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ON (f, g, u, v, λ)-STRUCTURES

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§ 0. Introduction.

Tashiro [10] has shown that hypersurfaces of an almost complex manifold carry almost contact structures. In particular, an odd-dimensional hypersphere in an even-dimensional Euclidean space carries an almost contact structure.

Blair, Ludden and one of the present authors [3] (see also, Ako [1], Blair and Ludden [2], Goldberg and Yano [4, 5], Okumura [7], Yano and Ishihara [13]) have studied submanifolds of codimension 2 of almost complex manifolds. These submanifolds admit, under certain conditions, what we call an (f, U, V, u, v, λ)-structure and, if the ambient space is an almost Hermitian manifold, the submanifolds admit what we call an (f, g, u, v, λ)-structure. In particular, an even-dimensional sphere of codimension 2 of an even-dimensional Euclidean space carries an (f, g, u, v, λ)-structure.

They also studied hypersurfaces of almost contact manifolds and found that the hypersurfaces also admit the same kind of structure (see also Okumura [8], Watanabe [11], Yamaguchi [12]).

The main purpose of the present paper is to study the (f, g, u, v, λ)-structure and to give characterizations of even-dimensional spheres.

In § 1, we define and discuss (f, U, V, u, v, λ)-structure and (f, g, u, v, λ)-structure.

In § 2, we prove that a totally umbilical submanifold of codimension 2 of a Kählerian manifold whose connection induced in the normal bundle is flat admits a normal (f, g, u, v, λ)-structure and that the vector fields U and V define infinitesimal conformal transformations of the submanifold.

In § 3, we prove that a hypersurface of a Sasakian manifold for which the tensor f and the second fundamental tensor h commute admits a normal (f, g, u, v, λ)-structure and that if the hypersurface is totally umbilical, then the vectors U and V define infinitesimal conformal transformations.

§ 4 is devoted to prove some identities valid in M with normal (f, g, u, v, λ)-structure for later use.

In § 5, we prove that if a manifold M with normal (f, g, u, v, λ)-structure satisfies $du=\phi f$ and $dv=f$ and if $\lambda(1-\lambda^2)$ is an almost everywhere non-zero function, then the vector fields U and V define infinitesimal conformal transformations.

In § 6, we prove a formula which gives the covariant derivative of f.

The last § 7 is devoted to prove two theorems which characterize even-dimensional spheres.

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§1. (f, U, V, u, v, λ)-structure.

Let M be an m-dimensional differentiable manifold of class C^∞. We assume that there exist on M a tensor field of type (1, 1), vector fields U and V, 1-forms u and v, and a function λ satisfying the conditions:

$$f^2X=-X+u(X)U+v(X)V \tag{1.1}$$

for any vector field X,

$$u\circ f=\lambda v, \qquad fU=-\lambda V, \tag{1.2}$$

$$v\circ f=-\lambda u, \qquad fV=\lambda U, \tag{1.3}$$

where 1-forms $u\circ f$ and $v\circ f$ are respectively defined by

$$(u\circ f)(X)=u(fX), \qquad (v\circ f)(X)=v(fX)$$

for any vector field X, and

$$u(U)=1-\lambda^2, \qquad u(V)=0, \tag{1.4}$$

$$v(U)=0, \qquad v(V)=1-\lambda^2. \tag{1.5}$$

In this case, we say that the manifold M has an (f, U, V, u, v, λ)-structure. Examples of manifolds with (f, U, V, u, v, λ)-structure will be given in §§2 and 3.

First of all, we prove

THEOREM 1.1. *A differentiable manifold with (f, U, V, u, v, λ)-structure is of even dimension.*

Proof. Let P be a point of M at which $\lambda^2\neq 1$. Then, from (1.4) and (1.5), we see that

$$U\neq 0, \qquad V\neq 0$$

at P. The vectors U and V are linearly independent. For, if there are two numbers a and b such that

$$aU+bV=0,$$

then evaluating u and v at $aU+bV$ and using (1.4) and (1.5), we obtain

$$u(aU+bV)=au(U)=a(1-\lambda^2)=0,$$

and

$$v(aU+bV)=bv(V)=b(1-\lambda^2)=0.$$

Thus we have $a=b=0$.

Thus U and V being linearly independent at P, we can choose m linearly independent vectors $X_1=U$, $X_2=V$, X_3, $\cdots$, X_m which span the tangent space $T_P(M)$

of M at P and such that $u(X_\alpha)=0$, $v(X_\alpha)=0$, for $\alpha=3, \cdots, m$. Consequently, we have from (1. 1),

$$f^2X_\alpha=-X_\alpha, \qquad \alpha=3, 4, \cdots, m,$$

which shows that f is an almost complex structure in the subspace V_P of $T_P(M)$ at P spanned by $X_3, \cdots, X_m$ and that V_P is even dimensional. Thus $T_P(M)$ is also even dimensional.

Next, let P be a point of M at which $\lambda^2=1$. In this case, we see, from (1. 4) and (1. 5), that

$$u(U)=0, \qquad u(V)=0,$$

$$v(U)=0, \qquad v(V)=0.$$

We also see, from (1. 2) and (1. 3), that

$$\text{if} \quad u\neq 0, \quad \text{then} \quad v\neq 0,$$

$$\text{if} \quad u=0, \quad \text{then} \quad v=0.$$

We first consider the case in which $u\neq 0$, $v\neq 0$. In this case, u and v are linearly independent. Because, if there are two numbers a and b such that

$$au+bv=0,$$

then, from (1. 2), (1. 3) and

$$(au+bv)\circ f=0,$$

we have

$$\lambda(bu-av)=0,$$

from which

$$bu-av=0,$$

λ being different from zero. Thus from $au+bv=0$ and $bu-av=0$ we have

$$(a^2+b^2)u=0,$$

from which $a=0$, $b=0$.

Thus, u and v being linearly independent at P, we can choose n linearly independent covectors $w_1=u$, $w_2=v$, $w_3, \cdots, w_m$ which span the cotangent space ${}^cT_P(M)$ of M at P. We denote the dual basis by $(X_1, X_2, \cdots, X_{m-1}, X_m)$.

If U and V are linearly independent at P, we can assume that

$$X_{m-1}=U, \qquad X_m=V.$$

Then we have

$$f^2X_\alpha=-X_\alpha+u(X_\alpha)U+v(X_\alpha)V=-X_\alpha, \qquad \alpha=3, 4, \cdots, m$$

which shows that f is an almost complex structure in the subspace V_P of $T_P(M)$ at P spanned by $X_3, \cdots, X_m$ and that V_P is even-dimensional and consequently $T_P(M)$ is also even- dimensional.

If U and V are linearly dependent, there exist two numbers a and b such that

$$aU+bV=0$$

and $a^2+b^2\neq 0$. Applying f to the equation above and using (1. 2) and (1. 3), we find

$$\lambda(-aV+bU)=0,$$

from which

$$bU-aV=0.$$

Thus, we must have

$$U=V=0.$$

Thus, from (1. 1), we have

$$f^2X=-X$$

for any vector X in $T_P(M)$. Thus $T_P(M)$ is even dimensional.

The case left to examine is the case in which $u=0, v=0$. But in this case also we have, from (1. 1), $f^2X=-X$ for any vector X in $T_P(M)$ and consequently $T_P(M)$ is even dimensional. Thus we have completed the proof of Theorem 1. 1.

DEFINITION. The structure (f, U, V, u, v, λ) is said to be *normal* if the Nijenhuis tensor N of f satisfies

$$S(X, Y)\equiv N(X, Y)+du(X, Y)U+dv(X, Y)V=0 \tag{1. 6}$$

for any vector field X and Y of M.

We consider a product manifold $M\times R^2$, where R^2 is a 2-dimensional Euclidean space. Then, (f, U, V, u, v, λ)-structure gives rise to an almost complex structure J on $M\times R^2$:

$$(J)=\begin{pmatrix} f & U & V \\ -u & 0 & -\lambda \\ -v & \lambda & 0 \end{pmatrix} \tag{1. 7}$$

as we can easily check using (1. 1)$\sim$(1. 5).

Computing the Nijenhuis tensor of J, we can easily prove

PROPOSITION 1. 2. *If J is integrable, then (f, U, V, u, v, λ)-structure is normal.*

We assume that, in M with (f, U, V, u, v, λ)-structure, there exists a positive definite Riemannian metric g such that

$$g(U, X)=u(X), \tag{1.8}$$

$$g(V, X)=v(X), \tag{1.9}$$

and

$$g(fX, fY)=g(X, Y)-u(X)u(Y)-v(X)v(Y) \tag{1.10}$$

for any vector fields X, Y of M. We call such a structure a metric (f, U, V, u, v, λ)-structure and denote it sometimes by (f, g, u, v, λ).

We prove

PROPOSITION 1. 3. *Let ω be a tensor field of type* (0. 2) *of M defined by*

$$\omega(X, Y)=g(fX, Y) \tag{1.11}$$

for any vector fields X and Y of M, then we have

$$\omega(X, Y)=-\omega(Y, X), \tag{1.2}$$

that is, ω is a 2-form.

Proof. From the definition (1. 11) of ω, we have

$$\omega(fX, fY)=g(f(fX), fY),$$

from which, using (1. 10),

$$\omega(fX, fY)=g(fX, Y)-u(fX)u(Y)-v(fX)v(Y),$$

or

$$\omega(fX, fY)=\omega(X, Y)-\lambda v(X)u(Y)+\lambda u(X)v(Y),$$

by virtue of (1. 2) and (1. 3).

On the other hand, using (1. 1), we have

$$\begin{aligned}\omega(fX, fY)&=g(f^2X, fY)\\&=g(-X+u(X)U+v(X)V, fY)\\&=-g(X, fY)+u(X)u(fY)+v(X)v(fY),\end{aligned}$$

by virtue of (1. 8) and (1. 9) and consequently

$$\omega(fX, fY)=-\omega(Y, X)+\lambda u(X)v(Y)-\lambda v(X)u(Y).$$

Thus we have

$$\omega(X, Y)=-\omega(Y, X).$$

§ 2. Submanifolds of codimension 2 of an almost Hermitian manifold.

In this section, we study submanifolds of codimension 2 of an almost Hermitian manifold as examples of the manifold with (f, g, u, v, λ)-structure.

Let $\tilde{M}$ be a $(2n+2)$-dimensional almost Hermitian manifold covered by a system of coordinate neighborhoods $\{\tilde{U}; y^\kappa\}$, where here and in this section the indices $\kappa, \lambda, \mu, \nu, \cdots$ run over the range $\{1, 2, \cdots, 2n+2\}$, and let $(F_\lambda{}^\kappa, G_{\mu\lambda})$ be the almost Hermitian structure, that is, let $F_\lambda{}^\kappa$ be the almost complex structure:

$$F_\alpha{}^\kappa F_\lambda{}^\alpha = -\delta_\lambda^\kappa, \tag{2. 1}$$

and $G_{\mu\lambda}$ a Riemannian metric such that

$$G_{\gamma\beta} F_\mu{}^\gamma F_\lambda{}^\beta = G_{\mu\lambda}. \tag{2. 2}$$

We denote by $\{{}_\mu{}^\kappa{}_\lambda\}$ the Christoffel symbols formed with $G_{\mu\lambda}$.

Let M be a $2n$-dimensional differentiable manifold which is covered by a system of coordinate neighborhoods $\{U; x^h\}$, where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2n\}$ and which is differentiably immersed in $\tilde{M}$ as a submanifold of codimension 2 by the equations

$$y^\kappa = y^\kappa(x^h). \tag{2. 3}$$

We put

$$B_i{}^\kappa = \partial_i y^\kappa, \qquad (\partial_i = \partial/\partial x^i) \tag{2. 4}$$

then $B_i{}^\kappa$ is, for each fixed i, a local vector field of $\tilde{M}$ tangent to M and vectors $B_i{}^\kappa$ are linearly independent in each coordinate neighborhood. $B_i{}^\kappa$ is, for each fixed κ, a local 1-form of M.

We choose two mutually orthogonal unit vectors C^κ and D^κ of $\tilde{M}$ normal to M in such a way that $2n+2$ vectors $B_i{}^\kappa, C^\kappa, D^\kappa$ give the positive orientation of M.

The transforms $F_\lambda{}^\kappa B_i{}^\lambda$ of $B_i{}^\lambda$ by $F_\lambda{}^\kappa$ can be expressed as linear combinations of $B_i{}^\kappa, C^\kappa$ and D^κ, that is,

$$F_\lambda{}^\kappa B_i{}^\lambda = f_i{}^h B_h{}^\kappa + u_i C^\kappa + v_i D^\kappa, \tag{2. 5}$$

where $f_i{}^h$ is a tensor field of type (1. 1) and u_i, v_i are 1-forms of M. Similarly the transform $F_\lambda{}^\kappa C^\lambda$ of C^λ by $F_\lambda{}^\kappa$ and the transform $F_\lambda{}^\kappa D^\lambda$ by $F_\lambda{}^\kappa$ can be written as

$$\begin{aligned} F_\lambda{}^\kappa C^\lambda &= -u^i B_i{}^\kappa + \lambda D^\kappa, \\ F_\lambda{}^\kappa D^\lambda &= -v^i B_i{}^\kappa - \lambda C^\kappa, \end{aligned} \tag{2. 6}$$

where

$$u^i = u_t g^{ti}, \qquad v^i = v_t g^{ti},$$

g_{ji} being the Riemannian metric on M induced from that of $\tilde{M}$.

$$g_{ji}=G_{\mu\lambda}B_j{}^\mu B_i{}^\lambda,$$

and λ is a function on M. The function λ seems to depend on the choice of normals C^κ and D^κ, but we can easily verify that λ is independent of the choise of normals and consequently that λ is a function globally defined on M.

Applying $F_\kappa{}^\mu$ again to (2. 5) and taking account of (2. 5) itself and (2. 6), we find

(2. 7) $$f_j{}^h f_i{}^j=-\delta_i^h+u_i u^h+v_i v^h$$

(2. 8) $$u_h f_i{}^h=\lambda v_i, \qquad v_h f_i{}^h=-\lambda u_i.$$

Applying $F_\kappa{}^\mu$ again to (2. 6) and taking account of (2. 5) and (2. 6) itself, we find

(2. 9) $$f_i{}^h u^i=-\lambda v^h, \qquad u_i u^i=1-\lambda^2, \qquad u_i v^i=0,$$

(2. 10) $$f_i{}^h v^i=\lambda u^h, \qquad v_i u^i=0, \qquad v_i v^i=1-\lambda^2.$$

On the other hand, we have, from (2. 2),

$$G_{\gamma\beta}F_\mu{}^\gamma F_\lambda{}^\beta B_j{}^\mu B_i{}^\lambda=G_{\mu\lambda}B_j{}^\mu B_i{}^\lambda,$$

from which

$$g_{kh}f_j{}^k f_i{}^h+u_j u_i+v_j v_i=g_{ji},$$

or

(2. 11) $$g_{kh}f_j{}^k f_i{}^h=g_{ji}-u_j u_i-v_j v_i.$$

Equations (2. 7), (2. 8), (2. 9), (2. 10) and (2. 11) show that a submanifold of codimension 2 of an almost Hermitian manifold admits a (f, g, u, v, λ)-structure.

We denote by $\{_j{}^h{}_i\}$ and ∇_i the Christoffel symbols formed with g_{ji} and the operator of covariant differentiation with respect to $\{_j{}^h{}_i\}$ respectively.

The so-called van der Waerden-Bortolotti covariant derivative of $B_i{}^\kappa$ is given by

(2. 11) $$\nabla_j B_i{}^\kappa=\partial_j B_i{}^\kappa+\{_\mu{}^\kappa{}_\lambda\}B_j{}^\mu B_i{}^\lambda-B_h{}^\kappa\{_j{}^h{}_i\}$$

and is orthogonal to M and consequently can be written as

(2. 12) $$\nabla_j B_i{}^\kappa=h_{ji}C^\kappa+k_{ji}D^\kappa,$$

which are equations of Gauss, where h_{ji} and k_{ji} are the second fundamental tensors of M with respect to the normals C^κ and D^κ respectively.

For the covariant derivatives of C^κ and D^κ along M, we have equations of Weingarten

(2. 13) $$\nabla_j C^\kappa=-h_j{}^i B_i{}^\kappa+l_j D^\kappa, \qquad \nabla_j D^\kappa=-k_j{}^i B_i{}^\kappa-l_j C^\kappa,$$

where

$$\nabla_j C^\kappa=\partial_j C^\kappa+\{_\mu{}^\kappa{}_\lambda\}B_j{}^\mu C^\lambda, \qquad \nabla_j D^\kappa=\partial_j D^\kappa+\{_\mu{}^\kappa{}_\lambda\}B_j{}^\mu D^\lambda,$$

$$h_j{}^i = h_{js}g^{si}, \qquad k_j{}^i = k_{js}g^{si}$$

and l_j is the so-called third fundamental tensor.

As we see from (2. 13), equations

$$\begin{aligned} {}'\nabla_j C^\kappa &= l_j D^\kappa, \\ {}'\nabla_j D^\kappa &= -l_j C^\kappa \end{aligned} \tag{2. 14}$$

define the connexion induced in the normal bundle. If this induced connexion is flat, then we can choose C^κ and D^κ in such a way that we have $l_j=0$.

Differentiating (2. 5) covariantly along M, we have, taking account of equations of Gauss and those of Weingarten,

$$\begin{aligned} &(\nabla_\mu F_\lambda{}^\kappa)B_j{}^\mu B_i{}^\lambda + F_\lambda{}^\kappa(h_{ji}C^\lambda + k_{ji}D^\lambda) \\ &= (\nabla_j f_i{}^h)B_h{}^\kappa + f_i{}^t(h_{jt}C^\kappa + k_{jt}D^\kappa) \\ &\quad + (\nabla_j u_i)C^\kappa + u_i(-h_j{}^h B_h{}^\kappa + l_j D^\kappa) \\ &\quad + (\nabla_j v_i)D^\kappa + v_i(-k_j{}^h B_h{}^\kappa - l_j C^\kappa), \end{aligned}$$

or

$$\begin{aligned} &(\nabla_\mu F_\lambda{}^\kappa)B_j{}^\mu B_i{}^\lambda - (h_{ji}u^h + k_{ji}v^h)B_h{}^\kappa - \lambda k_{ji}C^\kappa + \lambda h_{ji}D^\kappa \\ &= (\nabla_j f_i{}^h - h_j{}^h u_i - k_j{}^h v_i)B_h{}^\kappa \\ &\quad + (\nabla_j u_i + h_{jt}f_i{}^t - l_j v_i)C^\kappa \\ &\quad + (\nabla_j v_i + k_{jt}f_i{}^t + l_j u_i)D^\kappa. \end{aligned}$$

Thus, if $\tilde{M}$ is a Kählerian manifold, that is, if $\nabla_\mu F_\lambda{}^\kappa=0$, then we have

$$\nabla_j f_i{}^h = -h_{ji}u^h + h_j{}^h u_i - k_{ji}v^h + k_j{}^h v_i, \tag{2. 15}$$

$$\nabla_j u_i = -h_{jt}f_i{}^t - \lambda k_{ji} + l_j v_i, \tag{2. 16}$$

$$\nabla_j v_i = -k_{jt}f_i{}^t + \lambda h_{ji} - l_j u_i. \tag{2. 17}$$

Using (2. 15), (2. 16) and (2. 17) to compute

$$S_{ji}{}^h = N_{ji}{}^h + (\nabla_j u_i - \nabla_i u_j)u^h + (\nabla_j v_i - \nabla_i v_j)v^h,$$

we find

$$\begin{aligned} S_{ji}{}^h &= (f_j{}^r h_r{}^h - h_j{}^r f_r{}^h)u_i - (f_i{}^r h_r{}^h - h_i{}^r f_r{}^h)u_j \\ &\quad + (f_j{}^r k_r{}^h - k_j{}^r f_r{}^h)v_i - (f_i{}^r k_r{}^h - k_i{}^r f_r{}^h)v_j \\ &\quad + u^h(l_j v_i - l_i v_j) - v^h(l_j u_i - l_i u_j). \end{aligned}$$

Thus we have

PROPOSITION 3. 1. *Let M be a submanifold of codimension 2 of a Kählerian manifold whose connection induced in the normal bundle is flat. If f commutes with both of h and k, M admits a normal (f, g, u, v, λ)-structure.*

COROLLARY 3. 2 *A totally umbilical submanifold of codimension 2 of a Kählerian manifold whose connection induced in the normal bundle is flat admits a normal (f, g, u, v, λ)-structure.*

Corollary 3. 2. holds of course for a totally geodesic submanifold. A plane or a sphere of codimension 2 in an even-dimensional Euclidean space are examples for which the corollary holds.

For a totally umbilical submanifold whose connection induced in the normal bundle is flat, we have, for suitably chosen unit normals C and D,

$$h_{ji}=hg_{ji}, \qquad k_{ji}=kg_{ji}, \qquad l_j=0$$

and consequently (2. 16) and (2. 17) become

$$\nabla_j u_i=hf_{ji}-\lambda kg_{ji}, \tag{2. 18}$$

and

$$\nabla_j v_i=kf_{ji}+\lambda hg_{ji} \tag{2. 19}$$

respectively. These equations give

$$\nabla_j u_i+\nabla_i u_j=-2\lambda kg_{ji} \tag{2. 20}$$

and

$$\nabla_j v_i+\nabla_i v_j=2\lambda hg_{ji} \tag{2. 21}$$

which show that u^h and v^h define infinitesimal conformal transformations in M.

§ 3. Hypersurfaces of an almost contact metric manifold.

In this section, we study hypersurfaces of an almost contact metric manifold as examples of the manifold with (f, g, u, v, λ)-structure.

Let $\tilde{M}$ be a $(2n+1)$-dimensional almost contact metric manifold covered by a system of coordinate neighborhoods $\{\tilde{U}; y^\kappa\}$, where here and in this section, the indices $\kappa, \lambda, \mu, \nu, \cdots$ run over the range $\{1, 2, \cdots, 2n+1\}$ and let $(F_\lambda{}^\kappa, G_{\mu\lambda}, v_\lambda)$ be the almost contact metric structure, that is [9],

$$F_\mu{}^\kappa F_\lambda{}^\mu=-\delta_\lambda^\kappa+v_\lambda v^\kappa, \tag{3. 1}$$

$$v_\kappa F_\lambda{}^\kappa=0, \qquad F_\lambda{}^\kappa v^\lambda=0, \tag{3. 2}$$

$$v_\lambda v^\lambda=1 \tag{3. 3}$$

and

$$G_{\gamma\beta}F_\mu{}^\gamma F_\lambda{}^\beta = G_{\mu\lambda} - v_\mu v_\lambda. \tag{3.4}$$

Let M be a $2n$-dimensional differentiable manifold which is covered by a system of coordinate neighborhoods $\{U;\ x^h\}$, and which is differentiably immersed in $\tilde{M}$ as a hypersurface by the equations

$$y^\kappa = y^\kappa(x^h). \tag{3.5}$$

We put $B_i{}^\kappa = \partial_i y^\kappa$ and choose a unit vector C^κ of $\tilde{M}$ normal to M in such a way that $2n+1$ vectors $B_i{}^\kappa$ and C^κ give the positive orientation of M.

The transforms $F_\lambda{}^\kappa B_i{}^\lambda$ of $B_i{}^\lambda$ by $F_\lambda{}^\kappa$ can be expressed as linear combinations of $B_i{}^\kappa$ and C^κ, that is

$$F_\lambda{}^\kappa B_i{}^\lambda = f_i{}^h B_h{}^\kappa + u_i C^\kappa, \tag{3.6}$$

where $f_i{}^h$ is a tensor field of type (1, 1) and u_i is a 1-form of M. Similarly, the transform $F_\lambda{}^\kappa C^\lambda$ of C^λ by $F_\lambda{}^\kappa$ can be written as

$$F_\lambda{}^\kappa C^\lambda = -u^i B_i{}^\kappa, \tag{3.7}$$

where

$$u^i = u_j g^{ji},$$

g_{ji} being the Riemannian metric on M induced from that of $\tilde{M}$.

We put

$$v^\kappa = B_i{}^\kappa v^i + \lambda C^\kappa, \tag{3.8}$$

where v^i is a vector field of M and λ a function of M.

Applying $F_\kappa{}^\mu$ again to (3.6) and taking account of (3.6) itself, (3.7) and (3.8), we find

$$f_i{}^t f_t{}^h = -\delta_i^h + u_i u^h + v_i v^h, \tag{3.9}$$

$$u_t f_i{}^t = \lambda v_i. \tag{3.10}$$

Applying $F_\kappa{}^\mu$ again to (3.7) and taking account of (3.6), (3.7) and (3.8), we obtain

$$f_i{}^h u^i = -\lambda v^h, \tag{3.11}$$

$$u_i u^i = 1 - \lambda^2. \tag{3.12}$$

Finally applying $F_\kappa{}^\mu$ to (3.8), we find

$$f_i{}^h v^i = \lambda u^h, \tag{3.13}$$

$$u_i v^i = 0. \tag{3.14}$$

Since u^κ is a unit vector, we have, from (3. 8),

$$v_i v^i = 1 - \lambda^2. \tag{3. 15}$$

On the other hand, we have, from (3. 4)

$$G_{\gamma\beta} F_\mu{}^\gamma F_\lambda{}^\beta B_j{}^\mu B_i{}^\lambda = G_{\mu\lambda} B_j{}^\mu B_i{}^\lambda - u_\mu B_j{}^\mu u_\lambda B_i{}^\lambda,$$

from which

$$g_{kh} f_j{}^k f_i{}^h + u_j u_i = g_{ji} - v_j v_i,$$

that is

$$g_{kh} f_j{}^k f_i{}^h = g_{ji} - u_j u_i - v_j v_i. \tag{3. 16}$$

Equations (3. 9)~(3. 16) show that a hypersurface of an almost contact metric manifold admits a (f, g, u, v, λ)-structure.

For the hypersurface M, the equations of Gauss and those of Weingarten are

$$\nabla_j B_i{}^\kappa = h_{ji} C^\kappa, \tag{3. 17}$$

and

$$\nabla_j C^\kappa = -h_j{}^i B_i{}^\kappa \tag{3. 18}$$

respectively.

Differentiating (3. 6) covariantly along M, we have, taking account of (3. 17) and (3. 18),

$$(\nabla_\mu F_\lambda{}^\kappa) B_j{}^\mu B_i{}^\lambda + F_\lambda{}^\kappa h_{ji} C^\lambda$$
$$= (\nabla_j f_i{}^h) B_h{}^\kappa + f_i{}^t h_{jt} C^\kappa + (\nabla_j u_i) C^\kappa - u_i h_j{}^h B_h{}^\kappa$$

or

$$(\nabla_\mu F_\lambda{}^\kappa) B_j{}^\mu B_i{}^\lambda - h_{ji} u^h B_h{}^\kappa$$
$$= (\nabla_j f_i{}^h - h_j{}^h u_i) B_h{}^\kappa + (\nabla_j u_i + h_{jt} f_i{}^t) C^\kappa.$$

Thus, if $\tilde{M}$ is a Sasakian manifold, that is, if

$$\nabla_\mu F_\lambda{}^\kappa = -g_{\mu\lambda} v^\kappa + \delta_\mu^\kappa v_\lambda,$$

then we have

$$-g_{ji}(B_h{}^\kappa v^h + \lambda C^\kappa) + B_j{}^\kappa v_i - h_{ji} u^h B_h{}^\kappa$$
$$= (\nabla_j f_i{}^h - h_j{}^h u_i) B_h{}^\kappa + (\nabla_j u_i + h_{jt} f_i{}^t) C^\kappa,$$

from which

$$\nabla_j f_i{}^h = -h_{ji} u^h + h_j{}^h u_i - g_{ji} v^h + \delta_j^h v_i, \tag{3. 19}$$

$$\nabla_j u_i = -h_{jt} f_i{}^t - \lambda g_{ji}. \tag{3.20}$$

On the other hand, differentiating (3. 8) covariantly along M and taking account of (3. 17), (3. 18), and

$$\nabla_\lambda v^\kappa = F_\lambda{}^\kappa,$$

we find

$$F_\lambda{}^\kappa B_j{}^\lambda = h_{ji} v^i C^\kappa + B_i{}^\kappa \nabla_j v^i + (\nabla_j \lambda) C^\kappa + \lambda(-h_j{}^h B_h{}^\kappa),$$

or

$$f_j{}^h B_h{}^\kappa + u_j C^\kappa = (\nabla_j v^h - \lambda h_j{}^h) B_h{}^\kappa + (\nabla_j \lambda + h_{ji} v^i) C^\kappa,$$

from which

$$\nabla_j v^h = f_j{}^h + \lambda h_j{}^h, \tag{3.21}$$

or

$$\nabla_j v_i = f_{ji} + \lambda h_{ji} \tag{3.22}$$

and

$$\nabla_j \lambda = u_j - h_{ji} v^i. \tag{3.23}$$

Thus, computing $S_{ji}{}^h$ we obtain

$$S_{ji}{}^h = (f_j{}^t h_t{}^h - h_j{}^t f_t{}^h) u_i - (f_i{}^t h_t{}^h - h_i{}^t f_t{}^h) u_j. \tag{3.24}$$

Now we prove

PROPOSITION 4. 1. *In order that the induced (f, g, u, v, λ)-structure on a hypersurface of a Sasakian manifold be normal it is necessary and sufficient that f commutes with h.*

Proof. The sufficiency of the condition is trivially seen from (3. 24). So we prove the necessity of the condition.

Suppose that the (f, g, u, v, λ)-structure be normal, then we have, from $S_{ji}{}^h=0$,

$$(f_j{}^t h_t{}^h - h_j{}^t f_t{}^h) u_i = (f_i{}^t h_t{}^h - h_i{}^t f_t{}^h) u_j. \tag{3.25}$$

Thus, for some vector field w^h, we have

$$f_j{}^t h_t{}^h - h_j{}^t f_t{}^h = w^h u_j. \tag{3.26}$$

Since the covariant components of the tensor defined by the left hand members of the above equation are symmetric, it follows that w is proportional to u, that is,

$$f_j{}^t h_{th} + f_h{}^t h_{tj} = \alpha u_j u_h,$$

α being a function, from which, by transvection of g^{jh}, $\alpha=0$ or $u_j=0$. This, together with (3. 26), shows that f commutes with h.

It is known [12] that if f commutes with h and $\lambda^2 \neq 1$ almost everywhere, the hypersurface is totally umbilical. So we get

PROPOSITION 4. 2. *If the (f, g, u, v, λ)-structure induced on a hypersurface of a Sasakian manifold is normal, the hypersurface is totally umbilical.*

For a hypersurface with the induced normal (f, g, u, v, λ)-structure, we have from (3. 20),

$$\nabla_j u_i + \nabla_i u_j = -2\lambda g_{ji} \tag{3. 27}$$

and from (3. 22)

$$\nabla_j v_i + \nabla_i v_j = 2\lambda h g_{ji}, \tag{3. 28}$$

which show that u^h and v^h define infinitesimal conformal transformations in M.

§4. Identities in manifolds with normal (f, g, u, v, λ)-structure.

In this section we shall prove some identities in manifolds with normal (f, g, u, v, λ)-structure for later use.

Let M be a manifold with normal (f, g, u, v, λ)-structure. The structure being normal, we have

$$\begin{aligned} &f_j{}^t \nabla_t f_i{}^h - f_i{}^t \nabla_t f_j{}^h - (\nabla_j f_i{}^t - \nabla_i f_j{}^t) f_t{}^h \\ &\qquad + (\nabla_j u_i - \nabla_i u_j) u^h + (\nabla_j v_i - \nabla_i v_j) v^h = 0. \end{aligned} \tag{4. 1}$$

We first prove

LEMMA 4. 1. *In a manifold M with normal (f, g, u, v, λ)-structure, we have*

$$\begin{aligned} &\lambda(f_j{}^t u_{ti} - f_i{}^t u_{tj}) + f_j{}^t f_i{}^s v_{ts} - v_{ji} \\ &\qquad + (f_j{}^t u_i - f_i{}^t u_j) \nabla_t \lambda - \lambda\{(\nabla_j \lambda) v_i - (\nabla_i \lambda) v_j\} = 0, \end{aligned} \tag{4. 2}$$

and

$$\begin{aligned} &\lambda(f_j{}^t v_{ti} - f_i{}^t v_{tj}) - f_j{}^t f_i{}^s u_{ts} + u_{ji} + (f_j{}^t v_i - f_i{}^t v_j) \nabla_t \lambda \\ &\qquad + \lambda\{(\nabla_j \lambda) u_i - (\nabla_i \lambda) u_j\} = 0, \end{aligned} \tag{4. 3}$$

where

$$u_{ji} = \nabla_j u_i - \nabla_i u_j, \qquad v_{ji} = \nabla_j v_i - \nabla_i v_j.$$

Proof. Transvecting (4. 1) with v_h, we find

$$f_j{}^t(\nabla_t f_i{}^h)v_h - f_i{}^t(\nabla_t f_j{}^h)v_h + \lambda(\nabla_j f_i{}^t - \nabla_i f_j{}^t)u_t$$
$$+(1-\lambda^2)(\nabla_j v_i - \nabla_i v_j)=0$$

by virtue of (1. 3) and (1. 5), or

$$f_j{}^t\{\nabla_t(f_i{}^h v_h) - f_i{}^h \nabla_t v_h\} - f_i{}^t\{\nabla_t(f_j{}^h v_h) - f_j{}^h \nabla_t v_h\}$$
$$+\lambda\{\nabla_j(f_i{}^t u_t) - f_i{}^t \nabla_j u_t - \nabla_i(f_j{}^t u_t) + f_j{}^t \nabla_i u_t\} + (1-\lambda^2)(\nabla_j v_i - \nabla_i v_j) = 0,$$

from which

$$f_j{}^t\{-(\nabla_t\lambda)u_i - \lambda\nabla_t u_i - f_i{}^h\nabla_t v_h\} + f_i{}^t\{(\nabla_t\lambda)u_j + \lambda\nabla_t u_j + f_j{}^h\nabla_t v_h\}$$
$$+\lambda\{(\nabla_j\lambda)v_i + \lambda\nabla_j v_i - f_i{}^t\nabla_j u_t - (\nabla_i\lambda)v_j - \lambda\nabla_i v_j + f_j{}^t\nabla_i u_t\}$$
$$+(1-\lambda^2)(\nabla_j v_i - \nabla_i v_j)=0,$$

by virtue of (1. 2) and (1. 3), from which

$$\lambda\{f_j{}^t(\nabla_t u_i - \nabla_i u_t) - f_i{}^t(\nabla_t u_j - \nabla_j u_t)\} + f_j{}^t f_i{}^s(\nabla_t v_s - \nabla_s v_t) - (\nabla_j v_i - \nabla_i v_j)$$
$$+(f_j{}^t u_i - f_i{}^t u_j)\nabla_t\lambda - \lambda\{(\nabla_j\lambda)v_i - (\nabla_i\lambda)v_j\}=0,$$

which proves (4. 2)

Similarly, transvecting (4. 1) with u_h, we can prove (4. 3).

In order to get further results on manifolds with normal (f, g, u, v, λ)-structure, we put the condition

$$v_{ji}=2f_{ji}. \tag{4. 4}$$

As we have seen in the preceding section, for a hypersurface of Sasakian manifold, we have

$$\nabla_j v_i = f_{ji} + \lambda h_{ji}$$

and consequently the condition (4. 4) is always satisfied.

LEMMA 4. 2. *Let M be a manifold with normal (f, g, u, v, λ)-structure satisfying* (4. 4). *If the function $\lambda(1-\lambda^2)$ is almost everywhere non-zero, then we have*

$$u^t\nabla_t\lambda = 1-\lambda^2. \tag{4. 5}$$

Proof. Transvecting (4. 2) with $u^j v^i$ and using (1. 2) (1. 5), we find

$$\lambda(-\lambda u_{ji}v^j v^i - \lambda u_{ij}u^i u^j) + \lambda^2 v_{ts}u^t v^s - v_{ji}u^j v^i$$
$$-\lambda(1-\lambda^2)u^t\nabla_t\lambda - \lambda(1-\lambda^2)u^j\nabla_j\lambda = 0,$$

or, using $v_{ts}=2f_{ts}$,

$$2\lambda(1-\lambda^2)^2-2\lambda(1-\lambda^2)u^t\nabla_t\lambda=0,$$

which proves (4. 5)

LEMMA 4. 3. *Let M be a manifold with normal (f, g, u, v, λ)-structure satisfying* (4. 4), *then we have*

(4. 6) $$f_j{}^t\nabla_h f_{ti}-f_i{}^t\nabla_h f_{tj}=u_j(\nabla_i u_h)-u_i(\nabla_j u_h)+v_j(\nabla_i v_h)-v_i(\nabla_j v_h).$$

Proof. Since f_{ji} is given by

$$f_{ji}=\frac{1}{2}(\nabla_j v_i-\nabla_i v_j),$$

we have

(4. 7) $$\nabla_j f_{ih}+\nabla_i f_{hj}+\nabla_h f_{ji}=0.$$

On the other hand, (4. 1) can be written as

$$f_j{}^t\nabla_t f_{ih}-f_i{}^t\nabla_t f_{jh}+(\nabla_j f_{it}-\nabla_i f_{jt})f_h{}^t$$
$$+(\nabla_j u_i-\nabla_i u_j)u_h+(\nabla_j v_i-\nabla_i v_j)v_h=0,$$

and consequently

$$-f_j{}^t(\nabla_i f_{ht}+\nabla_h f_{ti})+f_i{}^t(\nabla_j f_{ht}+\nabla_h f_{tj})$$
$$+(\nabla_j f_{it}-\nabla_i f_{jt})f_h{}^t+(\nabla_j u_i-\nabla_i u_j)u_h+(\nabla_j v_i-\nabla_i v_j)v_h=0,$$

that is,

$$-\nabla_i(f_j{}^t f_{ht})-f_j{}^t\nabla_h f_{ti}+\nabla_j(f_i{}^t f_{ht})+f_i{}^t\nabla_h f_{tj}$$
$$+(\nabla_j u_i-\nabla_i u_j)u_h+(\nabla_j v_i-\nabla_i v_j)v_h=0.$$

Substituting

$$f_j{}^t f_{ht}=g_{jh}-u_j u_h-v_j v_h,$$

we obtain

$$u_j(\nabla_i u_h)+v_j(\nabla_i v_h)-f_j{}^t\nabla_h f_{ti}-u_i(\nabla_j u_h)-v_i(\nabla_j v_h)+f_i{}^t\nabla_h f_{tj}=0,$$

which gives (4. 6).

§ 5. Vector fields U and V.

In § 3, we have seen that a totally umbilical submanifold of codimension 2 of a Kählerian manifold whose connection induced on the normal bundle is flat admits a normal (f, g, u, v, λ)-structure and that the vector fields U and V define infinitesimal conformal transformations.

Also in §4, we have seen that a totally umbilical hypersurface of a Sasakian manifold admits a normal (f, g, u, v, λ)-structure and that the vector fields U and V define infinitesimal conformal transformations.

In this section, we prove that, under certain conditions, the vector fields U and V of a normal (f, g, u, v, λ)-structure both define infinitesimal conformal transformations.

In the sequel, we assume that

$$\nabla_j u_i - \nabla_i u_j = 2\phi f_{ji} \tag{5. 1}$$

and

$$\nabla_j v_i - \nabla_i v_j = 2 f_{ji}, \tag{5. 2}$$

where ϕ is a differentiable function on M.

LEMMA 5. 1. *Let M be a manifold with normal (f, g, u, v, λ)-structure satisfying* (5. 1) *and* (5. 2). *If the function $\lambda(1-\lambda^2)$ is almost everywhere non-zero, then we have*

$$v^t \nabla_t \lambda = -\phi(1-\lambda^2). \tag{5. 3}$$

Proof. Transvecting (4. 3) with $u^j v^i$ and using (1. 2)~(1. 5), we find

$$\lambda(-\lambda v_{ji} v^j v^i + \lambda v_{ji} u^j u^i) - \lambda^2 u_{ts} u^t v^s + u_{ji} u^j v^i$$
$$-\lambda(1-\lambda^2) v^t \nabla_t \lambda - \lambda(1-\lambda^2) v^i \nabla_i \lambda = 0,$$

or, using $u_{ts} = 2\phi f_{ts}$,

$$-2\lambda(1-\lambda^2)^2 \phi - 2\lambda(1-\lambda^2) v^i \nabla_i \lambda = 0,$$

which proves (5. 3).

LEMMA 5. 2. *Under the same assumptions as those in Lemma* 5. 1, *we have*

$$\nabla_i \lambda = u_i - \phi v_i. \tag{5. 4}$$

Proof. From (4. 2), (5. 1) and (5. 2), we have

$$2 f_j{}^t f_i{}^s f_{ts} - 2 f_{ji} + (f_j{}^t u_i - f_i{}^t u_j) \nabla_t \lambda - \lambda\{(\nabla_j \lambda) v_i - (\nabla_i \lambda) v_j\} = 0,$$

or

$$2\lambda(u_j v_i - u_i v_j) + (f_j{}^t u_i - f_i{}^t u_j) \nabla_t \lambda - \lambda\{(\nabla_j \lambda) v_i - (\nabla_i \lambda) v_j\} = 0.$$

Transvecting this equation with v^j, we find

$$-2\lambda(1-\lambda^2) u_i + \lambda u_i u^t \nabla_t \lambda - \lambda(v^j \nabla_j \lambda) v_i + \lambda(1-\lambda^2) \nabla_i \lambda = 0,$$

from which, substituting (4. 5) and (5. 3),

$$-2\lambda(1-\lambda^2)u_i+\lambda(1-\lambda^2)u_i+\lambda(1-\lambda^2)\phi v_i+\lambda(1-\lambda^2)\nabla_i\lambda=0,$$

which proves (5. 4).

LEMMA 5. 3. *Under the same assumptions as those in Lemma* 5. 1, ϕ *is constant.*

Proof. Differentiating (5. 4) covariantly, we have

$$\nabla_j\nabla_i\lambda=\nabla_j u_i-\phi\nabla_j v_i-v_i\nabla_j\phi,$$

from which, using (5. 1) and (5. 2),

$$v_j\nabla_i\phi=v_i\nabla_j\phi$$

which implies that

$$\nabla_i\phi=\alpha v_i$$

for some scalar function α.

Differentiating the equation above covariantly, we get

$$\nabla_j\nabla_i\phi=v_i\nabla_j\alpha+\alpha\nabla_j v_i,$$

from which, using (5. 1)

$$2\alpha f_{ji}=v_j\nabla_i\alpha-v_i\nabla_j\alpha.$$

Thus, if $n>2$, we have $\alpha=0$, because the rank of f_{ji} is almost everywhere maximum. This shows that ϕ is constant.

LEMMA 5. 4. *Under the same assumptions as those in Lemma* 5. 1, *we have*

$$(\nabla_j u_i+\nabla_i u_j)u^i=-2\lambda u_j \tag{5. 6}$$

and

$$(\nabla_j v_i+\nabla_i v_j)v^i=2\lambda\phi v_j. \tag{5. 7}$$

Proof. Differentiating

$$u_i u^i=1-\lambda^2$$

covariantly and using (5. 4), we find

$$2(\nabla_j u_i)u^i=-2\lambda(u_j-\phi v_j).$$

Substituting this into

$$2(\nabla_j u_i)u^i=\{(\nabla_j u_i+\nabla_i u_j)+(\nabla_j u_i-\nabla_i u_j)\}u^i,$$

or

$$2(\nabla_j u_i)u^i=(\nabla_j u_i+\nabla_i u_j)u^i+2\lambda\phi v_j,$$

we find

$$-2\lambda(u_j-\phi v_j)=(\nabla_j u_i+\nabla_i u_j)u^i+2\lambda\phi v_j,$$

which proves (5. 6).

Similarly, we can prove (5. 7).

THEOREM 5. 1. *Under the same assumptions as those in Lemma* 5. 1, *both of the vector fields* u^h *and* v^h *define infinitesimal conformal transformations.*

Proof. Transvecting (4. 6) with v^i and using (1. 3), we find

$$\begin{aligned}&f_j{}^t(\nabla_h f_{ti})v^i-\lambda u^t\nabla_h f_{tj}\\&=u_j(v^i\nabla_i u_h)+v_j(v^i\nabla_i v_h)-(1-\lambda^2)\nabla_j v_h,\end{aligned}$$

from which

$$\begin{aligned}&f_j{}^t\{\nabla_h(f_t{}^i v_i)-f_t{}^i\nabla_h v_i\}+\lambda\{\nabla_h(f_j{}^t u_t)-f_j{}^t\nabla_h u_t\}\\&=u_j(v^i\nabla_i u_h)+v_j(v^i\nabla_i v_h)-(1-\lambda^2)\nabla_j v_h,\end{aligned}$$

or, again using (1. 2) and (1. 3),

$$\begin{aligned}&-f_j{}^t\{(\nabla_h\lambda)u_t+\lambda\nabla_h u_t+f_t{}^i\nabla_h v_i\}\\&+\lambda\{(\nabla_h\lambda)v_j+\lambda\nabla_h v_j-f_j{}^t\nabla_h u_t\}\\&=u_j(v^i\nabla_i u_h)+v_j(v^i\nabla_i v_h)-(1-\lambda^2)\nabla_j v_h,\end{aligned}$$

that is,

$$\begin{aligned}&-2\lambda f_j{}^t\nabla_h u_t+(\delta_j^i-u_j u^i-v_j v^i)\nabla_h v_i+\lambda^2\nabla_h v_j\\&=u_j(v^i\nabla_i u_h)+v_j(v^i\nabla_i v_h)-(1-\lambda^2)\nabla_j v_h,\end{aligned}$$

or

$$\begin{aligned}&-2\lambda f_j{}^t\nabla_h u_t+(\nabla_h v_j+\nabla_j v_h)+\lambda^2(\nabla_h v_j-\nabla_j v_h)\\&=u_j v^i(\nabla_i u_h-\nabla_h u_j)+v_j v^i(\nabla_i v_h+\nabla_h v_i),\end{aligned}$$

or

$$\begin{aligned}&-2\lambda f_j{}^t\nabla_h u_t+(\nabla_h v_j+\nabla_j v_h)+2\lambda^2 f_{hj}\\&=2\lambda\phi u_j u_h+v_j v^i(\nabla_i v_h+\nabla_h v_i).\end{aligned}$$

Substituting

$$\begin{aligned}2\nabla_h u_t&=(\nabla_h u_t+\nabla_t u_h)+(\nabla_h u_t-\nabla_t u_h)\\&=\nabla_h u_t+\nabla_t u_h+2\phi f_{ht}\end{aligned}$$

and (5. 7) into the equation above, we find

$$-\lambda f_j{}^t(\nabla_h u_t+\nabla_t u_h)-2\lambda\phi(g_{jh}-u_j u_h-v_j v_h)$$

$$+(\nabla_h v_j+\nabla_j v_h)+2\lambda^2 f_{hj}$$

$$=2\lambda\phi u_j u_h+2\lambda\phi v_j v_h,$$

or

$$\nabla_h v_j+\nabla_j v_h=\lambda f_j{}^t(\nabla_h u_t+\nabla_t u_h)+2\lambda\phi g_{hj}-2\lambda^2 f_{hj} \tag{5.8}$$

Similarly, we have

$$\nabla_h u_j+\nabla_j u_h=-\lambda f_j{}^t(\nabla_h v_t+\nabla_t v_h)-2\lambda g_{hj}-2\lambda^2\phi f_{hj}. \tag{5.9}$$

Substituting (5. 8) into (5. 9), we obtain, using (5. 6),

$$(1-\lambda^2)(\nabla_h u_j+\nabla_j u_h)=-2\lambda(1-\lambda^2)g_{hj}-2\lambda^3 v_h v_j-\lambda^2 v^t(\nabla_h u_t+\nabla_t u_s)v_j. \tag{5.10}$$

Transvecting (5. 10) with v^j, we find

$$(1-\lambda^2)(\nabla_h u_j+\nabla_j u_h)v^j=-2\lambda(1-\lambda^2)v_h-2\lambda^3(1-\lambda^2)v_h-\lambda^2(1-\lambda^2)v^t(\nabla_h u_t+\nabla_t u_h),$$

or

$$(1+\lambda^2)(1-\lambda^2)(\nabla_h u_j+\nabla_j u_h)v^j=-2\lambda(1+\lambda^2)(1-\lambda^2)v_h,$$

from which

$$(\nabla_h u_j+\nabla_j u_h)v^j=-2\lambda v_h. \tag{5.11}$$

Substituting (5. 11) into (5. 10), we obtain

$$\nabla_j u_i+\nabla_i u_j=-2\lambda g_{ji}. \tag{5.12}$$

Substituting (5. 12) into (5. 8), we find

$$\nabla_j v_i+\nabla_i v_j=2\lambda\phi g_{ji}. \tag{5.13}$$

Equations (5. 12) and (5. 13) show that both of the vector fields u^h and v^h define infinitesimal conformal transformations.

Using (5. 12), (5. 3) and

$$\nabla_j u_i-\nabla_i u_j=2\phi f_{ji}, \qquad \nabla_j v_i-\nabla_i v_j=2f_{ji},$$

we have

$$\nabla_j u_i=-\lambda g_{ji}+\phi f_{ji}, \tag{5.14}$$

$$\nabla_j v_i=\lambda\phi g_{ji}+f_{ji}. \tag{5.15}$$

§ 6. Covariant derivative of 2-form f_{ji}.

THEOREM 6.1 *If a manifold with normal metric* (f, g, u, v, λ)*-structure satisfies* (5.1) *and* (5.2), *and if* $\lambda(1-\lambda^2)$ *is an almost everywhere non-zero function, then we have*

$$\nabla_j f_{ih} = -g_{ji}(\phi u_h + v_h) + g_{jh}(\phi u_i + v_i). \tag{6.1}$$

Proof. Substituting (5.14) and (5.15) into (4.6), we find

$$\begin{aligned} &f_j{}^t \nabla_h f_{ti} - f_i{}^t \nabla_h f_{tj} \\ &= u_j(-\lambda g_{ih} + \phi f_{ih}) - u_i(-\lambda g_{jh} + \phi f_{jh}) \\ &\quad + v_j(\lambda\phi g_{ih} + f_{ih}) - v_i(\lambda\phi g_{jh} + f_{jh}), \end{aligned}$$

or

$$\begin{aligned} &\nabla_h(f_j{}^t f_{ti}) - (\nabla_h f_j{}^t) f_{ti} - f_i{}^t \nabla_h f_{tj} \\ &= -\lambda(u_j - \phi v_j) g_{ih} + \lambda(u_i - \phi v_i) g_{jh} \\ &\quad + (\phi u_j + v_j) f_{ih} - (\phi u_i + v_i) f_{jh}, \end{aligned}$$

from which

$$\begin{aligned} &\nabla_h(-g_{ji} + u_j u_i + v_j v_i) + 2 f_i{}^t \nabla_h f_{tj} \\ &= -\lambda(u_j - \phi v_j) g_{ih} + \lambda(u_i - \phi v_i) g_{jh} \\ &\quad + (\phi u_j + v_j) f_{ih} - (\phi u_i + v_i) f_{jh}, \end{aligned}$$

or, using (5.14) and (5.15),

$$f_j{}^t \nabla_h f_{it} = \lambda(u_j - \phi v_j) g_{ih} + (\phi u_i + v_i) f_{jh}. \tag{6.2}$$

Transvecting (6.2) by $f_k{}^j$ and using (1.1), we find

$$\begin{aligned} &-\nabla_h f_{ik} + u_k u^t \nabla_h f_{it} + v_k v^t \nabla_h f_{it} \\ &= \lambda^2(\phi u_k + v_k) g_{ih} - (\phi u_i + v_i)(g_{hk} - u_h u_k - v_h v_k), \end{aligned}$$

or

$$\begin{aligned} &-\nabla_h f_{ik} + u_k\{\nabla_h(f_i{}^t u_t) - f_i{}^t \nabla_h u_t\} + v_k\{\nabla_h(f_i{}^t v_t) - f_i{}^t \nabla_h v_t\} \\ &= \lambda^2(\phi u_k + v_k) g_{ih} - (\phi u_i + v_i)(g_{hk} - u_h u_k - v_h v_k), \end{aligned}$$

from which, using (1.2) and (1.3),

$$\begin{aligned} &-\nabla_h f_{ik} + u_k\{(\nabla_h \lambda) v_i + \lambda(\nabla_h v_i) - f_i{}^t \nabla_h u_t\} \\ &-v_k\{(\nabla_h \lambda) u_i + \lambda(\nabla_h u_i) + f_i{}^t \nabla_h v_t\} \\ &= \lambda^2(\phi u_k + v_k) g_{ih} - (\phi u_i + v_i) g_{hk} + (\phi u_i + v_i)(u_h u_k + v_h v_k). \end{aligned}$$

Substituting (5. 4), (5. 14) and (5. 15) into this equation, we find

$$-\nabla_h f_{ik}+u_k\{(u_h-\phi v_h)v_i+\lambda(\lambda\phi g_{hi}+f_{hi})-f_i{}^t(-\lambda g_{ht}+\phi f_{ht})\}$$
$$-v_k\{(u_h-\phi v_h)u_i+\lambda(-\lambda g_{hi}+\phi f_{hi})+f_i{}^t(\lambda\phi g_{ht}+f_{ht})\}$$
$$=\lambda^2(\phi u_k+v_k)g_{ih}-(\phi u_i+v_i)g_{hk}+(\phi u_i+v_i)(u_h u_k+v_h v_k),$$

which proves (6. 1).

We have seen that if a manifold with normal (f, g, u, v, λ)-structure satisfies

$$\nabla_j u_i-\nabla_i u_j=2\phi f_{ji}, \qquad \nabla_j v_i-\nabla_i v_j=2f_{ji},$$

then

$$\nabla_j f_{ih}=-g_{ji}(\phi u_h+v_h)+g_{jh}(\phi u_i+v_i)$$

Conversely, we have

THEOREM 6. 2. *If a (f, g, u, v, λ)-structure satisfies* (5. 1), (5. 2) *and* (6. 1) *then the structure is normal.*

Proof. Substituting (6. 1) into

$$S_{ji}{}^h=f_j{}^t\nabla_t f_i{}^h-f_i{}^t\nabla_t f_j{}^h-(\nabla_j f_i{}^t-\nabla_i f_j{}^t)f_t{}^h$$
$$+(\nabla_j u_i-\nabla_i u_j)u^h+(\nabla_j v_i-\nabla_i v_j)v^h,$$

we have

$$S_{ji}{}^h=f_j{}^t\{-g_{ti}(\phi u^h+v^h)+\delta_t^h(\phi u_i+v_i)\}$$
$$-f_i{}^t\{-g_{tj}(\phi u^h+v^h)+\delta_t^h(\phi u_j+v_j)\}$$
$$-\{\delta_j^t(\phi u_i+v_i)-\delta_i^t(\phi u_j+v_j)\}f_t{}^h+2\phi f_{ji}u^h+2f_{ji}v^h$$
$$=0,$$

and consequently the structure is normal.

§7. Characterizations of even dimensional spheres.

We prove

THEOREM 7. 1. *Let M be a complete manifold with normal (f, g, u, v, λ)-structure satisfying* (5. 1) *and* (5. 2). *If $\lambda(1-\lambda^2)$ is an almost everywhere non-zero function and $n>2$ then M is isometric with an even dimensional sphere.*

Proof. Differentiating (5. 4) covariantly, we have

$$\nabla_j\nabla_i\lambda=\nabla_j u_i-\phi\nabla_j v_i,$$

ϕ being a constant, from which, using (5. 14) and (5. 15),

$$\nabla_j\nabla_i\lambda=-(1+\phi^2)\lambda g_{ji}. \tag{7.1}$$

Thus, λ being not identically zero, by a famous theorem of Obata [6], M is isometric with a sphere.

We next prove

THEOREM 7. 2. *Let M be a complete manifold with normal (f, g, u, v, λ)-structure satisfying*

$$\nabla_j v_i=f_{ji}. \tag{7.2}$$

If $\lambda(1-\lambda^2)$ is an almost everywhere non-zero function, then M is isometric with an even dimensional sphere.

Proof. Differentiating

$$v_i v^i=1-\lambda^2$$

covariantly and using (7. 2), we find

$$f_{ji}v^i=-\lambda\nabla_j\lambda$$

or

$$\lambda(\nabla_j\lambda-u_j)=0,$$

from which

$$\nabla_j\lambda=u_j.$$

This shows that

$$\nabla_j u_i-\nabla_i u_j=0.$$

Equation (7. 2) shows that

$$\nabla_j v_i-\nabla_i v_j=2f_{ji}.$$

Thus all the assumptions of Theorem 7. 1 are satisfied, and consequently M is isometric with an even dimensional sphere.

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