathrm{C},\gamma|$ *where* $1>\gamma>\alpha\geqq\beta>0$ *or also* $1>\gamma>\alpha>\beta\geqq 0$.

Further, if $\beta=0$, then the following theorems of Mohanty [8] also become the corollaries of our theorems.

THEOREM E. *If*

$$\int_0^\pi t^{-\alpha}|d\phi(t)|<\infty \qquad (1>\alpha>0);$$

then the series $\sum_{n=1}^\infty n^\alpha A_n(t)$ *is summable* $|\mathrm{C},\gamma|$, *for* $\gamma>\alpha$, *at the point* $t=\theta$.

THEOREM F. *If*

(i) $$\Psi(+0)=0,$$

and

(ii) $$\int_0^\pi t^{-\alpha}|d\psi(t)|<\infty \qquad (0<\alpha<1);$$

then the series $\sum_{n=1}^\infty n^\alpha B_n(t)$ *is summable* $|\mathrm{C},\gamma|$ *for* $\gamma>\alpha$, *at the point* $t=\theta$.

It is known that the conditions

$$\int_0^\pi t^{-\beta}|d\Phi_\beta(t)|<\infty \qquad \text{and} \qquad \varphi_\beta(+0)=0$$

are equivalent to the conditions, $\phi_\beta(t)$ is B.V. in $(0,\pi)$ and $\phi_\beta(+0)=0$. Hence the following theorems of Bosanquet [1] and Bosanquet and Hyslop [2] are the corollaries of our theorems for $\beta=\alpha$.

THEOREM G. *If $\phi_\beta(t)$ is* B.V. *in* $(0,\pi)$, *then the series $\sum_{n=1}^\infty A_n(t)$, at $t=\theta$, is summable $|C,\gamma|$, for $\gamma>\beta$.*

THEOREM H. *If $0<\beta<1$ and*

(i) $$\Psi_\beta(+0)=0,$$

(ii) $$\int_0^\pi t^{-\beta}|d\Psi_\beta(t)|<\infty,$$

then the series $\sum_{n=1}^\infty B_n(t)$, at $t=\theta$, is summable $|C,\gamma|$, for every $\gamma>\beta$.

5. For the proof of the theorems, we require the following lemmas.

LEMMA 1. *Uniformly*

$$|U_n(t)|\leqq\frac{K}{t}. \tag{5.1}$$

This can be easily proved.

LEMMA 2. *If t_n and u_n denote the Hausdorff means of the series $\sum_{n=1}^\infty a_n$ and sequence $\{na_n\}$ respectively, then for $n\geqq1$*

$$u_n=n(t_n-t_{n-1}). \tag{5.2}$$

This is known [4].

LEMMA 3. *If $g(x)$ and $h(x)$ are Lebesgue integrable in $(0,1)$, then for $\varepsilon>0$*

$$\int_0^1 g_\varepsilon^+(x)\,h(x)\,dx=\int_0^1 g(x)h_\varepsilon^-(x)dx. \tag{5.3}$$

This is due to Kuttner [5].

LEMMA 4. *For $\alpha-\beta>0$*

$$\int_0^x H(n,v,t)dv=O\left(\frac{n^{\alpha-\beta-1}}{t}\right), \tag{5.4}$$

$$\int_0^x J(n,v,t)\,dv=O\left(\frac{n^{\alpha-\beta}}{t}\right), \tag{5.5}$$

$$\int_0^x H(n, 1-v, t)dv = O\left(\frac{n^{\alpha-\beta-1}}{t}\right), \tag{5. 6}$$

and

$$\int_0^x J(n, 1-v, t)dv = O\left(\frac{n^{\alpha-\beta}}{t}\right), \tag{5. 7}$$

uniformly for x *in* $(0, 1)$.

The above estimates can be easily obtained from Tripathy [9] Lemma 4.

LEMMA 5. *If* $\alpha>0$, $\delta>0$ *and let* α, δ *be fixed, then for* $\alpha+\delta<1$

$$\int_0^x (x-u)^{\delta+\alpha-1} H(n, u, t)\, du = O\left(\frac{n^{-\beta-\delta}}{t^{\alpha+\delta}}\right), \tag{5. 8}$$

$$\int_0^x (x-u)^{\alpha+\delta-1} J(n, u, t)\, du = O\left(\frac{n^{1-\beta-\delta}}{t^{\alpha+\delta}}\right); \tag{5. 9}$$

$$\int_x^1 (x-u)^{\alpha+\delta-2} H(n, u, t)du = O\left(\frac{n^{-\beta-\delta}}{t^{\alpha+\delta}}\right); \tag{5. 10}$$

and

$$\int_x^1 (x-u)^{\alpha+\delta-1} J(n, u, t)\, du = O\left(\frac{n^{1-\beta-\delta}}{t^{\alpha+\delta}}\right) \tag{5. 11}$$

uniformly for $0 \leqq x \leqq 1$.

Proof of (5. 8). We have

$$\begin{aligned}
&\int_0^x (x-u)^{\alpha+\delta-1} H(n, u, t)du \\
&= \left(\int_0^{x-1/nt} + \int_{x-1/nt}^x\right)(x-u)^{\alpha+\delta-1} H(n, u, t)du \\
&= P_1 + P_2, \quad \text{say.}
\end{aligned}$$

By the aid of lemma 4 and by the second mean value theorem, we have

$$\begin{aligned}
P_1 &= \int_0^{x-1/nt} (x-u)^{\delta+\alpha-1} H(n, u, t)du \\
&= (nt)^{1-\alpha-\delta} \int_0^{x-1/nt} H(n, u, t)du \\
&= O(nt)^{1-\alpha-\delta} O\left(\frac{n^{\alpha-\beta-1}}{t}\right) = O\left(\frac{n^{-\beta-\delta}}{t^{\alpha+\delta}}\right).
\end{aligned}$$

Since

$$H(n, x, t) \leqq n^{\alpha-\beta} \sum_{\nu=1}^{n} \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu} = n^{\alpha-\beta}$$

then

$$\begin{aligned} P_2 &= \int_{x-1/nt}^{x} (x-u)^{\alpha+\delta-1} H(n, u, t) du \\ &= O(n^{\alpha-\beta}) \int_{x-1/nt}^{x} (x-u)^{\alpha+\delta-1} du \\ &= O(n^{\alpha-\beta})(nt)^{-\alpha-\delta} = O\left(\frac{n^{-\beta-\delta}}{t^{\alpha+\delta}}\right). \end{aligned}$$

Similarly the other estimates can be proved.

Proof of Theorem I. We shall prove this theorem for the case $\alpha > \beta$. In view of the lemma 2, the series $\sum_{n=1}^{\infty} n^{\alpha-\beta} A_n(t)$ is summable $|\mathrm{H}, \mu_n|$, if

$$I = \sum_{n=1}^{\infty} \frac{1}{n} \left| \sum_{\nu=1}^{n} \binom{n}{\nu} (\Delta^{n-\nu} \mu_\nu) \nu^{\alpha-\beta+1} A_\nu(t) \right| < \infty.$$

Since (H, μ_n) is conservative, we have

$$\begin{aligned} I &= \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^1 d\chi(x) \sum_{\nu=1}^{n} \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu} \nu^{\alpha-\beta+1} A_\nu(t) \right| \\ &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^1 d\chi(x) \sum_{\nu=1}^{n} \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu} \nu^{\alpha-\beta+1} \int_0^{\pi} \phi(t) \cos \nu t dt \right| \\ &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^{\pi} \phi(t) G(n, t) dt \right| \\ &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^{\pi} G(n, t) \left\{ \frac{1}{\Gamma(1-\beta)} \int_0^t (t-u)^{-\beta} d\Phi_\beta(u) \right\} dt \right| \\ &= \frac{2}{\pi \Gamma(1-\beta)} \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^{\pi} d\Phi_\beta(u) \int_u^{\pi} (t-u)^{-\beta} G(n, t) dt \right| \\ &= \frac{2}{\pi \Gamma(1-\beta)} \int_0^{\pi} |d\Phi_\beta(u)| \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_u^{\pi} (t-u)^{-\beta} G(n, t) dt \right|. \end{aligned}$$

To prove the theorem, we have to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| \int_u^{\pi} (t-u)^{-\beta} G(n, t) dt \right| = O(u^{-\alpha}).$$

Now

$$\sum_{n=1}^{\infty}\frac{1}{n}\left|\int_u^\pi (t-u)^{-\beta}G(n,t)dt\right|$$
$$=\sum_{n=1}^{\infty}\frac{1}{n}\left|\int_u^\pi (t-u)^{-\beta}\left\{\int_0^1 d\chi(x)\operatorname{Re}J(n,x,t)\right\}dt\right|$$
$$=\sum_{n\leqq 1/u}+\sum_{n>1/u}=M_1+M_2,\quad \text{say.}$$

Since

$$|J(n,x,t)|\leqq n^{\alpha-\beta+1}\sum_{\nu=1}^{n}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}=n^{\alpha-\beta+1},$$

we have

$$M_1=\sum_{n\leqq 1/u}\frac{1}{n}\left|\int_u^\pi (t-u)^{-\beta}\left\{\int_0^1 d\chi(x)\operatorname{Re}J(n,x,t)\right\}dt\right|$$
$$\leqq \sum_{n\leqq 1/u}\frac{1}{n}\left|\int_u^{n+1/n} (t-u)^{-\beta}\left\{\int_0^1 d\chi(x)\operatorname{Re}J(n,x,t)\right\}dt\right|$$
$$+\sum_{n\leqq 1/u}\frac{1}{n}\left|\int_{u+1/n}^{\pi} (t-u)^{-\beta}\left\{\frac{d}{dt}\int_1^0 d\chi(x)\operatorname{Im}H(n,x,t)\right\}dt\right|$$
$$=\sum_{n\leqq 1/u}\frac{1}{n}\left|\int_u^{u+1/n} (t-u)^{-\beta}O(n^{\alpha-\beta+1})\left\{\int_0^1 d\chi(x)\right\}dt\right|$$
$$+\sum_{n\leqq 1/u}\frac{1}{n}n^{\beta}\left|\int_0^1 d\chi(x)\operatorname{Im}H(n,x,t)\right|$$
$$=O\left(\sum_{n\leqq 1/u}n^{-1}n^{\alpha-\beta+1}n^{\beta-1}\left|\int_0^1 d\chi(x)\right|\right)+\sum_{n\leqq 1/u}n^{\beta-1}\cdot O(n^{\alpha-\beta})\left|\int_0^1 d\chi(x)\right|$$
$$=O\left(\sum_{n\leqq 1/u}n^{\alpha-1}\right)=O\left(\int_0^{1/u}y^{\alpha-1}dy\right)=O(u^{-\alpha}).$$

If (a) $\chi(x)=g^-_{\alpha+\delta+1}(x)+C$, then

$$M_2=\sum_{n>1/u}\frac{1}{n}\left|\int_u^\pi (t-u)^{-\beta}\left\{\int_0^1 d\chi(x)\operatorname{Re}J(n,x,t)\right\}dt\right|$$
$$\leqq \sum_{n>1/u}\frac{1}{n}\left|\int_u^{u+1/n} (t-u)^{-\beta}\left\{\int_0^1 d\chi(x)E_1(n,x,t)\right\}dt\right|$$
$$+\sum_{n>1/u}\frac{1}{n}\left|\int_{u+1/n}^{\pi} (t-u)^{-\beta}\left\{\frac{d}{dt}\int_0^1 d\chi(x)F(n,x,t)\right\}dt\right|$$
$$=M_{2.1}+M_{2.2},\quad \text{say.}$$

By using the lemma 3, we have

$$M_{2.1}=\sum_{n>1/u}\frac{1}{n}\left|\int_u^{u+1/n}(t-u)^{-\beta}\left\{\int_0^1 g^-_{\alpha+\delta}(x)E_1(n,x,t)dx\right\}dt\right|$$

$$=\sum_{n>1/u}\frac{1}{n}\left|\int_u^{u+1/n}(t-u)^{-\beta}\left\{\int_0^1 g(x)E^+_{1\alpha+\delta}(n,x,)dx\right\}dt\right|.$$

Since $E^+_{1\alpha+\delta}(n, x, t)$ is the $(\alpha+\delta)$th forward fractional integral of $E_1(n, x, t)$ regarded as function of x and

$$E^+_{1\alpha+\delta}(n,x,t)=\frac{1}{\Gamma(\alpha+\delta)}\int_0^x(x-u)^{\delta+\alpha-1}E_1(n,u,t)du$$

$$=\frac{1}{\Gamma(\alpha+\delta)}\int_0^x(x-u)^{\delta+\alpha-1}\operatorname{Re}J(n,u,t)du$$

$$=O\left(\frac{h^{1-\beta-\delta}}{t^{\alpha+\delta}}\right)\quad\text{by lemma 5.}$$

Hence

$$M_{2.1}=\sum_{n>1/u}\frac{1}{n}\left|\int_u^{u+1/n}(t-u)^{-\beta}O\left(\frac{n^{1-\beta-\delta}}{t^{\alpha+\delta}}\right)\left\{\int_0^1 g(x)dx\right\}dt\right|$$

$$=O\left(\frac{1}{u^{\alpha+\delta}}\right)\sum_{n>1/u}n^{-\beta-\delta}\left|\int_u^{u+1/n}(t-u)^{-\beta}dt\right|\quad\left(\text{since }\int_0^1 g(x)dx\text{ is finite}\right)$$

$$=O(u^{-\alpha-\delta})\sum_{n>1/u}n^{-\delta-1}$$

$$=O(n^{-\alpha-\delta})\int_{1/u}^\infty y^{-\delta-1}dy=O(u^{-\alpha}).$$

Using the second mean value theorem and lemmas 3 and 5, we have

$$M_{2.2}=\sum_{n>1/u}n^{-1}\left|\int_{u+1/n}^{\pi}(t-u)^{-\beta}\frac{d}{dt}\left\{\int_0^1 g^-_{\alpha+\delta}(x)F(n,x,t)dx\right\}dt\right|$$

$$=\sum_{n>1/u}n^{-1+\beta}\left|\int_{u+1/n}^{\pi}\frac{d}{dt}\left\{\int_0^1 g(x)F^+_{\alpha+\delta}(n,x,t)dx\right\}dt\right|$$

$$=\sum_{n>1/u}n^{-1+\beta}\left|\int_0^1 g(x)F^+_{\alpha+\delta}(n,x,u)dx\right|$$

$$=\sum_{n>1/u}n^{-1+\beta}O\left\{\frac{n^{-\beta-\delta}}{u^{\alpha+\delta}}\right\}\left|\int_0^1 g(x)dx\right|$$

$$=O(u^{-\alpha-\delta})\sum_{n>1/u}n^{-1-\delta}=O(u^{-\alpha-\delta})\int_{1/u}^\infty y^{-1-\delta}dy=O(u^{-\alpha}).$$

If (b) $\chi(x)=g^{+}_{1+\alpha+\delta}(x)+C$, then using the similar arguments as used in the case (a), we can prove with the help of (5.10) and (5.11) of lemma 5 that

$$M_2=O(u^{-\alpha}).$$

This completes the proof of the theorem I.

Proof of the theorem II. The series $\sum_{n=1}^{\infty} n^{\alpha-\beta}B_n(t)$ is summable $|\mathrm{H}, \mu_n|$, if

$$L=\sum_{n=1}^{\infty}\frac{1}{n}\left|\sum_{\nu=1}^{n}\binom{n}{\nu}\Delta_{\mu\nu}^{n-\nu}\nu^{\alpha-\beta+1}B_\nu(t)\right|<\infty.$$

since the method (H, μ_n) is conservative, then

$$L=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left|\int_0^1 d\chi(x)\sum_{\nu=1}^{n}\binom{n}{\nu}\Delta_{\mu\nu}^{n-\nu}\nu^{\alpha-\beta+1}\int_0^\pi \phi(t)\sin\nu t dt\right|$$

$$=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left|\int_0^\pi \phi(t)G_1(n, t)dt\right|$$

$$=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left|\int_0^\pi G_1(n, t)\left\{\frac{1}{\Gamma(1-\beta)}\int_0^t (t-u)^{-\beta}d\Psi_\beta(u)\right\}dt\right|$$

$$=\frac{2}{\pi\Gamma(1-\beta)}\sum_{n=1}^{\infty}\frac{1}{n}\left|\int_0^\pi d\Psi_\beta(u)\int_u^\pi (t-u)^{-\beta}G_1(n, t)dt\right|$$

$$=\frac{2}{\pi\Gamma(1-\beta)}\left|\int_0^\pi |d\Psi_\beta(u)|\sum_{n=1}^{\infty}\frac{1}{n}\int_u^\pi (t-u)^{-\beta}G_1(n, t)\,dt\right|.$$

To prove the theorem, it is sufficient to prove that

$$N=\sum_{n=1}^{\infty}\frac{1}{n}\left|\int_u^\pi (t-u)^{-\beta}G_1(n, t)\,dt\right|=O(u^{-\alpha}).$$

Now

$$N=\sum_{n=1}^{\infty}\frac{1}{n}\left|\int_u^\pi (t-u)^{-\beta}\left\{\int_0^1 d\chi(x)\,\mathrm{Im}\,J(n, x, t)\right\}dt\right|$$

$$=\sum_{n\leq 1/u}+\sum_{n>1/u}=N_1+N_2,\quad \text{say.}$$

Proceeding in a similar way as in the proof of M_1 and M_2, we can prove that

$$N_1=O(u^{-\alpha})$$

and

$$N_2=O(u^{-\alpha}).$$

Hence

$$L=\frac{2}{\pi\Gamma(1-\beta)}\int_0^\pi |d\Psi_\beta(u)|O(u^{-\alpha})$$

$$=O(1)\int_0^\pi u^{-\alpha}|d\Psi_\beta(u)|=O(1).$$

This completes the proof of the theorem II.

The author is highly grateful to Dr. P. L. Sharma for his kind advice and suggestions during the preparation of this paper.

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