KŌDAI MATH. SEM. REP.
20 (1967), 155—158

ON RIGID ANALYTIC MAPPINGS AMONG SURFACES $\{e^w = f(z)\}$

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1. Introduction. Let R be an open Riemann surface (z, w) defined by

$$e^w = f(z)$$

with an entire function $f(z)$ which has no zeros other than an infinite number of simple zeros. For the topological structure of the surface R we can refer to a paper due to Heins [1]. Let $\mathfrak{p}_R$ be the projection map $(z, w) \to z$. Let S be another such surface defined by $e^W = g(Z)$. Consider a non-trivial analytic mapping φ of R into S, which satisfies the following rigidity condition:

$$\mathfrak{p}_S \varphi(p) = \mathfrak{p}_S \varphi(q) \qquad \text{whenever} \qquad \mathfrak{p}_R p = \mathfrak{p}_R q.$$

Let D_R be the domain in which $f(z) \neq 0$ and E_R the set of zeros of $f(z)$. Evidently $D_R = \{|z| < \infty\} - E_R$. Let $h(z) = \mathfrak{p}_S \circ \varphi \circ \mathfrak{p}_R^{-1}(z)$, then $h(z)$ is a single-valued regular function in D_R, whose image $h(D_R)$ lies in D_S. In the present paper we shall prove the following theorems.

THEOREM 1. *Let φ be a non-trivial rigid analytic mapping of R into S, then the corresponding h is a polynomial and φ is onto.*

THEOREM 2. *Let φ be a non-trivial rigid analytic mapping of R into itself, then the corresponding h is of the following form $e^{2\pi i r} z + \beta$ with a suitable rational number r and φ reduces to a one-to-one conformal mapping of R onto itself.*

2. Proof of Theorem 1. Assume that $h(z)$ has an essential singularity at a point z^* of E_R. Then in an arbitrary small neighborhood of the point z^* $h(z)$ takes every value infinitely often excepting at most two. Hence all the points of E_S excepting at most two are taken by $h(z)$. This contradicts the into-ness of $h(z)$. Thus there is no essential singularity of h at E_R. The same is true for $z = \infty$. Hence $h(z)$ must be a rational function of z in the z-sphere. Next we prove that $h(z)$ is a polynomial of z.

Assume that $h(z)$ has a pole at some point in the finite z-plane. Then a fixed neighborhood of this point is mapped around $Z = \infty$ by $h(z)$ and its image by $h(z)$ contains a neighborhood of $Z = \infty$, which contains at least a point of E_S. This contradicts the into-ness of $h(z)$ in D_R. This implies that $h(z)$ is a polynomial.

Received September 25, 1967.

Next we prove the onto-ness of φ. Suppose that φ is not onto. Evidently $h(z)$ is a mapping of the z-sphere onto the Z-sphere with a constant finite valence. If $h(z)$ is a mapping of D_R onto D_S, then there is a point P of S such that φ does not cover P but h does cover its projection $\mathfrak{p}_S$P. Then there is a point Q such that $\varphi(q)=$Q, $\mathfrak{p}_S$Q$=\mathfrak{p}_S$P. On S we join P and Q by a suitable curve C and make its projection $\mathfrak{p}_S C$. $\mathfrak{p}_S C$ is a closed curve joining $\mathfrak{p}_S$P with itself. There is a curve c which starts from $\mathfrak{p}_R q$ and ends to a point t whose image by h is $\mathfrak{p}_S$P. Now t does not belong to E_R by the onto-ness of h. Then we can construct the curve $\tilde{c}$ whose projection is c and whose starting point is q. Then φ can be continued along $\tilde{c}$ to the end point. This means that P is covered by $\varphi(R)$. If $h(z)$ is a mapping of D_R into D_S, then there is a point z^* of E_R such that $Z^*\equiv h(z^*)\notin E_S$. Consider the set of counter-images $\{z_\mu^*\}_{\mu-1}$, $z_\mu^*=h^{-1}(Z^*)$, $z_1{}^*=z^*$ of Z^*. Really this is a finite set. All the z_μ^* $(\mu=1, \cdots, \nu)$ must belong to E_R. Consider the set of small neighborhoods n_j of z_j^* such that $h(n_j)$ covers just the same neighborhood $N(Z^*)-Z^*$ of Z^* and $n_j\cap n_k=\phi$ for $j\neq k$. We can make $N(Z^*)$ a sufficiently small disc. Then consider $\mathfrak{p}_S^{-1}N(Z^*)=\tilde{N}(Z^*)$. This consists of an infinite number of disjoint discs $K_1, K_2, \cdots$. Since φ is analytic, $\varphi\circ\mathfrak{p}_R^{-1}(n_j)$ must be connected. Hence $\varphi\circ\mathfrak{p}_R^{-1}(n_j)$ lies in a single K_j. There remains still an infinite number of discs. Take such a disk K_n. If every point of K_n is not covered by $\varphi(R)$, then this point must be a point of S over a point in $h(E_R)$. Then we can find a point which is near from that point and is covered by $\varphi(R)$. We have already taken all the counter-images $h^{-1}(N(Z))$, ν in number. If there is another disc lying over $N(Z^*)$ which has a point covered by $\nu(R)$, the number of $h^{-1}(N(Z^*))$ must be greater than ν. This contradicts the definition of ν.

3. **Proof of Theorem 2.** By theorem 1 $h(z)$ must be a polynomial and a mapping of D_R onto itself. Let d be the degree of $h(z)$. Suppose $d\geqq 2$. Consider the solutions of $h(z)=z_j$, $z_j\in E_R$. Then every solution belongs to E_R. If $|z_j|\leqq R_0$ for a sufficiently large R_0, the solution satisfies the same inequality. Making these processes for some z_j successively then, the successive solutions make a bounded infinite set. This implies that E has at least one cluster point in a bounded part of the z-sphere. This is a contradiction. Hence $d=1$, that is, $h(z)=\alpha z+\beta$. If $\alpha\neq e^{2\pi i r}$ with any rational number r, we make the iterations of h. Then we have some cluster point of E_R in a bounded part of the z-plane. This is a contradiction.

4. We do not have any effective method in order to decide whether there is a non-rigid analytic mapping of R into S or not. Now we shall consider a simple case. Let R be the surface (z, w) defined by $e^w=z^3+az+b$ with two constant a and b. We here assume that $a\neq 0$. Let S be the surface (Z, W) defined by $e^W=Z^3+AZ+B$ with $B\neq 0$. Then R and S are two three-sheeted algebroid surfaces over the w-plane and the W-plane, respectively. Let $\mathfrak{p}_S^*\circ\varphi\circ\mathfrak{p}_R^{*-1}=h^*$ is a single-valued function [3], [6]. However the rigidity in this sense is not the rigidity defined in No. 1. Anyhow we have the following condition:

$$D_S\circ h^*(w)=D_R[f_1{}^3+af_1f_2{}^2-(e^w-b)f_2{}^3]^2,$$

where $D_S=27(e^w-B)^2+4A^3$, $D_R=27(e^w-b)^2+4a^3$. Hence

$$27(e^{h^*(w)}-B)^2+4A^3=[27(e^w-b)^2+4a^3][f_1{}^3+af_1f_2{}^2-(e^w-b)f_2{}^3]^2.$$

Assume $a=0$. Then $A=0$ and vice versa. In this case we have $a=0$, $A=0$. Then R and S are regularly branched three-sheeted. By an earlier result in [4] there exists a suitable entire function $f(w)$ satisfying either $e^{h^*(w)}-B=f(w)^3(e^w-b)$ or $e^{h^*(w)}-B=f(w)^3(e^w-b)^2$. In the second case we have a contradiction by considering the set of simple zeros. In the first case by [2] or [5][1)] we have $h^*(w)=\alpha w+\beta$, $|\alpha|=1$. Consider the sets of zeros of e^w-b and $e^{\alpha w}-e^{-\beta}B$, that is, $\{\log b+2n\pi i\}$, $\{\alpha^{-1}(-\beta+\log B+2n\pi i)\}$. These two sets must be coincide with each other. Hence

$$\alpha(\log b+2n\pi i)+\beta=\log B+2m\pi i.$$

Thus $\alpha=\pm1$.

Assume $aA\neq0$. If $27B^2+4A^3=0$ and $27b^2+4a^3=0$, then we have

$$27e^{h^*(w)}(e^{h^*(w)}-2B)=27e^w(e^w-2b)(f_1{}^3+af_1f_2{}^2-(e^w-b)f_2{}^3)^2.$$

By this equation we have $h^*(w)=\alpha w+\beta$, $|\alpha|=1$. Then by the same argument we have $\alpha=\pm1$. If $27B^2+4A^3=0$ and $27b^2+4a^3\neq0$, then

$$27e^{h^*(w)}(e^{h^*(w)}-2B)=[27(e^w-b)^2+4a^3][f_1{}^3+af_1f_2{}^2-(e^w-b)f_2{}^3]^2.$$

By this equation we have $h^*(w)=\alpha w+\beta$. The set of zeros of $(e^w-b+2ia^{3/2}/3\sqrt{3})(e^w-b-2ia^{3/2}/3\sqrt{3})$ coincides with that of $e^{\alpha w}-2Be^{-\beta}$. Then $|\alpha|=2$. This implies that the distance of two successive zeros must be equal to π. But this is not the case unless $b=0$. This is a contradiction. If $27B^2+4A^3\neq0$ and $27b^2+4a^3=0$, then

$$27(e^{h^*(w)}-B)^2+4A^3=27e^w(e^w-2b)[f_1{}^3+af_1f_2{}^2-(e^w-b)f_2{}^3]^2.$$

By this equation we have $h^*(w)=\alpha w+\beta$. Consider the set of zeros of e^w-2b. Then $|\alpha|=1/2$. In order that the minimum distance of two zeros of $(e^{\alpha w+\beta}-B)^2-4A^3/27$ is equal to 2π, B must be equal to zero, which is a contradiction. If $(27B^2+4A^3)(27b^2+4a^3)\neq0$, then $h^*(w)=\alpha w+\beta$, $|\alpha|=1$. In this case we have $\alpha=\pm1$.

Summing up these results we have the desired rigidity of φ with respect to $\mathfrak{p}_R$ and $\mathfrak{p}_S$. Indeed $\mathfrak{p}_Rp=\mathfrak{p}_Rq$, $p=(z, w)$, $q=(z, w')$ imply $w'=w+2n\pi i$ and $\mathfrak{p}_S\varphi(p)=\mathfrak{p}_S\varphi(q)$, $\varphi(p)=(Z, W)$, $\varphi(q)=(Z, W')$ imply $W'=W+2m\pi i$. And further $h^*(w)=\pm w+\beta$ implies $W'-W=\pm(w'-w)$ and hence $W'-W=\pm2n\pi i$ whenever $w'-w=2n\pi i$. This is nothing but the rigidity of φ with respect to $\mathfrak{p}_R$ and $\mathfrak{p}_S$.

THEOREM 3. *Let R and S be three-sheeted surfaces defined by*

$$y^3+ay+b=e^x \quad \textit{and} \quad Y^3+AY+B=e^X, \quad Bb\neq0,$$

1) In [5] we proved several estimations of the N-function of a composed function. In these estimations we used the second fundamental theorem erroneously. Main theorem was proved in [2] correctly. In our present case we can use our estimations in [5]. Indeed $|w_j-w_k|\geqq2\pi$ for any two roots of $e^w-b=0$.

respectively. If there is a non-trivial analytic mapping φ of R into S, then φ is rigid in the sense of No. 1.

REFERENCES

[1] HEINS, M., Riemann surfaces of infinite genus. Ann. of Math. **55** (1952), 296–317.
[2] HIROMI, G., AND H. MUTŌ, On the existence of analytic mappings, I. Kōdai Math. Sem. Rep. **19** (1967), 236–244.
[3] HIROMI, G., AND H. MUTŌ, On the existence of analytic mappings, II. Kōdai Math. Sem. Rep. **19** (1967), 439–450.
[4] MUTŌ, H., On the existence of analytic mappings. Kōdai Math. Sem. Rep. **18** (1966), 24–35.
[5] OZAWA, M., On the existence of analytic mappings, II. Kōdai Math. Sem. Rep. **18** (1966), 1–7.
[6] OZAWA, M., On analytic mappings among three-sheeted surfaces. Kōdai Math. Sem. Rep. **20** (1968), 146–154.

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