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ON SOME CONFORMAL EQUIVALENCE CONDITIONS OF COMPACT RIEMANN SURFACES

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The purpose of this paper is to obtain some conditions for two compact Riemann surfaces to be conformally equivalent. We shall mention our results by use of the Douglas-Dirichlet functional and harmonic mappings.

Let R and S be compact Riemann surfaces of genus g, and let $\eta=\rho(w)|dw|^2$ be a conformal metric on S, where $\rho(w)$ is positive and continuous with respect to each local parameter w on S. We call η a *normalized conformal metric* on S, if it satisfies

$$\iint_S \rho(w)dudv=1.$$

Let f be an orientation-preserving homeomorphism of R onto S. We assume that f is L_2-derivable, that is, $w=f(z)$ has generalized partial derivatives which are square integrable, where $w=f(z)$ is a local representation of f for local parameters z and w on R and S, respectively. Since f is orientation-preserving, we have

$$\left|\frac{\partial f}{\partial z}\right|^2-\left|\frac{\partial f}{\partial \bar{z}}\right|^2\geqq 0$$

almost everywhere in each parametric disk on R. Furthermore, it is known that f is a measurable mapping, and

$$\text{mes } f(E)=\iint_E\left(\left|\frac{\partial f}{\partial z}\right|^2-\left|\frac{\partial f}{\partial \bar{z}}\right|^2\right)dxdy$$

for any measurable set E on R (cf. [3]). The integral

$$I_\eta[f]=\iint_R \rho(f(z))\left(\left|\frac{\partial f}{\partial z}\right|^2+\left|\frac{\partial f}{\partial \bar{z}}\right|^2\right)dxdy$$

is called the *Douglas-Dirichlet integral.* If $\eta=\rho(w)|dw|^2$ is a normalized conformal metric on S, we have

$$I_\eta[f]-1=2\iint_R \rho(f(z))\left|\frac{\partial f}{\partial \bar{z}}\right|^2dxdy,$$

since

$$\iint_R \rho(f(z))\left(\left|\frac{\partial f}{\partial z}\right|^2-\left|\frac{\partial f}{\partial \bar{z}}\right|^2\right)dxdy=1.$$

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Consequently, when η is normalized, $I_\eta[f]\geqq 1$ for any mapping f, and equality holds if and only if f is a conformal mapping. So, the following question arises.

PROBLEM 1. Let Ω be a certain family of normalized conformal metrics on S, and let $\mathfrak{F}$ be a certain family of homeomorphisms of R onto S. We suppose that $\inf I_\eta[f]=1$ for all $\eta\in\Omega$ and for all $f\in\mathfrak{F}$. Then, are R and S conformally equivalent?

For a normalized conformal metric $\eta=\rho(w)|dw|^2$ on S, an orientation-preserving and L_2-derivable homeomorphism f of R onto S is called a harmonic mapping relative to η, if the quadratic differential

$$\left\{\rho(f(z))\frac{\partial f}{\partial z}\overline{\frac{\partial f}{\partial \bar{z}}}\right\}dz^2$$

on R is analytic (cf. [4], [6]). When a normalized conformal metric on S and a homotopy class α of orientation-preserving homeomorphisms of R onto S are arbitrarily given, there always exists a harmonic mapping relative to η which belongs to α (cf. [6]). We denote it by f_η, and we set

$$\varphi_\eta(z)=\rho(f_\eta(z))\frac{\partial f_\eta}{\partial z}\overline{\frac{\partial f_\eta}{\partial \bar{z}}}.$$

The quadratic differential $\varphi_\eta(z)dz^2$ on R is said to be attached to the harmonic mapping f_η. Clearly, f_η is conformal if and only if $\varphi_\eta(z)\equiv 0$. In the paper [6], it is proved that a harmonic mapping f_η is obtained as a homeomorphism which minimizes the Douglas-Dirichlet functional $I_\eta[f]$ in a family $\mathfrak{F}_{\gamma,M}$ of all orientation-preserving homeomorphisms f of R onto S satisfying the following conditions:

(i) f belongs to the homotopy class α,

(ii) f and f^{-1} are L_2-derivable,

(iii) $$\iint_R \rho(f(z))\left(\left|\frac{\partial f}{\partial z}\right|^2+\left|\frac{\partial f}{\partial \bar{z}}\right|^2\right)dxdy\leqq K+K^{-1}$$

for the maximal dilatation K of a fixed quasiconformal mapping belonging to α,

(iv) $$\iint_S \lambda(f^{-1}(w))\left(\left|\frac{\partial f^{-1}}{\partial w}\right|^2+\left|\frac{\partial f^{-1}}{\partial \bar{w}}\right|^2\right)dudv\leqq M(K+K^{-1})$$

for a positive constant M and a conformal metric $\gamma=\lambda(z)|dz|^2$ on R. In this paper, by a harmonic mapping f_η we shall mean the harmonic mapping which minimizes $I_\eta[f]$ in a certain family $\mathfrak{F}_{\gamma,M}$. Evidently,

$$(1)\qquad 1\leqq I_\eta[f_\eta]\leqq K+K^{-1},$$

and $I_\eta[f_\eta]=1$ if and only if f_η is conformal. Hence, we can consider the another problem:

PROBLEM 2. Let Ω be a certain family of normalized conformal metrics on S, and suppose that $\inf I_\eta[f_\eta]=1$ for all $\eta\in\Omega$. Then, are R and S conformally equivalent?

It is our aim to obtain some results about these problems.

When $g\geqq 2$, the universal covering surfaces of R and S are conformally equivalent to unit disks $U=[|z|<1\}$ and $V=\{|w|<1\}$, respectively. From now on, we consider U and V the universal covering surfaces of R and S, respectively, and denote by G and H the groups of cover transformations of U and V over R and S, respectively. G and H are properly discontinuous groups of linear transformations, and each element of them has no fixed point in U or V if it is not an identity. When a normalized conformal metric $\eta=\rho(w)|dw|^2$ is given, we can define a continuous function $\rho(w)$ on V such that

$$\rho(B(w))|B'(w)|^2=\rho(w) \qquad \text{for all} \quad B\in H.$$

If we set

$$m_\eta=\inf_{w\in V}\rho(w),$$

then m_η is positive obviously. For a positive constant δ, we denote by Ω_δ the family of all normalized conformal metrics η on S satisfying $m_\eta\geqq\delta$, and for a positive constant M, we denote by Ω_M^* the family of all normalized conformal metrics η on S satisfying $||\varphi_\eta||/m_\eta\leqq M$, where φ_η is the attached quadratic differential to f_η, and

$$||\varphi_\eta||=\iint_R|\varphi_\eta(z)|dxdy.$$

In view of $||\varphi_\eta||\leqq(1/2)I_\eta[f_\eta]$, (1) implies

$$||\varphi_\eta||\leqq\frac{1}{2}(K+K^{-1})$$

for all normalized conformal metrics η on S. So, for an arbitrary $\delta>0$, there exists a constant $M>0$ such as $\Omega_\delta\subset\Omega_M^*$.

A homeomorphism f of R onto S can be extended to a homeomorphism $w=f(z)$ of U onto V. Since $w=B(f(z))$ is also an extension of f for every $B\in H$, the extension of f is not unique. We know that there exists an isomorphism σ of G onto H for any extended homeomorphism $w=f(z)$ such that

$$f(A(z))=A^\sigma(f(z)) \qquad \text{for all} \quad A\in G,$$

where A^σ denotes the image of A by σ. Let f_1 and f_2 be two homeomorphisms of R onto S and let $w=f_i(z)$ be extensions of f_i $(i=1, 2)$. We denote by σ_i the isomorphisms of G onto H such that $f_i(A(z))=A^{\sigma_i}(f_i(z))$ for all $A\in G$ $(i=1, 2)$. It is well known that f_1 is homotopic to f_2 if and only if there exists an element $B\in H$ such that $A^{\sigma_1}=B\circ A^{\sigma_2}\circ B^{-1}$ for all $A\in G$ (cf. [1]).

We shall prove the following lemma about a family of harmonic mappings.

LEMMA. *Let M be a positive constant and let $\mathfrak{H}$ be a family of homeomorphisms $w=f_\eta(z)$ of U onto V for all $\eta\in\Omega_M^*$, where each $w=f_\eta(z)$ is an arbitrary extension of a harmonic mapping f_η in a fixed homotopy class. Then, $\mathfrak{H}$ is a normal family on U.*

Proof. It is sufficient to show that $\mathfrak{H}$ is equicontinuous on $|z|\leqq r_0$ for any r_0 with $0<r_0<1$. We fix an r such as $r_0<r<1$. By extending an attached quadratic differential $\varphi_\eta(z)dz^2$ to f_η, we can define an analytic function $\varphi_\eta(z)$ on U satisfying

$$\varphi_\eta(A(z))A'(z)^2=\varphi_\eta(z) \qquad \text{for all} \quad A\in G.$$

Since $||\varphi_\eta||/m_\eta\leqq M$ for all $\eta\in\Omega_M^*$, we see

$$m_\eta^{-1}\iint_P |\varphi_\eta(z)|dxdy\leqq M,$$

where P is a normal polygon of G. Consequently, functions $\varphi_\eta(z)/m_\eta$ are uniformly bounded on $|z|\leqq r$ for all $\eta\in\Omega_M^*$, because each $\varphi_\eta(z)$ is analytic on U, and $|z|\leqq r$ intersects only a finite number of normal polygons of G. By the inequality

$$|\varphi_\eta(z)|\geqq m_\eta\left|\frac{\partial f_\eta}{\partial \bar{z}}\right|^2$$

we see that functions $\partial f_\eta/\partial\bar{z}$ are uniformly bounded on $|z|\leqq r$ for all $\eta\in\Omega_M^*$. By means of generalized Green's formula, the following relation is derived;

$$f_\eta(z)=\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f_\eta(\zeta)}{\zeta-z}d\zeta-\frac{1}{\pi}\iint_{|\zeta|<r}\frac{f_{\eta,\bar\zeta}(\zeta)}{\zeta-z}d\xi d\eta \tag{2}$$

for all z with $|z|<r$ (cf. [3]). By this integral formula, and in view of the fact that functions $f_{\eta,\bar\zeta}$ are uniformly bounded on $|\zeta|<r$, we can easily obtain inequalities

$$|f_\eta(z_1)-f_\eta(z_2)|\leqq C(r_0)|z_1-z_2||\log|z_1-z_2||$$

for all $\eta\in\Omega_M^*$, and for arbitrary two points z_1 and z_2 in $|z|\leqq r_0$, where $C(r_0)$ is a constant dependent only on r_0. Therefore, the family $\mathfrak{H}$ is equicontinuous on $|z|\leqq r_0$.

As a result concerning Problem 2, we shall prove the following theorem by use of the above lemma.

THEOREM 1. *Let R and S be two compact Riemann surfaces which are topologically equivalent, and suppose that for a constant M*

$$\inf_{\eta\in\Omega_M^*}\frac{I_\eta[f_\eta]-1}{m_\eta}=0,$$

where f_η is a harmonic mapping relative to η in a fixed homotopy class. Then, R and S are conformally equivalent.

Proof. By assumption, there exist a sequence $\eta_n \in \Omega_M^*$ and a sequence f_{η_n} of harmonic mappings in the fixed homotopy class α, such that

$$\lim_{n\to\infty}\frac{I_{\eta_n}[f_{\eta_n}]-1}{m_{\eta_n}}=0.$$

We put $\eta_n=\rho_n(w)|dw|^2$, and denote by P and Q fixed normal polygons of G and H, respectively. Their closures $\bar{P}$ and $\bar{Q}$ are compact, for R and S are compact. By $w=f_{\eta_n}(z)$ we denote the extension of f_{η_n} such as $f_{\eta_n}(0)\in\bar{Q}$. From

$$I_{\eta_n}[f_{\eta_n}]-1=2\iint_P \rho_n(f_{\eta_n}(z))\left|\frac{\partial f_{\eta_n}}{\partial\bar{z}}\right|^2 dxdy$$

it follows that

$$\iint_P\left|\frac{\partial f_{\eta_n}}{\partial\bar{z}}\right|^2 dxdy \leqq \frac{1}{2}\,\frac{I_{\eta_n}[f_{\eta_n}]-1}{m_{\eta_n}},$$

hence

$$\lim_{n\to\infty}\iint_P\left|\frac{\partial f_{\eta_n}}{\partial\bar{z}}\right|^2 dxdy=0.$$

Thus we may assume that

$$\lim_{n\to\infty}\frac{\partial f_{\eta_n}}{\partial\bar{z}}=0 \qquad \text{a. e. on } U \tag{3}$$

by taking a subsequence if necessary. By Lemma there exists a subsequence of $\{f_{\eta_n}(z)\}$ which converges uniformly in the wider sense on U. Let $f(z)$ be the limit function. We may assume that

$$f_{\eta_n}(z)\to f(z) \qquad \text{uniformly in the wider sense on } U. \tag{4}$$

Now we fix an r with $0<r<1$. Since $\eta_n\in\Omega_M^*$, the sequence $\{\partial f_{\eta_n}/\partial\bar{z}\}$ is uniformly bounded on $|z|<r$. Hence, by (3) and Lebesgue's dominated-convergence theorem, we find

$$\lim_{n\to\infty}\iint_{|\zeta|<r}\frac{f_{\eta_n,\bar{\zeta}}(\zeta)}{\zeta-z}\,d\xi d\eta=0 \tag{5}$$

for every z with $|z|<r$. By (2), (4) and (5) we obtain

$$f(z)=\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(\zeta)}{\zeta-z}\,d\zeta$$

on $|z|<r$. This implies that $f(z)$ is analytic on $|z|<r$. Since r is arbitrary, $f(z)$ is analytic on U.

Conditions $f_{\eta_n}(0)\in\bar{Q}$ yield $f(0)\in\bar{Q}$. Therefore, we can conclude that $f(z)$ is not constant. In fact, if $f(z)$ is a constant, we can show that its absolute value is 1.

Suppose that $f(z)$ is a constant c with $|c|<1$, and let σ_n be an isomorphism of G onto H such that

$$f_{\eta_n}(A(z))=A^{\sigma_n}(f_{\eta_n}(z)) \qquad \text{for all } A\in G. \tag{6}$$

We fix an element A of G which is not an identity. By choosing a subsequence if necessary, we may assume that $A^{\sigma_n}(w)$ tends to a function $B(w)$ uniformly in the wider sense on V. If $B(w)$ is a constant, we may set $B(w)=e^{i\theta}$, where θ is a real constant. Letting n tend to infinity in (6), we have $c=e^{i\theta}$, which is a contradiction. Consequently, $B(w)$ must be a linear transformation which maps V onto itself. Then by the discontinuity of H, A^{σ_n} are identical with B for all sufficiently large n. Therefore, by letting n tend to infinity in (6), we have $c=B(c)$. This shows that an element B of H which is not an identity has a fixed point c in V, which is a contradiction.

Since A^{σ_n} are identical with an element B of H for each element A of G if n is sufficiently large, we can define the correspondence σ: $A\rightarrow B$. By use of the maximum principle, we see that $|f(z)|<1$ on every compact subset of U. Therefore, it is easily proved that σ is an isomorphism of G onto H. Furthermore, it follows from (6) that

$$f(A(z))=A^{\sigma}(f(z)) \qquad \text{for all } A\in G. \tag{7}$$

Now we shall show that $w=f(z)$ is a mapping of U onto V. For every $w_0\in V$ and for every n, there exist $w_n\in\overline{f_{\eta_n}(P)}$ and $T_n\in H$ such as $w_n=T_n(w_0)$, because the set $f_{\eta_n}(P)$ is a fundamental domain of H. By the maximum principle, we see that the set $\overline{f(P)}=f(\bar{P})$ is compact. Accordingly, the sequence $\{w_n\}$ is contained in a compact subset of V, for the set $\overline{f_{\eta_n}(P)}$ tends to the compact set $\overline{f(P)}$. Hence, if we denote by w^* an accumulation point of $\{w_n\}$, then $|w^*|<1$, and w^* belongs to to $f(\bar{P})$. We may assume that w_n tends to w^*. Moreover, we may assume that $T_n(w)$ tends to a function $T(w)$ uniformly in the wider sense on V. Then, $w^*=T(w_0)$, consequently, $T(w)$ is non-constant. Therefore, $T(w)$ is a linear transformation which maps V onto itself. By the discontinuity of H, T_n are equal to T for all sufficiently large n. Hence, T belongs to H. Let A be an element of G such as $A^{\sigma}=T^{-1}$, and let z^* be a point of $\bar{P}$ such as $w^*=f(z^*)$. If we set $z_0=A(z^*)$, then, by (7) we obtain $f(z_0)=w_0$. Thus $w=f(z)$ is a mapping of U onto V.

In order to prove the univalence of $f(z)$, we suppose that $w_0=f(z_1)=f(z_2)$ for two distinct points z_1, z_2 in U. Since $f(z)$ is analytic, if we take a sufficiently small disk D about w_0, there exist two disjoint connected components N_1 and N_2 of the inverse image $f^{-1}(D)$ which contain z_1 and z_2, respectively. Since $f_{\eta_n}(z)$ tends to $f(z)$ uniformly on $\bar{N}_j$, the image of the boundary of N_j by f_{η_n} tends to the boundary of D $(j=1, 2)$. As a consequence, the image $f_{\eta_n}(N_j)$ contains w_0 for a sufficiently large n. Therefore, there exist two points $\zeta_j\in N_j$ $(j=1, 2)$ such as $f_{\eta_n}(\zeta_j)=w_0$. Evidently $\zeta_1\neq\zeta_2$, and so, this contradicts with the univalence of $f_{\eta_n}(z)$.

We have just proved that $w=f(z)$ is a conformal mapping of U onto V satisfying (7). Therefore, it induces a conformal mapping f of R onto S which belongs to the homotopy class α. Thus R and S are conformally equivalent.

Immediately we obtain the following corollary, since for any $\delta>0$ there exists a constant $M>0$ such as $\Omega_\delta\subset\Omega_M^*$.

COROLLARY. *Under the same conditions as Theorem* 1, *if*

$$\inf_{\eta\in\Omega_\delta} I_\eta[f_\eta]=1$$

for a constant $\delta>0$, *then* R *and* S *are conformally equivalent.*

We will incidentally state the following result.

THEOREM 2. *Let* Ω *be the family of all normalized conformal metrics on* S, *and suppose that*

$$\inf_{\eta\in\Omega}\frac{||\varphi_\eta||}{m_\eta}=0,$$

where φ_η *is the attached quadratic differential to a harmonic mapping* f_η *belonging to a fixed homotopy class. Then,* R *and* S *are conformally equivalent.*

Proof. We take a sequence $\eta_n\in\Omega$ such that $||\varphi_{\eta_n}||/m_{\eta_n}$ tends to zero. Evidently, all η_n belong to Ω_M^* for a constant M. Under the same notation as in the proof of Theorem 1, we may assume, by Lemma, that $f_{\eta_n}(z)$ converges uniformly in the wider sense on U. The sequence $\{\partial f_{\eta_n}/\partial\bar{z}\}$ is uniformly bounded on $|z|<r$ for any r with $0<r<1$, and we can assume that it tends to zero almost everywhere on U. Accordingly, the proof left over follows the same lines.

Concerning Problem 1, we have the following theorem.

THEOREM 3. *Let* $\mathfrak{F}$ *be the family of all orientation-preserving homeomorphisms of* R *onto* S *which are* L_2*-derivable and belong to a fixed homotopy class. If*

$$\inf_{\eta\in\Omega_\delta,\, f\in\mathfrak{F}} I_\eta[f]=1$$

for a positive constant δ, *then* R *and* S *are conformally equivalent.*

Proof. If we take sequences $\eta_n\in\Omega_\delta$ and $f_n\in\mathfrak{F}$, such that $I_{\eta_n}[f_n]$ tends to 1, we obtain

$$\lim_{n\to\infty}\iint_P \rho_n(f_n(z))\left|\frac{\partial f_n}{\partial\bar{z}}\right|^2 dxdy=0, \tag{8}$$

where $\eta_n=\rho_n(w)|dw|^2$.

Let $\{A_1, B_1, \cdots, A_g, B_g\}$ be a canonical homology basis on R. If we set $A_j^*=f(A_j)$, $B_j^*=f(B_j)$ $(j=1, 2, \cdots, g)$ for a fixed mapping f belonging to $\mathfrak{F}$, then $\{A_1^*, B_1^*, \cdots, A_g^*, A_g^*\}$ is a canonical homology basis on S. We denote by $\omega_j=\theta_j(z)dz$ $(j=1, 2, \cdots, g)$

and $\omega_j^* = \theta_j^*(w)dw$ $(j=1, 2, \cdots, g)$ the normalized bases of the linear spaces of all abelian differentials of the first kind on R and S belonging to the canonical homology bases above mentioned, respectively. By definition, they satisfy

$$\int_{A_k} \omega_j = \delta_{jk}, \qquad \int_{A_k^*} \omega_j^* = \delta_{jk} \qquad (j, k=1, 2, \cdots, g).$$

Furthermore, π_{jk} and π_{jk}^* denote the periods of ω_j and ω_j^* over B_k and B_k^*, respectively. We denote by $\alpha_j^{(n)}$ the transplant of ω_j^* by the mapping f_n, that is,

$$\alpha_j^{(n)} = \theta_j^*(f_n(z))\left(\frac{\partial f_n}{\partial z} dz + \frac{\partial f_n}{\partial \bar{z}} d\bar{z}\right).$$

Since all f_n belong to the same homotopy class, by Riemann's period relation we find

$$\begin{aligned}
\pi_{jk}^* - \pi_{jk} &= \int_{B_k^*} \omega_j^* - \int_{B_k} \omega_j \\
&= \int_{f_n(B_k)} \omega_j^* - \int_{B_k} \omega_j \\
&= \int_{B_k} \alpha_j^{(n)} - \int_{B_k} \omega_j \\
&= \sum_{\nu=1}^{g} \left[\int_{A_\nu} \omega_k \int_{B_\nu} \alpha_j^{(n)} - \int_{A_\nu} \alpha_j^{(n)} \int_{B_\nu} \omega_k\right] \\
&= \iint_R \alpha_j^{(n)} \wedge \omega_k.
\end{aligned}$$

Thus we have obtained relations

$$\pi_{jk}^* - \pi_{jk} = 2i \iint_P \theta_k(z)\theta_j^*(f_n(z)) \frac{\partial f_n}{\partial \bar{z}} dxdy \qquad (j, k=1, 2, \cdots, g). \tag{9}$$

Here, we remark that $|\theta_j^*(w)|^2/\rho_n(w)$ are automorphic functions on V with respect to the group H, and that they are uniformly bounded on $\bar{Q}$ for all n, since $\bar{Q}$ is compact and $\rho_n(w) \geqq \delta$ on V for all n. Because of automorphic property, they are uniformly bounded on V. Therefore, by Schwarz' inequality it follows from (9) that

$$|\pi_{jk}^* - \pi_{jk}| \leqq C\left[\iint_P \rho_n(f_n(z)) \left|\frac{\partial f_n}{\partial \bar{z}}\right|^2 dxdy\right]^{1/2},$$

where C is a constant independent of n. Hence, by (8) we get

$$\pi_{jk}^* = \pi_{jk} \qquad (j, k=1, 2, \cdots, g).$$

Thus, by using Torelli's theorem (cf. [2], [5]), we can conclude that R and S are conformally equivalent.

When $g=1$, we can prove in the same way all the results we have mentioned by taking the whole planes as U and V. Furthermore, when R and S are compact bordered Riemann surfaces which are topologically equivalent, we can also prove the similar results by taking their doubles.

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