KŌDAI MATH. SEM. RFP.
21 (1969), 335—349

INTEGRAL FORMULAS FOR CLOSED HYPERSURFACES

BY KENTARO YANO AND MARIKO TANI

§ 0. Introduction.

Liebmann [7] and Süss [9] proved that only ovaloid with constant mean curvature of a Euclidean space is a sphere. To prove this, we need an integral formula of Minkowski. So that to generalize the theorem above to the case of closed hypersurfaces of a Riemannian manifold, we must first of all obtain an integral formula for closed hypersurfaces of a Riemannian manifold. In the case of hypersurfaces of a Euclidean space, the so-called position vector plays an important rôle. So, to obtain the integral formulas for closed hypersurfaces in a Riemannian manifold, we assume the existence of a certain vector field, for example, a conformal Killing vector field or a concurrent vector field in a Riemannian manifold.

The study in this direction has been done by Hsiung [2], [3], [4], Katsurada [5], [6], Shahin [8], Tani [10] and Yano [11], [12].

Let V be a closed and orientable hypersurface of an $(n+1)$-dimensional Euclidean space E and denote by g, h and M_l the first fundamental tensor, the second fundamental tensor and the l-th mean curvature of the hypersurface respectively. Let $X(x^h)$ be the position vector from a fixed point O in E to a point P on the hypersurface V, where x^h are parameters on the hypersurface and N the unit normal to the hypersurface, and put $\alpha=X\cdot N$, $X_i=\partial X/\partial x^i$ and $z_i=X\cdot X_i$.

Shahin [8] recently proved the integral formulas

$$m\int_V \alpha^{m-1}h_{ji}z^j z^i dV - n\int_V \alpha^m(1+\alpha M_1)dV=0,$$

$$m\int_V \alpha^{m-1}M_n g_{ji}z^j z^i dV - n\int_V \alpha^m(M_{n-1}+\alpha M_n)dV=0,$$

for an arbitrary m for which α^{m-1} and α^m have meaning, dV being the volume element of V.

These formulas generalize those of Chern [1], Hsiung [2], [3], [4] and Shahin [8].

The main purpose of the present paper is to obtain a series of integral formulas the first and the last of which are those given by Shahin and to generalize this to the case of hypersurfaces of a Riemannian manifold.

Received March 17, 1969.

§ 1. Preliminaries.

We consider an orientable differentiable hypersurface V covered by a system of coordinate neighborhoods $\{U; x^h\}$ and imbedded differentiablly in an $(n+1)$-dimensional Euclidean space referred to a rectangular coordinate system, where and in the sequel the indices $h, i, j, \cdots$ take the values $1, 2, \cdots, n$. If we denote by X the position vector from a fixed point O to a point P of the hypersurface, then the hypersurface V is represented by

$$X=X(x^h). \tag{1.1}$$

If we put

$$X_i=\partial_i X, \qquad \partial_i=\partial/\partial x^i, \tag{1.2}$$

then X_i are n linearly independent vectors tangent to the hypersurface V. We suppose that the coordinates x^h are chosen in such a way that the vectors $X_1, X_2, \cdots, X_n$ give the positive orientation of V. Then

$$g_{ji}=X_j \cdot X_i \tag{1.3}$$

give the components of the metric tensor of V with respect to the system of coordinate neighborhoods $\{U; x^h\}$, where the dot denotes the inner product of vectors in E. We choose the unit normal vector N in such a way that the vectors $N, X_1, X_2, \cdots, X_n$ give the positive orientation of E. Then we have

$$X_i \cdot N=0, \qquad N \cdot N=1. \tag{1.4}$$

We denote by ∇_i the operator of covariant differentiation with respect to the metric tensor g_{ji}. Then the equations of Gauss of the hypersurface V are written as

$$\nabla_j X_i=h_{ji}N, \tag{1.5}$$

where h_{ji} are the components of the second fundamental tensor and those of Weingarten as

$$\nabla_j N=-h_j{}^i X_i, \tag{1.6}$$

where

$$h_j{}^i=h_{jt}g^{ti},$$

g^{ti} being the contravariant components of the metric tensor.

Using the Ricci identities

$$\nabla_k \nabla_j X_i-\nabla_j \nabla_k X_i=-K_{kji}{}^h X_h,$$

and

$$\nabla_k \nabla_j N-\nabla_j \nabla_k N=0,$$

we obtain the equations of Gauss

$$K_{kji}{}^h=h_k{}^h h_{ji}-h_j{}^h h_{ki} \tag{1.7}$$

and those of Codazzi

$$\nabla_k h_{ji} - \nabla_j h_{ki} = 0, \tag{1.8}$$

where $K_{kji}{}^h$ are the components of the curvature tensor of V.

The principal curvatures of the hypersurface V are roots of the equation

$$|h_i{}^h - k\delta_i^h| = 0. \tag{1.9}$$

We denote them by $k_1, k_2, \cdots, k_n$ and put

$$\begin{aligned} s_0 &= 1, \qquad & s_l &= \sum_{i_1 < \cdots < i_l} k_{i_1} k_{i_2} \cdots k_{i_l}, \\ p_0 &= n, \qquad & p_l &= \sum_i (k_i)^l \\ & & & (l = 1, 2, \cdots, n). \end{aligned} \tag{1.10}$$

From (1.9) and (1.10), we have

$$p_l = h_{i_2}{}^{i_1} h_{i_3}{}^{i_2} \cdots h_{i_l}{}^{i_{l-1}} h_{i_1}{}^{i_l}. \tag{1.11}$$

It is well known that $s_1, \cdots, s_n$ and $p_1, \cdots, p_n$ are related by Newton's formulas

$$\begin{aligned} & p_1 - s_1 = 0, \\ & p_2 - s_1 p_1 + 2s_2 = 0, \\ & p_3 - s_1 p_2 + s_2 p_1 - 3s_3 = 0, \\ & \cdots\cdots\cdots, \\ & p_n - s_1 p_{n-1} + s_2 p_{n-2} - \cdots + (-1)^{n-1} s_{n-1} p_1 + (-1)^n n s_n = 0. \end{aligned} \tag{1.12}$$

Representing s_l in terms of $p_1, p_2, \cdots, p_l$, we obtain

$$\begin{aligned} & s_1 = p_1, \\ & s_2 = \frac{1}{2!}(-p_2 + p_1^2), \\ & s_3 = \frac{1}{3!}(2p_3 - 3p_1 p_2 + p_1^3), \\ & s_4 = \frac{1}{4!}(-6p_4 + 8p_1 p_3 - 6p_1^2 p_2 + 3p_2^2 + p_1^4), \\ & \cdots\cdots\cdots, \\ & s_l = \sum_{\substack{t_1 + 2t_2 + \cdots + lt_l = l \\ 0 \leqq t_i}} \frac{(-1)^{t_1 + t_2 + \cdots + t_l + l}}{(t_1!) \cdots (t_l!) 2^{t_2} \cdots l^{t_l}} p_1^{t_1} p_2^{t_2} \cdots p_l^{t_l}, \\ & \cdots\cdots\cdots, \\ & s_n = \sum_{\substack{t_1 + 2t_2 + \cdots + nt_n = n \\ 0 \leqq t_i}} \frac{(-1)^{t_1 + t_2 + \cdots + t_n + n}}{(t_1!) \cdots (t_n!) 2^{t_2} \cdots n^{t_n}} p_1^{t_1} p_2^{t_2} \cdots p_n^{t_n}. \end{aligned} \tag{1.13}$$

We introduce here the notations

(1.14) $$h_{(0)i}{}^{h}=\delta_{i}^{h}, \qquad h_{(l)i}{}^{h}=h_{i_1}{}^{h}h_{i_2}{}^{i_1}\cdots h_{i}{}^{i_{l-1}} \qquad l(=1, 2, \cdots, n),$$

(1.15) $$z_{(l)}{}^{h}=h_{(l)i}{}^{h}z^{i}=h_{i_1}{}^{h}h_{i_2}{}^{i_1}\cdots h_{i}{}^{i_{l-1}}z^{i} \qquad (l=0, 1, 2, \cdots, n),$$

and

(1.16) $$\binom{n}{l}M_{l}=s_{l},$$

where $\binom{n}{l}$ are binomial coefficients. The M_l is the l-th mean curvature of V. From (1.15) we see that

(1.17) $$z_{(l)}{}^{h}=h_{i}{}^{h}z_{(l-1)}{}^{i} \qquad (l=1, 2, \cdots, n).$$

Since g_{ji} is positive definite and h_{ji} is symmetric in j and i, we can assume that, at a fixed point of V, we have

$$(g_{ji})=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & \cdots\cdots\cdots & & \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \qquad (h_{ji})=\begin{pmatrix} k_1 & 0 & \cdots & 0 \\ 0 & k_1 & \cdots & 0 \\ & \cdots\cdots\cdots & & \\ 0 & 0 & \cdots & k_n \end{pmatrix},$$

and consequently

$$(h_{i}{}^{h})=\begin{pmatrix} k_1 & 0 & \cdots & 0 \\ 0 & k_1 & \cdots & 0 \\ & \cdots\cdots\cdots & & \\ 0 & 0 & \cdots & k_n \end{pmatrix}$$

and

$$(h_{(l)i}{}^{h})=\begin{pmatrix} k_1^{l} & 0 & \cdots & 0 \\ 0 & k_2^{l} & \cdots & 0 \\ & \cdots\cdots\cdots & & \\ 0 & 0 & \cdots & k_n^{l} \end{pmatrix}.$$

Now, $k_1, k_2, \cdots, k_n$ satisfy the equation

$$t^{n}-s_{1}t^{n-1}+s_{2}t^{n-2}-\cdots+(-1)^{n-1}s_{n-1}t+(-1)^{n}s_{n}=0,$$

and consequently, we have

$$h_{(n)i}{}^{h}-s_{1}h_{(n-1)i}{}^{h}+s_{2}h_{(n-2)i}{}^{h}-\cdots+(-1)^{n-1}s_{n-1}h_{i}{}^{h}+(-1)^{n}s_{n}\delta_{i}^{h}=0,$$

or

(1.18) $$h_{(n)ji}-s_{1}h_{(n-1)ji}+s_{2}h_{(n-2)ji}-\cdots+(-1)^{n-1}s_{n-1}h_{ji}+(-1)^{n}s_{n}g_{ji}=0.$$

In the sequel, we need the expression for $\nabla_{i}z_{(l)}{}^{i}$. We have

$$
\begin{aligned}
\nabla_i z_{(l)}{}^i &= \nabla_i(h_{i_1}{}^i h_{i_2}{}^{i_1} \cdots h_{i_{l-1}}{}^{i_{l-2}} h_{i_l}{}^{i_{l-1}} z^{i_l}) \\
&= (\nabla_{i_1} h_i{}^i) h_{i_2}{}^{i_1} \cdots h_{i_{l-1}}{}^{i_{l-2}} h_{i_l}{}^{i_{l-1}} z^{i_l} \\
&\quad + h_{i_1}{}^i (\nabla_{i_2} h_i{}^{i_1}) \cdots h_{i_{l-1}}{}^{i_{l-2}} h_{i_l}{}^{i_{l-1}} z^{i_l} \\
&\quad + \cdots \\
&\quad + h_{i_1}{}^i h_{i_2}{}^{i_1} \cdots (\nabla_{i_{l-1}} h_i{}^{i_{l-2}}) h_{i_l}{}^{i_{l-1}} z^{i_l} \\
&\quad + h_{i_1}{}^i h_{i_2}{}^{i_1} \cdots h_{i_{l-1}}{}^{i_{l-2}} (\nabla_{i_l} h_i{}^{i_{l-1}}) z^{i_l} \\
&\quad + h_{i_1}{}^i h_{i_2}{}^{i_1} \cdots h_{i_{l-1}}{}^{i_{l-2}} h_{i_l}{}^{i_{l-1}} (\nabla_i z^{i_l})
\end{aligned}
$$

by virtue of equations (1. 8) of Codazzi and consequently

$$
\begin{aligned}
\nabla_i z_{(l)}{}^i &= (\nabla_i p_1) z_{(l-1)}{}^i + \frac{1}{2}(\nabla_i p_2) z_{(l-2)}{}^i + \cdots \\
&\quad + \frac{1}{l-1}(\nabla_i p_{l-1}) z_{(1)}{}^i + \frac{1}{l}(\nabla_i p_l) z^i + h_{(l)h}{}^i (\nabla_i z^h) \qquad (1.\ 19) \\
&\qquad (l=0, 1, 2, \cdots, n).
\end{aligned}
$$

§ 2. Integral formulas for hypersurfaces of a Euclidean space.

We consider a compact and orientable hypersurface V of an $(n+1)$-dimensional Euclidean space E and put

$$\alpha = X \cdot N, \qquad z_i = X \cdot X_i. \tag{2. 1}$$

We then have

$$\nabla_j \alpha = -h_{ji} z^i, \tag{2. 2}$$

$$\nabla_j z_i = g_{ji} + \alpha h_{ji} \tag{2. 3}$$

by virtue of equations of Gauss and Weingarten, where $z^i = z_j g^{ji}$ and consequently we have, from (1. 19),

$$
\begin{aligned}
\nabla_i z_{(l)}{}^i &= (\nabla_i p_1) z_{(l-1)}{}^i + \frac{1}{2}(\nabla_i p_2) z_{(l-2)}{}^i + \cdots \\
&\quad + \frac{1}{l-1}(\nabla_i p_{l-1}) z_{(1)}{}^i + \frac{1}{l}(\nabla_i p_l) z^i + p_l + \alpha p_{l+1}.
\end{aligned} \tag{2. 4}
$$

For $l=0, 1, 2$, (2. 4) gives

$$\nabla_i z^i = n + \alpha p_1, \tag{2. 5}$$

$$\nabla_i z_{(1)}{}^i = (\nabla_i p_1) z^i + p_1 + \alpha p_2, \tag{2. 6}$$

$$\nabla_i z_{(2)}{}^i = (\nabla_i p_1) z_{(1)}{}^i + \frac{1}{2}(\nabla_i p_2) z^i + p_2 + \alpha p_3 \tag{2. 7}$$

respectively.

Now, we have

$$\nabla_i(\alpha^m z^i)=-m\alpha^{m-1}h_{ji}z^j z^i+\alpha^m(n+\alpha p_1)$$

by virtue of (2. 2) and (2. 5), from which, integrating over V,

$$m\int_V \alpha^{m-1}h_{ji}z^j z^i dV-\int_V \alpha^m(n+\alpha p_1)dV=0, \tag{2. 8}$$

or substituting $p_1=s_1=nM_1$,

$$m\int_V \alpha^{m-1}h_{ji}z^j z^i dV-n\int_V \alpha^m(1+\alpha M_1)dV=0, \tag{2. 9}$$

where dV is the volume element of V, which is a formula proved by Shahin [8].

We also have

$$\nabla_i(\alpha^m z_{(1)}{}^i)=-m\alpha^{m-1}h_{(2)ji}z^j z^i+\alpha^m\{(\nabla_i p_1)z^i+p_1+\alpha p_2\} \tag{2. 10}$$

and

$$\nabla_i(\alpha^m p_1 z^i)=-m\alpha^{m-1}p_1 h_{ji}z^j z^i+\alpha^m\{(\nabla_i p_1)z^i+p_1(n+\alpha p_1)\} \tag{2. 11}$$

by virtue of (2. 2), (2. 5) and (2. 6). Integrating $-(2.10)+(2.11)$ over V, we find

$$m\int_V \alpha^{m-1}(h_{(2)ji}-p_1 h_{ji})z^j z^i dV+\int_V \alpha^m\{(n-1)p_1+\alpha(p_1{}^2-p_2)\}dV=0, \tag{2. 12}$$

or

$$m\int_V \alpha^{m-1}(h_{(2)ji}-s_1 h_{ji})z^j z^i dV+\int_V \alpha^m\{(n-1)s_1+2\alpha s_2\}dV=0 \tag{2. 13}$$

by virtue of (1. 13), or again

$$m\int_V \alpha^{m-1}(h_{(2)ji}-s_1 h_{ji})z^j z^i dV+2\binom{n}{2}\int_V \alpha^m(M_1+\alpha M_2)dV=0 \tag{2. 14}$$

by virtue of (1. 16).

We also have

$$\begin{aligned}\nabla_i(\alpha^m z_{(2)}{}^i)&=-m\alpha^{m-1}h_{(3)ji}z^j z^i\\&\quad+\alpha^m\left\{(\nabla_i p_1)z_{(1)}{}^i+\frac{1}{2}(\nabla_i p_2)z^i+p_2+\alpha p_3\right\},\end{aligned} \tag{2. 15}$$

$$\begin{aligned}\nabla_i(\alpha^m p_1 z_{(1)}{}^i)&=-m\alpha^{m-1}p_1 h_{(2)ji}z^j z^i\\&\quad+\alpha^m[(\nabla_i p_1)z_{(1)}{}^i+p_1\{(\nabla_i p_1)z^i+p_1+\alpha p_2\}]\end{aligned}$$

that is,

$$\begin{aligned}\nabla_i(\alpha^m p_1 z_{(1)}{}^i)&=-m\alpha^{m-1}p_1 h_{(2)ji}z^j z^i\\&\quad+\alpha^m\left\{(\nabla_i p_1)z_{(1)}{}^i+\frac{1}{2}(\nabla_i p_1{}^2)z^i+p_1(p_1+\alpha p_2)\right\}\end{aligned} \tag{2. 16}$$

and

$$
\nabla_i\left\{\frac{1}{2}\alpha^m(p_1{}^2-p_2)z^i\right\}=-\frac{1}{2}m\alpha^{m-1}(p_1{}^2-p_2)h_{ji}z^jz^i
$$

(2.17)

$$
+\alpha^m\left[\frac{1}{2}\{\nabla_i(p_1{}^2-p_2)\}z^i+\frac{1}{2}(p_1{}^2-p_2)(n+\alpha p_1)\right]
$$

by virtue of (1.17), (2.5), (2.6) and (2.7).

Integrating $-(2.15)+(2.16)-(2.17)$ over V, we find

$$
m\int_V \alpha^{m-1}\left\{h_{(3)ji}-p_1h_{(2)ji}+\frac{1}{2}(p_1{}^2-p_2)h_{ji}\right\}z^jz^i dV
$$

(2.18)

$$
-\frac{1}{2}\int_V \alpha^m\{(n-2)(p_1{}^2-p_2)+\alpha(p_1{}^3-3p_1p_2+2p_3)\}dV=0,
$$

or

$$
m\int_V \alpha^{m-1}(h_{(3)ji}-s_1h_{(2)ji}+s_2h_{ji})z^jz^i dV-\int_V \alpha^m\{(n-2)s_2+3\alpha s_3\}dV=0 \tag{2.19}
$$

by virtue of (1.13), or again

$$
m\int_V \alpha^{m-1}(h_{(3)ji}-s_1h_{(2)ji}+s_2h_{ji})z^jz^i dV-3\binom{n}{3}\int_V \alpha^m(M_2+\alpha M_3)dV=0 \tag{2.20}
$$

by virtue of (1.16).

To obtain integral formula for the most general case, we compute

$$
\nabla_i(\alpha^m z_{(l)}{}^i)=-m\alpha^{m-1}h_{(l+1)ji}z^jz^i
$$

(2.21)

$$
+\alpha^m\left\{(\nabla_i p_1)z_{(l-1)}{}^i+\frac{1}{2}(\nabla_i p_2)z_{(l-2)}{}^i+\cdots\right.
$$

$$
\left.+\frac{1}{l-1}(\nabla_i p_{l-1})z_{(1)}{}^j+\frac{1}{l}(\nabla_i p_l)z^i+p_l+\alpha p_{l+1}\right\},
$$

$$
\nabla_i(\alpha^m p_1 z_{(l-1)}{}^i)=-m\alpha^{m-1}p_1h_{(l)ji}z^jz^i
$$

(2.22)

$$
+\alpha^m\left[(\nabla_i p_1)z_{(l-1)}{}^i+p_1\left\{(\nabla_i p_1)z_{(l-2)}{}^i+\frac{1}{2}(\nabla_i p_2)z_{(l-2)}{}^i+\cdots\right.\right.
$$

$$
\left.\left.+\frac{1}{l-2}(\nabla_i p_{l-2})z_{(1)}{}^i+\frac{1}{l-1}(\nabla_i p_{l-1})z^i+p_{l-1}+\alpha p_l\right\}\right],
$$

$$
\begin{aligned}
\nabla_i\left\{\frac{1}{2}\alpha^m(p_1{}^2-p_2)z_{(l-2)}{}^i\right\}=&-\frac{1}{2}m\alpha^{m-1}(p_1{}^2-p_2)h_{(l-1)ji}z^jz^i\\
&+\frac{1}{2}\alpha^m[\{\nabla_i(p_1{}^2-p_2)\}z_{(l-2)}{}^i\\
&+(p_1{}^2-p_2)\left\{(\nabla_i p_1)z_{(l-3)}{}^i+\frac{1}{2}(\nabla_i p_2)z_{(l-4)}{}^i+\cdots\right.\\
&\left.\left.+\frac{1}{l-3}(\nabla_i p_{l-3})z_{(1)}{}^i+\frac{1}{l-2}(\nabla_i p_{l-2})z^i+p_{l-2}+\alpha p_{l-1}\right\}\right],
\end{aligned}\tag{2. 23}
$$

$$
\begin{aligned}
&\nabla_i\left\{\frac{1}{3!}\alpha^m(p_1{}^3-3p_1p_2+2p_3)z_{(l-3)}{}^i\right\}\\
&=-\frac{1}{3!}m\alpha^{m-1}(p_1{}^3-3p_1p_2+2p_3)h_{(l-2)ji}z^jz^i\\
&+\frac{1}{3!}\alpha^m\Big[\{\nabla_i(p_1{}^3-3p_1p_2+2p_3)\}z_{(l-3)}{}^i\\
&\qquad+(p_1{}^3-3p_1p_2+2p_3)\left\{(\nabla_i p_1)z_{(l-4)}{}^i+\frac{1}{2}(\nabla_i p_2)z_{(l-5)}{}^i+\cdots\right.\\
&\qquad\left.\left.+\frac{1}{l-4}(\nabla_i p_{l-4})z_{(1)}{}^i+\frac{1}{l-3}(\nabla_i p_{l-3})z^i+p_{l-3}+\alpha p_{l-2}\right\}\right],
\end{aligned}\tag{2. 24}
$$

$$\cdots\cdots\cdots\cdots\cdots,$$

$$
\begin{aligned}
&\nabla_i\sum_{\substack{t_1+2t_2+\cdots+lt_l=l\\0\leqq t_i}}\frac{(-1)^{t_1+\cdots+t_l+l}}{(t_1!)\cdots(t_l!)2^{t_2}\cdots l^{t_l}}\alpha^m p_1{}^{t_1}\cdots p_l{}^{t_l}z^i\\
&=\sum_{\substack{t_1+2t_2+\cdots+lt_l=l\\0\leqq t_i}}\frac{(-1)^{t_1+\cdots+t_l+l}}{(t_1!)\cdots(t_l!)2^{t_2}\cdots l^{t_l}}[-m\alpha^{m-1}p_1{}^{t_1}\cdots p_l{}^{t_l}h_{ji}z^jz^i\\
&\qquad+\alpha^m\{(\nabla_i p_1{}^{t_1}\cdots p_l{}^{t_l})z^i+p_1{}^{t_1}\cdots p_l{}^{t_l}(n+\alpha p_1)\}].
\end{aligned}\tag{2. 25}
$$

Integrating $-(2.\ 21)+(2.\ 22)-(2.\ 23)+(2.\ 24)-\cdots+(-1)^l(2.\ 25)$, we find

$$
\begin{aligned}
&m\int_V\alpha^{m-1}\{h_{(l+1)ji}-s_1h_{(l)ji}+s_2h_{(l-1)ji}-s_3h_{(l-2)ji}+\cdots\\
&\qquad+(-1)^{l-1}s_{l-1}h_{(2)ji}+(-1)^ls_lh_{ji}\}z^jz^idV\\
&-\int_V\alpha^m\{(p_l+\alpha p_{l+1})-s_1(p_{l-1}+\alpha p_l)+s_2(p_{l-2}+\alpha p_{l-1})\\
&\qquad-s_3(p_{l-3}+\alpha p_{l-2})+\cdots+(-1)^ls_l(n+\alpha p_1)\}dV=0,
\end{aligned}\tag{2. 26}
$$

or, by (1. 12),

$$
\begin{aligned}
&m\int_V \alpha^{m-1}\{h_{(l+1)ji}-s_1h_{(l)ji}+s_2h_{(l-1)ji}-s_3h_{(l-2)ji}+\cdots \\
&\qquad +(-1)^{l-1}s_{l-1}h_{(2)ji}+(-1)^l s_l h_{ji}\}z^jz^i dV \\
&+(-1)^l\int_V \alpha^m\{(n-l)s_l+\alpha(l+1)s_{l+1}\}dV=0,
\end{aligned} \tag{2.27}
$$

or again, by (1. 16),

$$
\begin{aligned}
&m\int_V \alpha^{m-1}\{h_{(l+1)ji}-s_1h_{(l)ji}+s_2h_{(l-1)ji}-s_3h_{(l-2)ji}+\cdots \\
&\qquad +(-1)^{l-1}s_{l-1}h_{(2)ji}+(-1)^l s_l h_{ji}\}z^jz^i dV \\
&+(-1)^{l+1}(l+1)\binom{n}{l+1}\int_V \alpha^m(M_l+\alpha M_{l+1})dV=0.
\end{aligned} \tag{2.28}
$$

In particular, for $l=n-1$, we have

$$
\begin{aligned}
&m\int_V \alpha^{m-1}\{h_{(n)ji}-s_1h_{(n-1)ji}+s_2h_{(n-2)ji}-s_3h_{(n-3)ji}+\cdots \\
&\qquad +(-1)^{n-2}s_{n-2}h_{(2)ji}+(-1)^{n-1}s_{n-1}h_{ji}\}z^jz^i dV \\
&+(-1)^n n\int_V \alpha^m(M_{n-1}+\alpha M_n)dV=0,
\end{aligned}
$$

or

$$
m\int_V \alpha^{m-1}M_n g_{ji}z^jz^i dV-n\int_V \alpha^m(M_{n-1}+\alpha M_n)dV=0, \tag{2.29}
$$

by virtue of (1. 18), which is a formula obtained by Shahin [8].

§ 3. Integral formulas for hypersurfaces of a Riemannian manifold.

We consider a compact and orientable hypersurface V covered by a system of coordinate neighborhoods $\{U; x^h\}$ of an $(n+1)$-dimensional orientable Riemannian manifold M with the metric tensor G and assume that the hypersurface V admits a concurrent vector field Z.

We denote by X_i the n vectors $\partial_i=\partial/\partial x^i$ tangent to the hypersurface V and assume that the vectors $X_1, X_2, \cdots, X_n$ give the positive orientation of V. We choose the unit normal vector N of V in such a way that the $n+1$ vectors $N, X_1, \cdots, X_n$ give the positive orientation of the Riemannian manifold M. Then the components of the metric tensor of V are given by

$$
g_{ji}=G(X_j, X_i). \tag{3.1}
$$

We also have

$$
G(X_i, N)=0, \qquad G(N, N)=1 \tag{3.2}
$$

along the hypersurface V.

Then the equations of Gauss and Weingarten can be written as

(3. 3) $$\nabla_j X_i = h_{ji} N$$

and

(3. 4) $$\nabla_j N = -h_j{}^i X_i,$$

where ∇_j denotes the operator of the so-called van der Waerden-Bortolotti covariant differentiation along the hypersurface.

We now assume that there exists a concurrent vector field along the hypersurface V, that is, a vector field Z such that

(3. 5) $$\nabla_j Z = X_j$$

along the hypersurface V. If we put

(3. 6) $$Z = z^i X_i + \alpha N,$$

we have, from (3. 3), (3. 4) and (3. 5),

(3. 7) $$\nabla_j \alpha = -h_{ji} z^i$$

and

(3. 8) $$\nabla_j z^h = \delta_j^h + \alpha h_j{}^h,$$

where $h_j{}^h = h_{ji} g^{ih}$.

From the Ricci identity

$$\nabla_k \nabla_j X_i - \nabla_j \nabla_k X_i = K(X_k, X_j) X_i - R_{kji}{}^h X_h,$$

where K is the curvature tensor of M and $R_{kji}{}^h$ that of V, we have equations of Gauss

(3. 9) $$K(X_k, X_j, X_i, X_h) = R_{kjih} - h_{kh} h_{ji} + h_{jh} h_{ki}$$

and those of Codazzi

(3. 10) $$K(X_k, X_j, X_i, N) = \nabla_k h_{ji} - \nabla_j h_{ki}.$$

For the sake of simplicity, we put in the sequel

(3. 11) $$K_{kji} = K(X_k, X_j, X_i, N)$$

and

(3. 12) $$K_k = g^{ji} K(X_k, X_j, X_i, N).$$

From (3. 10) and (3. 11), we have

(3. 13) $$K_{kji} = \nabla_k h_{ji} - \nabla_j h_{ki}$$

or

(3. 14) $$K_{kj}{}^i = \nabla_k h_j{}^i - \nabla_j h_k{}^i$$

or

(3. 15) $$\nabla_k h_j{}^i = \nabla_j h_k{}^i - K_{jk}{}^i.$$

Also, from (3. 10) and (3. 12), we have

$$K_k=\nabla_k h_i^i-\nabla_i h_k^i \tag{3.16}$$

or

$$\nabla_i h_k^i=\nabla_k h_i^i-K_k. \tag{3.17}$$

From (3. 8), (3. 15) and (3. 16), we have

$$\nabla_i z^i=n+\alpha p_1, \tag{3.18}$$

$$\begin{aligned}\nabla_i z_{(1)}{}^i&=\nabla_i(h_j{}^i z^j)\\&=(\nabla_i p_1-K_i)z^i+p_1+\alpha p_2,\end{aligned} \tag{3.19}$$

$$\begin{aligned}\nabla_i z_{(2)}{}^i&=\nabla_i(h_t{}^i h_j{}^t z^j)\\&=(\nabla_t h_i^i-K_t)h_j{}^t z^j\\&\quad+h_t{}^i(\nabla_j h_i{}^t-K_{ji}{}^t)z^j+h_t{}^i h_j{}^t(\delta_i^j+\alpha h_i{}^j)\\&=(\nabla_i p_1-K_i)z_{(1)}{}^i+\left(\frac{1}{2}\nabla_i p_2-K_{is}{}^r h_r{}^s\right)z^i+p_2+\alpha p_3.\end{aligned} \tag{3.20}$$

In general, we have

$$\begin{aligned}\nabla_i z_{(l)}{}^i&=\nabla_i(h_{i_1}{}^i h_{i_2}{}^{i_1}\cdots h_{i_{l-1}}{}^{i_{l-2}}h_{i_l}{}^{i_{l-1}}z^{i_l})\\&=(\nabla_{i_1}h_i^i-K_{i_1})h_{i_2}{}^{i_1}\cdots h_{i_{l-1}}{}^{i_{l-2}}h_{i_l}{}^{i_{l-1}}z^{i_l}\\&\quad+h_{i_1}{}^i(\nabla_{i_2}h_i{}^{i_1}-K_{i_2 i}{}^{i_1})\cdots h_{i_{l-1}}{}^{i_{l-2}}h_{i_l}{}^{i_{l-1}}z^{i_l}\\&\quad+\cdots\\&\quad+h_{i_1}{}^i h_{i_2}{}^{i_1}\cdots(\nabla_{i_{l-1}}h_i{}^{i_{l-2}}-K_{i_{l-1}i}{}^{i_{l-2}})h_{i_l}{}^{i_{l-1}}z^{i_l}\\&\quad+h_{i_1}{}^j h_{i_2}{}^{i_1}\cdots h_{i_{l-1}}{}^{i_{l-2}}(\nabla_{i_l}h_i{}^{i_{l-1}}-K_{i_l i}{}^{i_{l-1}})z^{i_l}\\&\quad+h_{i_1}{}^i h_{i_2}{}^{i_1}\cdots h_{i_{l-1}}{}^{i_{l-2}}h_{i_l}{}^{i_{l-1}}(\delta_i^{i_l}+\alpha h_i{}^{i_l})\end{aligned}$$

by virtue of (3. 8), (3. 15) and (3. 17) and consequently

$$\begin{aligned}\nabla_i z_{(l)}{}^i&=(\nabla_i p_1-K_i)z_{(l-1)}{}^i+\left(\frac{1}{2}\nabla_i p_2-K_{is}{}^r h_r{}^s\right)z_{(l-2)}{}^i\\&\quad+\cdots\\&\quad+\left(\frac{1}{l-1}\nabla_i p_{l-1}-K_{is}{}^r h_{(l-2)r}{}^s\right)z_{(1)}{}^i\\&\quad+\left(\frac{1}{l}\nabla_i p_l-K_{is}{}^r h_{(l-1)r}{}^s\right)z^i+p_l+\alpha p_{l+1}.\end{aligned} \tag{3.21}$$

Thus we have

$$\nabla_i(\alpha^m z^i)=-m\alpha^{m-1}h_{ji}z^j z^i+\alpha^m(n+\alpha p_1),$$

from which, integrating over V,

$$m\int_V \alpha^{m-1} h_{ji} z^j z^i dV - \int_V \alpha^m (n+\alpha p_1) dV = 0, \tag{3.22}$$

or

$$m\int_V \alpha^{m-1} h_{ji} z^j z^i dV - n\int_V \alpha^m (1+\alpha M_1) dV = 0. \tag{3.23}$$

We also have

$$\nabla_i(\alpha^m z_{(1)}{}^i) = -m\alpha^{m-1} h_{(2)ji} z^j z^i + \alpha^m \{(\nabla_i p_1 - K_i) z^i + p_1 + \alpha p_2\} \tag{3.24}$$

and

$$\nabla_i(\alpha^m p_1 z^i) = -m\alpha^{m-1} p_1 h_{ji} z^j z^i + \alpha^m \{(\nabla_i p_1) z^i + p_1(n+\alpha p_1)\} \tag{3.25}$$

by virtue of (3. 17) and (3. 18). Integrating $-(3.24)+(3.25)$ over V, we find

$$\begin{aligned} &m\int_V \alpha^{m-1}(h_{(2)ji} - p_1 h_{ji}) z^j z^i dV \\ &\quad + \int_V \alpha^m \{(n-1)p_1 + \alpha(p_1{}^2 - p_2) + K_i z^i)\} dV = 0 \end{aligned} \tag{3.26}$$

or

$$\begin{aligned} &m\int_V \alpha^{m-1}(h_{(2)ji} - s_1 h_{ji}) z^j z^i dV \\ &\quad + 2\binom{n}{2}\int_V \alpha^m (M_1 + \alpha M_2) dV + \int_V \alpha^m K_i z^i dV = 0. \end{aligned} \tag{3.27}$$

We also have

$$\begin{aligned} \nabla_i(\alpha^m z_{(2)}{}^i) &= -m\alpha^{m-1} h_{(3)ji} z^j z^i \\ &\quad + \alpha^m \left\{(\nabla_i p_1 - K_i) z_{(1)}{}^i + \left(\frac{1}{2}\nabla_i p_2 - K_{is}{}^r h_r{}^s\right) z^i + p_2 + \alpha p_3\right\}, \end{aligned} \tag{3.28}$$

$$\begin{aligned} \nabla_i(\alpha^m p_1 z_{(1)}{}^i) &= -m\alpha^{m-1} p_1 h_{(2)ji} z^j z^i \\ &\quad + \alpha^m \left\{(\nabla_i p_1) z_{(1)}{}^i + \left(\frac{1}{2}\nabla_i p_1{}^2 - p_1 K_i\right) z^i + p_1(p_1 + \alpha p_2)\right\}, \end{aligned} \tag{3.29}$$

$$\begin{aligned} \nabla_i\left\{\frac{1}{2}\alpha^m (p_1{}^2 - p_2) z^i\right\} &= -\frac{1}{2} m\alpha^{m-1} (p_1{}^2 - p_2) h_{ji} z^j z^i \\ &\quad + \alpha^m \left[\frac{1}{2}\{\nabla_i (p_1{}^2 - p_2)\} z^i + \frac{1}{2}(p_1{}^2 - p_2)(n + \alpha p_1)\right] \end{aligned} \tag{3.30}$$

by virtue of (3. 7), (3. 18), (3. 19) and (3. 20).

Integrating $-(3.28)+(3.29)-(3.30)$ over V, we find

$$(3.31)\quad \begin{aligned} & m\int_V \alpha^{m-1}\left\{h_{(3)ji}-p_1h_{(2)ji}+\frac{1}{2}(p_1^2-p_2)h_{ji}\right\}z^jz^idV \\ & -\frac{1}{2}\int_V \alpha^m\{(n-2)(p_1^2-p_2)+\alpha(p_1^3-3p_1p_2+2p_3)\}dV \\ & +\int_V \alpha^m(K_iz_{(1)}^i-p_1K_iz^i+K_{is}{}^rh_r{}^sz^i)dV=0 \end{aligned}$$

or

$$(3.32)\quad \begin{aligned} & m\int_V \alpha^{m-1}\{h_{(3)ji}-s_1h_{(2)ji}+s_2h_{ji}\}z^jz^idV \\ & -3\binom{n}{3}\int_V \alpha^m(M_2+\alpha M_3)dV+\int_V \alpha^m(K_iz_{(1)}^i-s_1K_iz^i+K_{is}{}^rh_r{}^sz^i)dV=0. \end{aligned}$$

More generally, we have

$$(3.33)\quad \begin{aligned} \nabla_i(\alpha^mz_{(l)}^i) = & -m\alpha^{m-1}h_{(l+1)ji}z^jz^i \\ & +\alpha^m\left\{(\nabla_ip_1-K_i)z_{(l-1)}^i+\left(\frac{1}{2}\nabla_ip_2-K_{is}{}^rh_r{}^s\right)z_{(l-2)}^i\right. \\ & +\cdots \\ & +\left(\frac{1}{l-1}\nabla_ip_{l-1}-K_{is}{}^rh_{(l-2)r}{}^s\right)z_{(1)}^i \\ & \left.+\left(\frac{1}{l}\nabla_ip_l-K_{is}{}^rh_{(l-1)r}{}^s\right)z^i+p_l+\alpha p_{l+1}\right\}, \end{aligned}$$

$$(3.34)\quad \begin{aligned} \nabla_i(\alpha^mp_1z_{(l-1)}^i) = & -m\alpha^{m-1}p_1h_{(l)ji}z^jz^i \\ & +\alpha^m\Big[(\nabla_ip_1)z_{(l-1)}^i \\ & +p_1\left\{(\nabla_ip_1-K_i)z_{(l-2)}^i+\left(\frac{1}{2}\nabla_ip_2-K_{is}{}^rh_r{}^s\right)z_{(l-3)}^i\right. \\ & +\cdots \\ & +\left(\frac{1}{l-2}\nabla_ip_{l-2}-K_{is}{}^rh_{(l-3)r}{}^s\right)z_{(1)}^i \\ & \left.+\left(\frac{1}{l-1}\nabla_ip_{l-1}-K_{is}{}^rh_{(l-2)r}{}^s\right)z^i+p_{l-1}+\alpha p_l\right\}\Big], \end{aligned}$$

$$(3.35)\quad \begin{aligned} \nabla_i\left\{\frac{1}{2}\alpha^m(p_1^2-p_2)z_{(l-2)}^i\right\} = & -\frac{1}{2}m\alpha^{m-1}(p_1^2-p_2)h_{(l-1)ji}z^jz^i \\ & +\frac{1}{2}\alpha^m\Big[\{\nabla_i(p_1^2-p_2)\}z_{(l-2)}^i \\ & +(p_1^2-p_2)\left\{(\nabla_ip_1-K_i)z_{(l-3)}^i+\left(\frac{1}{2}\nabla_ip_1-K_{is}{}^rh_r{}^s\right)z_{(l-4)}^i\right. \end{aligned}$$

$$
\begin{aligned}
&+\cdots\\
&+\left(\frac{1}{l-3}\nabla_i p_{l-3}-K_{is}{}^r h_{(l-4)r}{}^s\right)z_{(1)}{}^i\\
&+\left(\frac{1}{l-2}\nabla_i p_{l-2}-K_{is}{}^r h_{(l-3)r}{}^s\right)z^i+p_{l-2}+\alpha p_{l-1}\Big\}\Big],
\end{aligned}
$$

$$
\begin{aligned}
&\nabla_i\left\{\frac{1}{3!}\alpha^m(p_1{}^3-3p_1p_2+2p_3)z_{(l-3)}{}^i\right\}\\
&=-\frac{1}{3!}m\alpha^{m-1}(p_1{}^3-3p_1p_2+2p_3)h_{(l-2)ji}z^jz^i\\
&+\frac{1}{3!}\alpha^m\Big[\{\nabla_i(p_1{}^3-3p_1p_2+2p_3)\}z_{(l-3)}{}^i\\
&+(p_1{}^3-3p_1p_2+2p_3)\Big\{(\nabla_i p_1-K_i)z_{(l-4)}{}^i+\left(\frac{1}{2}\nabla_i p_1-K_{is}{}^r h_r{}^s\right)z_{(l-5)}{}^i \qquad (3.36)\\
&+\cdots\\
&+\left(\frac{1}{l-4}\nabla_i p_{l-4}-K_{is}{}^r h_{(l-5)r}{}^s\right)z_{(1)}{}^i\\
&+\left(\frac{1}{l-3}\nabla_i p_{l-3}-K_{is}{}^r h_{(l-4)r}{}^s\right)z^i+p_{l-3}+\alpha p_{l-2}\Big\}\Big],\\
&\cdots\cdots\cdots\cdots\cdots,
\end{aligned}
$$

$$
\begin{aligned}
&\nabla_i\sum_{\substack{t_1+2t_2+\cdots+lt_l=l\\0\leqq t_i}}\frac{(-1)^{t_1+\cdots+t_l+l}}{(t_1!)\cdots(t_l!)2^{t_2}\cdots l^{t_l}}\alpha^m p_1{}^{t_1}\cdots p_l{}^{t_l}z^i\\
&=\sum_{\substack{t_1+2t_2+\cdots+lt_l=l\\0\leqq t_i}}\frac{(-1)^{t_1+\cdots+t_l+l}}{(t_1!)\cdots(t_l!)2^{t_2}\cdots l^{t_l}}[-m\alpha^{m-1}p_1{}^{t_1}\cdots p_l{}^{t_l}h_{ji}z^jz^i \qquad (3.37)\\
&\qquad+\alpha^m\{(\nabla_i p_1{}^{t_1}\cdots p_l{}^{t_l})z^i+p_1{}^{t_1}\cdots p_l{}^{t_l}(n+\alpha p_1)\}].
\end{aligned}
$$

Integrating $-(3.33)+(3.34)-(3.35)+(3.36)-\cdots$ over V, we find

$$
\begin{aligned}
&m\int_V\alpha^{m-1}\{h_{(l+1)ji}-s_1h_{(l)ji}+s_2h_{(l-1)ji}-s_3h_{(l-2)ji}+\cdots\\
&\qquad+(-1)^{l-1}s_{l-1}h_{(2)ji}+(-1)^l s_l h_{ji}\}z^jz^i dV\\
&+(-1)^{l+1}(l+1)\binom{n}{l+1}\int_V\alpha^m(M_l+dM_{l+1})dV\\
&+\int_V[\alpha^m K_i(z_{(l-1)}{}^i-s_1z_{(l-2)}{}^i+s_2z_{(l-3)}{}^i-\cdots) \qquad (3.38)\\
&+K_{is}{}^r h_r{}^s(z_{(l-2)}{}^i-s_1z_{(l-3)}{}^i+s_2z_{(l-4)}{}^i-\cdots)\\
&+K_{is}{}^r h_{(2)r}{}^s(z_{(l-3)}{}^i-s_1z_{(l-4)}{}^i+s_2z_{(l-5)}{}^i-\cdots)\\
&+\cdots\cdots\cdots\cdots]dV=0.
\end{aligned}
$$

BIBLIOGRAPHY

[1] CHERN, S. S., Some formulas in the theory of surfaces. Bol. Soc. Mat. Mexicana **10** (1953), 30–40.
[2] HSIUNG, C. C., Some integral formulas for closed hypersurfaces. Math. Scand. **2** (1954), 286–294.
[3] HSIUNG, C. C., Some integral formulas for closed hypersurfaces in Riemannian space. Pacific J. Math. **6** (1956), 291–299.
[4] HSIUNG, C. C., AND J. K. SHAHIN, Affine differential geometry of closed hypersurfaces. Proc. London Math. Soc. (3) **17** (1967), 715–735.
[5] KATSURADA, Y., Generalized Minkowski formulas for closed hypersurfaces in a Riemann space. Annali di Mat. **57** (1962), 283–294.
[6] KATSURADA, Y., On a certain property of closed hyperface in an Einstein space. Comment. Math. Helv. **38** (1964), 165–171.
[7] LIEBMANN, H., Über die Verbiegung der geschlossenen Flächen positiver Krümmung. Math. Ann. **53** (1900), 91–112.
[8] SHAHIN, J. K., Some integral formulas for closed hypersurfaces in Euclidean space. Proc. Amer. Math. Soc. **19** (1968), 609–613.
[9] SÜSS, W., Zur relativen Differentialgeometrie. Tôhoku Math. J. **31** (1929), 202–209.
[10] TANI, M., On hypersurfaces with constant k-th mean curvature. Kōdai Math. Sem. Rep. **20** (1968), 94–102.
[11] YANO, K., Closed hypersurface with constant mean curvature in a Riemannian manifold. J. Math. Soc. Japan **17** (1965), 333–340.
[12] YANO, K., Notes on hypersurfaces in a Riemannian manifold. Canad. J. Math. **19** (1967), 439–445.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.