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NOTES ON KÄHLERIAN METRICS WITH VANISHING BOCHNER CURVATURE TENSOR

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As well known, an infinitesimal projective or conformal transformation in a Kählerian space M^{2n} is necessarily an isometry provided that M^{2n} is compact.[1] Though the lack of projective transformations in M^{2n} is filled up by holomorphically projective ones,[2] we do not know what transformations in M^{2n} correspond to conformal ones in a Riemannian space. In this point of view it is significant to study problems related to Bochner curvature tensor introduced by S. Bochner.[3] In fact, it is known that Bochner curvature tensor is constructed formally by modifying Weyl's conformal one on taking account of the formal resemblance between Weyl's projective curvature tensor and holomorphically projective one.[4] Thus the problem seems to reduce to what transformations of Kählerian spaces leave invariant Bochner curvature tensor, and especially it is a problem how we can get a Kählerian metric with vanishing Bochner curvature tensor from a flat Kählerian one.

As a contribution to this problem we shall give in this paper examples of Kählerian metrics with vanishing Bochner curvature tensor.

Preliminary facts will be given in §1 following to Yano-Bochner's notation. Bochner curvature tensor vanishes identically for spaces of constant holomorphic curvature, but the converse is not true. In §2 we shall show that if M^{2n} with vanishing Bochner curvature tensor is a locally product space of Kählerian spaces and not flat it is a locally product space of two Kählerian spaces of constant holomorphic curvature, which corresponds to the case of Riemannian space with vanishing conformal curvature tensor.

On the other hand, the metric tensor $g_{\alpha\beta*}$ of a Kählerian space can be expressed in the form

$$g_{\alpha\beta*}=\frac{\partial^2\phi}{\partial z^\alpha\partial \bar{z}^\beta}$$

with respect to local complex coordinates $\{z^\alpha\}$, $\alpha=1,\cdots,n$, where $\phi(z,\bar{z})$ is a real valued holomorphic function of $\{z^\alpha,\bar{z}^\alpha\}$. Sang Seup Eum [2] determined ϕ for the non-flat metric of constant holomorphic curvature in the complex number space

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1) Tashiro [8], Yano [9].
2) Ishihara [3], Tachibana and Ishihara [6], Ishihara and Tachibana [4].
3) Bochner [1], Yano and Bochner [10].
4) Tashiro [7], Tachibana and Ishihara [5].

under the assumption that ϕ is a function $f(t)$ of $t=\sum_{\alpha=1}^{n} z^\alpha \bar{z}^\alpha$. The result is Fubinian, i.e.,

$$f(t)=\frac{1}{c}\log\left(\frac{c}{b}t+1\right)+k,$$

where $b>0$, $c\neq 0$ and k are constant.

We shall follow his method in §3 to get metrics of vanishing Bochner curvature tensor in the complex number space.

1. Preliminaries. We agree to adopt the summation convention and the following ranges of indices throughout the paper:

$$1\leqq i, j, k, \cdots \leqq 2n,$$

$$1\leqq \alpha, \beta, \gamma, \cdots \leqq n, \qquad \alpha^*=n+\alpha.$$

Consider an n complex dimensional Kählerian space M^{2n} with metric

$$ds^2=g_{jk}dz^j dz^k, \tag{1.1}$$

where $\{z^\alpha\}$ is a local complex coordinate and $z^{\alpha^*}=\bar{z}^\alpha$ (=conjugate of z^α). As the metric is Kählerian, g_{jk} satisfy the following conditions:

$$\begin{aligned} &g_{\alpha\beta}=g_{\alpha^*\beta^*}=0,\\ &g_{\alpha\beta^*}=g_{\beta^*\alpha}=\bar{g}_{\alpha^*\beta}=\bar{g}_{\beta\alpha^*} \end{aligned} \tag{1.2}$$

and (1.1) becomes

$$ds^2=2g_{\alpha\beta^*}dz^\alpha dz^{\beta^*}.$$

g^{jk} satisfy the corresponding equations to (1.2). The Christoffel symbols Γ^i_{jk} vanish except

$$\Gamma^\alpha_{\beta\gamma}=g^{\alpha\varepsilon^*}\frac{\partial g_{\beta\varepsilon^*}}{\partial z^\gamma} \tag{1.3}$$

and their conjugates. As to the curvature tensor $R^i{}_{jkl}$, only the components of the form $R^\alpha{}_{\beta\gamma\delta^*}$ and $R^\alpha{}_{\beta\gamma^*\delta}$ and their conjugates can be different from zero, and

$$R^\alpha{}_{\beta\gamma\delta^*}=-\frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial z^{\delta^*}} \tag{1.4}$$

hold good. The Ricci tensor $R_{jk}=R^i{}_{jki}$ satisfies

$$\begin{aligned} &R_{\beta\gamma}=R_{\beta^*\gamma^*}=0,\\ &R_{\beta\gamma^*}=R^\alpha{}_{\beta\gamma^*\alpha}=-R^\alpha{}_{\beta\alpha\gamma^*} \end{aligned} \tag{1.5}$$

and the scalar curvature $R=g^{jk}R_{jk}$ is $R=2g^{\alpha\beta^*}R_{\alpha\beta^*}$.

A Kählerian space is called a space of constant holomorphic curvature if its curvature tensor satisfies

$$R_{\alpha\beta^*\gamma\delta^*}=\frac{R}{2n(n+1)}(g_{\alpha\beta^*}g_{\gamma\delta^*}+g_{\alpha\delta^*}g_{\gamma\beta^*}).$$

Bochner introduced a curvature tensor which is called Bochner one and given by

$$\begin{aligned}K_{\alpha\beta^*\gamma\delta^*}=R_{\alpha\beta^*\gamma\delta^*}&-\frac{1}{n+2}(g_{\alpha\beta^*}R_{\gamma\delta^*}+g_{\alpha\delta^*}R_{\gamma\beta^*}+g_{\gamma\delta^*}R_{\alpha\beta^*}+g_{\gamma\beta^*}R_{\alpha\delta^*})\\&+\frac{R}{2(n+1)(n+2)}(g_{\alpha\beta^*}g_{\gamma\delta^*}+g_{\alpha\delta^*}g_{\gamma\beta^*}).\end{aligned}\tag{1. 6}$$

In a former paper, one of the authors gave its components K_{ijkh} with respect to real local coordinates and got some theorems about spaces with vanishing K_{ijkh}.[5] Bochner curvature tensor vanishes identically for a space of constant holomorphic curvature, and Kählerian spaces with vanishing Bochner curvature tensor are more general than ones of constant holomorphic curvature.

2. Locally product Kählerian metrics with vanishing Bochner curvature tensor.

In this section we shall admit the followihg ranges of indices keeping the notation in §1.

$$1\leqq a, b, c, \cdots \leqq p, \qquad a^*=a+n,$$

$$p+1\leqq r, s, \cdots\leqq n, \qquad r^*=r+n.$$

Consider a Kählerian metric (1. 1) of the form

$$ds^2=ds_1{}^2+ds_2{}^2 \tag{2. 1}$$

where

$$ds_1{}^2=2g_{ab^*}dz^a dz^{b^*}, \qquad ds_2{}^2=2g_{rs^*}dz^r dz^{s^*}$$

are Kählerian metrics of dimensions p and $n-p$.

For a metric of this type we have

$$R_{ab^*rs^*}=0, \tag{2. 2}$$

because of

$$R_{ab^*rs^*}=-R_{b^*ars^*}=-g_{b^*c}R^c{}_{ars^*}=-g_{b^*c}\frac{\partial\Gamma^c_{ar}}{\partial z^{s^*}}=0.$$

5) Tachibana [5].

Now assume that our metric (2. 1) has the vanishing Bochner curvature tensor. Then from (2. 2) and

$$0=K_{ab^*rs^*}=R_{ab^*rs^*}-\frac{1}{n+2}(g_{ab^*}R_{rs^*}+g_{rs^*}R_{ab^*})+\frac{R}{2(n+1)(n+2)}g_{ab^*}g_{rs^*},$$

it follows that

$$R_{ab^*}=\frac{1}{2}\left(\frac{R}{n+1}-\frac{R_2}{n-p}\right)g_{ab^*}, \tag{2. 3}$$

$$R_{rs^*}=\frac{1}{2}\left(\frac{R}{n+1}-\frac{R_1}{p}\right)g_{rs^*} \tag{2. 4}$$

hold good, where R_1 or R_2 denotes the scalar curvature of ds_1 or ds_2 respectively.

Substituting (2. 3) into $K_{ab^*cd^*}=0$ we have

$$R_{ab^*cd^*}=\frac{1}{n+2}\left(\frac{R}{2(n+1)}-\frac{R_2}{n-p}\right)(g_{ab^*}g_{cd^*}+g_{ad^*}g_{cb^*}).$$

As $R_{ab^*cd^*}$ is just the curvature tensor of ds_1, the last equation expresses ds_1 to be of constant holomorphic curvature and hence from (2. 3) we get

$$\frac{R_1}{p}+\frac{R_2}{n-p}=\frac{R}{n+1}. \tag{2. 5}$$

On the other hand, we have

$$R_1+R_2=R. \tag{2. 6}$$

Thus by eliminating R from (2. 5) and (2. 6) we can obtain

$$\frac{R_1}{p(p+1)}+\frac{R_2}{(n-p)(n-p+1)}=0. \tag{2. 7}$$

Similarly ds_2 is of constant holomorphic curvature. Consequently if (2. 1) has the vanishing Bochner curvature tensor, ds_1 and ds_2 are metrics of constant holomorphic curvature with the scalar curvature satisfying (2. 7).

Conversely, consider two Kählerian metrics ds_1 and ds_2 of constant holomorphic curvature satisfying (2. 7). Then we can easily see that the metric $ds^2=ds_1{}^2+ds_2{}^2$ has the vanishing Bochner curvature tensor.

As a non-flat space of constant holomorphic curvature is never a locally product space of Kählerian spaces, we know that a non-flat metric of vanishing Bochner curvature tensor can not be a direct sum of more than two Kählerian metrics.

3. Metrics with vanishing Bochner curvature tensor. Let C^{n+1} be the complex number space with complex coordinate $\{z_\alpha\}$.[6] A real valued holomorphic

6) In this section we shall denote coordinates by z_α instead of z^α.

function $\phi=\phi(z, \bar{z})$ of $\{z_\alpha, \bar{z}_\alpha\}$ gives a Kählerian metric

$$g_{\alpha\beta^*}=\frac{\partial^2\phi}{\partial z_\alpha \partial \bar{z}_\beta}$$

to C^{n+1} or its subdomain.

Our problem of this section is to find a function $\phi=f(t)$ of $t=\sum_{\alpha=1}^{n} z_\alpha \bar{z}_\alpha$ so that the corresponding Kählerian metric has the vanishing Bochner curvature tensor. First we shall represent $K_{\alpha\beta^*\gamma\delta^*}$ in terms of f and its derivatives.

We have

$$g_{\alpha\beta^*}=f'\delta_{\alpha\beta}+f''\bar{z}_\alpha z_\beta, \tag{3. 1}$$

where dashes mean differentiation with respect to t.

As the metric is positive definite, $f'>0$ holds good in a domain containing $t=0$.

$g^{\alpha\beta^*}$ are given by

$$g^{\alpha\beta^*}=\frac{1}{f'}\left(\delta_{\alpha\beta}-\frac{f''}{f'+tf''}z_\alpha\bar{z}_\beta\right)$$

and hence the domain we shall consider hereafter is one satisfying $f'+tf''>0$.

From (1. 3) it holds that

$$\Gamma^\alpha_{\beta\gamma}=\frac{f''}{f'}(\bar{z}_\beta\delta_{\alpha\gamma}+\bar{z}_\gamma\delta_{\alpha\beta})+\sigma z_\alpha\bar{z}_\beta\bar{z}_\gamma,$$

where we have put

$$\sigma(t)=\frac{f'f'''-2f''^2}{f'(f'+tf'')}. \tag{3. 2}$$

Some computations and (1. 4) show the following equations:

$$\begin{aligned} R^\alpha{}_{\beta\gamma\delta^*}=&\frac{f'f'''-f''^2}{f'^2}z_\delta(\bar{z}_\beta\delta_{\alpha\gamma}+\bar{z}_\gamma\delta_{\alpha\beta})+\frac{f''}{f'}(\delta_{\beta\delta}\delta_{\alpha\gamma}+\delta_{\gamma\delta}\delta_{\alpha\beta})\\ &+\sigma' z_\alpha\bar{z}_\beta\bar{z}_\gamma z_\delta+\sigma z_\alpha(\bar{z}_\beta\delta_{\gamma\delta}+\bar{z}_\gamma\delta_{\beta\delta}),^{7)} \end{aligned} \tag{3. 3}$$

and

$$R_{\beta\delta^*}=-R^\alpha{}_{\beta\alpha\delta^*}=\lambda\bar{z}_\beta z_\delta+\mu\delta_{\beta\delta},$$

where λ and μ are functions defined by

$$\lambda=-\frac{(n+1)(f'f'''-f''^2)}{f'^2}-\sigma' t-\sigma,$$

7) Eum [2].

$$\mu=-\frac{(n+1)f''}{f'}-\sigma t. \tag{3. 4}$$

The scalar curvature R is

$$R=\frac{2}{f'}\left(\lambda t+n\mu-\frac{tf''(\lambda t+\mu)}{f'+tf''}\right).$$

For convenience sake we shall put

$$\Delta=\frac{R}{2(n+1)(n+2)}.$$

From (3. 3) it follows

$$\begin{aligned}R_{\alpha\beta^*\gamma\delta^*}=&-f''(\delta_{\alpha\delta}\delta_{\beta\gamma}+\delta_{\gamma\delta}\delta_{\beta\alpha})-\frac{f'f'''-f''^2}{f'}z_\delta(\bar{z}_\alpha\delta_{\beta\gamma}+\bar{z}_\gamma\delta_{\beta\alpha})\\&-\left\{(f'+tf'')\sigma+\frac{f''^2}{f'}\right\}z_\beta(\bar{z}_\gamma\delta_{\alpha\delta}+\bar{z}_\alpha\delta_{\gamma\delta})\\&-\left\{(f'+tf'')\sigma'+\frac{2f''(f'f'''-f''^2)}{f'^2}\right\}\bar{z}_\alpha z_\beta\bar{z}_\gamma z_\delta.\end{aligned}$$

Substituting these equations into (1. 6) we can get

$$\begin{aligned}K_{\alpha\beta^*\gamma\delta^*}=&A(\delta_{\alpha\delta}\delta_{\beta\gamma}+\delta_{\gamma\delta}\delta_{\alpha\beta})+B\bar{z}_\alpha z_\beta\bar{z}_\gamma z_\delta\\&+Cz_\delta(\bar{z}_\alpha\delta_{\beta\gamma}+\bar{z}_\gamma\delta_{\beta\alpha})+Dz_\beta(\bar{z}_\gamma\delta_{\alpha\delta}+\bar{z}_\alpha\delta_{\gamma\delta}),\end{aligned} \tag{3. 5}$$

where A, B, C and D are as follows:

$$A=-f''-\frac{2\mu f'}{n+2}+f'^2\Delta,$$

$$B=-(f'+tf'')\sigma'-\frac{2f''(ff'''-f''^2)}{f'^2}-\frac{4\lambda f}{n+2}+2f''^2\Delta,$$

$$C=-\frac{f'f'''-f''^2}{f'}-\frac{\lambda f'+\mu f''}{n+2}+f'f''\Delta,$$

$$D=-(f'+tf'')\sigma-\frac{f''^2}{f'}-\frac{\lambda f'+\mu f''}{n+2}+f'f''\Delta.$$

Now we assume that $n\geqq 4$ and $K_{\alpha\beta^*\gamma\delta^*}$ vanish identically. Consider (3. 5) with indices $\alpha, \beta, \gamma, \delta$ which are different to one another. Then we have

$$0=K_{\alpha\beta^*\gamma\delta^*}=B\bar{z}_\alpha z_\beta \bar{z}_\gamma z_\delta,$$

from which we know that B vanishes identically, because it vanishes at any point not on the coordinate planes. Next, considering the case of indices $\alpha\neq\delta$ and $\gamma\neq\delta$ we have

$$0=K_{\alpha\beta^*\gamma\delta^*}=Cz_\delta(\bar{z}_\alpha\delta_{\beta\gamma}+\bar{z}_\gamma\delta_{\beta\alpha}),$$

from which $C=0$ holds good for the similar reason. $D=0$ follows similarly. Thus we have

$$0=K_{\alpha\beta^*\gamma\delta^*}=A(\delta_{\alpha\delta}\delta_{\beta\gamma}+\delta_{\gamma\delta}\delta_{\alpha\beta})$$

and hence for $\alpha\neq\beta$, $0=K_{\alpha\beta^*\beta\alpha^*}=A$. Consequently we obtain $A=B=C=D=0$.

Next we shall look for the differential equation for f satisfying $A=0$. From the definition of A it follows that

$$f''+\frac{2\mu f'}{n+2}=\frac{f'}{(n+1)(n+2)}\left\{\lambda t+n\mu-\frac{tf''(\lambda t+\mu)}{f'+tf''}\right\}$$

or equivalently

$$\begin{aligned}&(n+1)(n+2)(f'+tf'')f''+2(n+1)\mu f'(f'+tf'')\\ &=\lambda tf'^2+(n-1)\mu tf'f''+n\mu f'^2.\end{aligned}$$

We substitute (3.4) into the last equation to get

$$2\sigma f''=\sigma' f'.$$

By integration it holds that

$$\sigma=af'^2 \tag{3.6}$$

or equivalently

$$f'f'''-2f''^2=af'^3(f'+tf''), \tag{3.7}$$

where a is any constant, and the case $a=0$ reduces to one of Eum.

Before solving (3.7) we shall show that $B=C=D=0$ are consequence of $A=0$. From $A=0$ we have

$$f'^2\Delta=f''+\frac{2\mu f'}{n+2}=-\frac{nf''+2f'\sigma t}{n+2}. \tag{3.8}$$

On the other hand it follows from (3.7) that

$$f'f'''-f''^2=f''^2+af'^3\,(f'+tf''), \tag{3.9}$$

The calculation of $C=0$ is as follows:

$$C=-\frac{f'f'''-f''^2}{f'}-\frac{\lambda f'+\mu f''}{n+2}+ff''\Delta$$

$$=-\frac{f'f'''-f''^2}{f'}+\frac{f'}{n+2}\left\{\frac{(n+1)(f'f'''-f''^2)}{f'^2}+\sigma' t+\sigma\right\}$$

$$+\frac{f''}{n+2}\left\{\frac{(n+1)f''}{f'}+\sigma t\right\}+f'f''\Delta.$$

Substituting (3. 8) and (3. 9) into the last equation, and taking account of $\sigma=af'^2$ and $\sigma'=2af'f''$ we can get $C=0$. $D=0$, comparing with $C=0$, follows from

$$(f'+tf'')\sigma+\frac{f''^2}{f'^2}=\frac{f'f'''-f''^2}{f'}$$

which is proved easily. We can show $B=0$ from $A=0$ in a similar way.

We shall solve (3. 7) and get exact forms of f. As (3. 7) is

$$\frac{f'f'''-2f''^2}{f'^3}=a(tf')',$$

we have by integration

$$\frac{f''}{f'^2}=atf'+k_1, \qquad k_1=\text{const.} \tag{3.10}$$

In the following we shall find f satisfying $f''(0)=0$.[8]

From (3. 10) we have $f'^{-3}f''=at$ and by integration $f'^{-2}=b_1-at^2$, where b_1 is constant. As $f'(0)>0$, b_1 must be positive and we have

$$f'=\frac{1}{\sqrt{b^2-at^2}}, \qquad b>0. \tag{3.11}$$

The metric with $a=0$ being flat, we shall exclude this case and consider the following cases with $a\neq 0$ in which b and c mean positive constant and k any constant.

Case I. $a=c^2$.

$$f'=\frac{1}{\sqrt{b^2-c^2t^2}}, \qquad f=\frac{1}{c}\sin^{-1}\left(\frac{c}{b}t\right)+k.$$

Case II. $a=-c^2$.

$$f'=\frac{1}{\sqrt{b^2+c^2t^2}}, \qquad f=\frac{1}{c}\mathrm{sh}^{-1}\left(\frac{c}{b}t\right)+k.$$

As a conclusion we get the following results.

In order that $f(t)$ gives a Kählerian metric satisfying $f''(0)=0$ which has the vanishing Bochner curvature tensor and is not flat, it is necessary and sufficient

8) Eum's one stated at Introduction is excluded by this condition.

that f is one of the following two functions:

$$f=\frac{1}{c}\sin^{-1}\left(\frac{c}{b}t\right)+k, \tag{3.12}$$

$$f=\frac{1}{c}\operatorname{sh}^{-1}\left(\frac{c}{b}t\right)+k, \tag{3.13}$$

where b and c are positive constant and k any constant.

The scalar curvature of (3.11) is obtained from

$$\Delta=\frac{R}{2(n+1)(n+2)}=-\frac{at}{\sqrt{b^2-at^2}}$$

which follows from (3.8). Hence Δ becomes

$$\Delta=\begin{cases} -\dfrac{ct}{\sqrt{b^2-c^2t^2}}\leqq 0, \\[2ex] \dfrac{ct}{\sqrt{b^2+c^2t^2}}\geqq 0 \end{cases}$$

corresponding to (3.12) and (3.13).

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