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ON FINITE MODIFICATIONS OF TWO- OR THREE-SHEETED COVERING OPEN RIEMANN SURFACES

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§1. Introduction.

1. Let $\mathfrak{M}(R)$ be the family of non-constant analytic functions meromorphic on an open Riemann surface R. We denote by $P(f)$ the number of exceptional values of $f\in\mathfrak{M}(R)$ and put $P(R)=\sup\{P(f)|f\in\mathfrak{M}(R)\}$. The quantity $P(R)$ was introduced by Ozawa [6] in discussing the existence of analytic mappings between Riemann surfaces.

2. In [10], Ozawa introduced the notion of a finite modification $\tilde{R}$ of R and obtained two interesting results on analytic mappings of R into $\tilde{R}$.

Let R and $\tilde{R}$ be two ultrahyperelliptic surfaces defined by two equations $y^2=G(z)$ and $y^2=\tilde{G}(z)$, where $G(z)$ and $\tilde{G}(z)$ are two entire functions having no zero other than an infinite number of simple zeros respectively. If $G(z)$ and $\tilde{G}(z)$ have the same zeros for $|z|\geqq r_0$ for a suitable r_0, then we call $\tilde{R}$ as a *finite modification of* R (cf. Ozawa [6]). Let S be another ultrahyperelliptic surface and $\tilde{S}$ be its finite modification.

In the present paper we shall consider the following two problems:

(A) What is $P(\tilde{R})$, if $P(R)=4$?

(B) When is there any analytic mapping of R (or $\tilde{R}$) into $\tilde{S}$, if there exists an analytic mapping of R into S?

We shall discuss the problem (A) in **4—6** and the problem (B) with respect to R and S with $P(R)=P(S)=4$ in **7—9**.

In **10—12** we shall consider the similar problems (A), (B) for regularly branched three-sheeted covering Riemann surfaces.

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§2. Lemma.

3. In order to discuss the problems (A) and (B), we shall use a lemma proved in our previous paper [5], that is,

LEMMA A. *Let $a_0(z), a_1(z), \cdots, a_n(z)$ be meromorphic functions and $g_1(z), \cdots, g_n(z)$ be entire functions. Further suppose that*

$$T(r, a_j)=o(m(r, e^{g_\nu}))$$

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and

$$T(r, a_j)=o(m(r, e^{g_1-g_\nu})),\ j=0, 1, \cdots, n;\ \nu=k,\ k+1, \cdots, n,$$

outside a set of finite measure. If $a_1(z)\not\equiv 0$ *and the identity*

$$\sum_{\nu=1}^{n} a_\nu(z)e^{g_\nu(z)}=a_0(z)$$

holds, then we have

$$\sum_{\nu=1}^{k-1} c_\nu a_\nu(z)e^{g_\nu(z)}+c_0a_0(z)=0,$$

where $c_1=1$ *and* c_ν, $\nu=0, 2, 3, \cdots, k-1$, *are suitable constants.*

§ 3. Ultrahyperelliptic surfaces.

4. Picard's constant. Now we shall consider the problem (A). Let R be an ultrahyperelliptic surface with $P(R)=4$ defined by the equation $y^2=G(z)$, where $G(z)$ is an entire function having no zero other than an infinite number of simple zeros. Then by virtue of Ozawa's theorem [7], we have

$$F(z)^2G(z)=(e^{H(z)}-\alpha)(e^{H(z)}-\beta),\qquad \alpha\beta(\alpha-\beta)\neq 0,\ H(0)=0, \tag{4.1}$$

where $F(z)$ is a suitable entire function and $H(z)$ is a non-constant entire function. Let $\tilde{R}$ be a finite modification of R defined by the equation $y^2=\tilde{G}(z)$ with $\tilde{G}(z)=Q(z)G(z)$, where $Q(z)$ has the following form:

$$\prod_{j=1}^{\lambda}(z-a_j)\prod_{j=1}^{\mu}\frac{1}{z-b_j},\qquad \lambda+\mu\geq 1, \tag{4.2}$$

where a_j and b_j are mutually distinct constants and their moduli are less than r_0. First we shall prove the following theorems:

THEOREM 1. *Let R be an ultrahyperelliptic surface and $\tilde{R}$ be its finite modification. If $P(R)=4$, then we have $P(\tilde{R})=2$ or* 3.

THEOREM 2. *Let R be an ultrahyperelliptic surface with $P(R)=4$ defined by the equation $y^2=G(z)$, where $G(z)$ satisfies the equation* (4.1). *If $H(z)$ is a polynomial, then we have $P(\tilde{R})=2$ for its finite modification $\tilde{R}$.*

5. *Proof of Theorem* 1. In order to prove $P(\tilde{R})=2$ or 3, from Ozawa's theorem [7], it is sufficient to show the impossibility of an identity of the form

$$\begin{aligned} &f(z)^2(e^{H(z)}-\alpha)(e^{H(z)}-\beta)=P(z)(e^{L(z)}-\gamma)(e^{L(z)}-\delta), \\ &P(z)=1/Q(z),\quad L(z)\not\equiv\text{const.},\quad L(0)=0,\quad \gamma\delta(\gamma-\delta)\neq 0, \end{aligned} \tag{5.1}$$

where $L(z)$ is an entire function and $f(z)$ is a meromorphic function which has zeros and poles possibly at the multiple zeros of $(e^L-\gamma)(e^L-\delta)$ and $Q(e^H-\alpha)(e^H-\beta)$. We claim from the reasoning in [3] that

$$\begin{aligned} &m(r, e^H)\sim m(r, e^L), \\ &T(r, f'/f)=o(m(r, e^H))=o(m(r, e^L)) \end{aligned} \tag{5.2}$$

outside a set of finite measure. By differentiating both sides of (5.1), setting $\zeta_1=-(\alpha+\beta)$, $\zeta_2=\alpha\beta$, $\eta_1=-(\gamma+\delta)$, $\eta_2=\gamma\delta$ and eliminating f^2, we obtain

$$
\begin{aligned}
&(2f'/f+2H'-2L'-P'/P)e^{2H+2L}+\eta_1(2f'/f+2H'-L'-P'/P)e^{2H+L}\\
&+\zeta_1(2f'/f+H'-2L'-P'/P)e^{H+2L}+\eta_2(2f'/f+2H'-P'/P)e^{2H}\\
&+\zeta_1\eta_1(2f'/f+H'-L'-P'/P)e^{H+L}+\zeta_2(2f'/f-2L'-P'/P)e^{2L}\\
&+\zeta_1\eta_2(2f'/f+H'-P'/P)e^{H}+\zeta_2\eta_1(2f'/f-L'-P'/P)e^{L}\\
&+\zeta_2\eta_2(2f'/f-P'/P)=0.
\end{aligned}
\tag{5.3}
$$

Here we remark from (5.2) that all functions $\nu H(z)+\mu L(z)$, $|\nu|\neq|\mu|$; $\nu,\mu=\pm1,\pm2$, are not constants and further satisfy $T(r,a)=o(m(r,e^{\nu H+\mu L}))$ outside a set of finite measure, where $a(z)$ is a meromorphic function satisfying $T(r,a)=o(m(r,e^{H}))$ outside a set of finite measure.

In the first place assume that $a_1(z)\equiv2f'(z)/f(z)+2H'(z)-2L'(z)-P'(z)/P(z)\not\equiv0$. From (5.2) and by the above remark we can apply lemma A to the identity (5.3). Therefore lemma A leads to

$$a_1(z)e^{2H(z)+2L(z)}+c_2a_2(z)e^{H(z)+L(z)}+c_0a_0(z)=0,$$

where c_2, c_0 are constants and $a_2(z)=\zeta_1\eta_1(2f'(z)/f(z)+H'(z)-L'(z)-P'(z)/P(z))$, $a_0(z)=\zeta_2\eta_2(2f'(z)/f(z)-P'(z)/P(z))$. The above identity and the fact $T(r,a_j)=o(m(r,e^{H}))\,(j=1,2)$ outside a set of finite measure offer $m(r,e^{H+L})=o(m(r,e^{H}))$ outside a set of finite measure. Since $\eta_2\neq0$, writing the identity (5.3) in the form

$$
\begin{aligned}
&\eta_2(2f'/f+2H'-P'/P)e^{2H}+[\eta_1(2f'/f+2H'-L'-P'/P)e^{H+L}\\
&+\zeta_1\eta_2(2f'/f+H'-P'/P)]e^{H}+[(2f'/f+2H'-2L'-P'/P)e^{2H+2L}\\
&+\zeta_1\eta_1(2f'/f+H'-L'-P'/P)e^{H+L}+\zeta_2\eta_2(2f'/f-P'/P)]\\
&+[\zeta_1(2f'/f+H'-2L'-P'/P)e^{2H+2L}+\zeta_2\eta_1(2f'/f-L'-P'/P)e^{H+L}]e^{-H}\\
&+[\zeta_2(2f'/f-2L'-P'/P)e^{2H+2L}]e^{-2H}=0,
\end{aligned}
$$

the impossibility of Borel's identity gives

$$2f'(z)/f(z)+2H'(z)-P'(z)/P(z)=0, \text{ that is, } f(z)^2=dP(z)e^{-2H(z)},$$

where d is a non-zero constant, which contradicts the simplicity of zeros and poles of $P(z)$.

Next assume that $a_1(z)\equiv0$. Then we get $f(z)^2=dP(z)e^{-2H(z)+2L(z)}$, which is a contradiction, because of the same reason. Thus we have proved theorem 1.

6. *Proof of Theorem* 2. By virtue of theorem 1 it is sufficient to prove $P(\tilde{R})\neq3$. We assume that $P(\tilde{R})=3$. Then from Hiromi and Ozawa [3] $\tilde{G}(z)$ satisfies

$$
\begin{aligned}
&\tilde{F}(z)^2\tilde{G}(z)=1-2\beta_1e^{L_1}-2\beta_2e^{L_2}+\beta_1^2e^{2L_1}-2\beta_1\beta_2e^{L_1+L_2}+\beta_2^2e^{2L_2},\\
&L_1(z)\not\equiv\text{const.},\quad L_2(z)\not\equiv\text{const.},\quad L_1(0)=L_2(0)=0,\quad \beta_1\beta_2\neq0,
\end{aligned}
\tag{6.1}
$$

with three suitable entire functions $\tilde{F}$, L_1 and L_2 and two constants β_1 and β_2. By substituting $\tilde{G}(z)$ into (6.1) we obtain

$$
\begin{aligned}
(6.2) \qquad & f(z)^2Q(z)(e^H-\alpha)(e^H-\beta) \\
&=1-2\beta_1e^{L_1}-2\beta_2e^{L_2}+\beta_1^2e^{2L_1}-2\beta_1\beta_2e^{L_1+L_2}+\beta_2^2e^{2L_2},
\end{aligned}
$$

where $f(z)=\tilde{F}(z)/F(z)$.

First we shall verify that $L_1(z)$ and $L_2(z)$ are polynomials. We denote the right side term of (6.2) by $\tilde{g}(z)$. Then $f_0=(1/2)(1+\beta_1e^{L_1}-\beta_2e^{L_2})+(i/2)\sqrt{\tilde{g}}$ is a regular function on $\tilde{R}$, and hence f_0 belongs to $\mathfrak{M}(\tilde{R})$. Its defining equation is

$$F(z,f_0)=f_0^2-(1+\beta_1e^{L_1}-\beta_2e^{L_2})f_0+\beta_1e^{L_1}=0.$$

$F(z,0)=\beta_1e^{L_1}$ and $F(z,1)=\beta_2e^{L_2}$ show that $f_0\neq 0,1,\infty$ on $\tilde{R}$, that is, $P(f_0)=3$. Let $N(r,\tilde{R})$ be the quantity $N(r,\mathfrak{X})$ defined by Selberg [11]. Then it is easily verified that $N(r,\tilde{R})\sim m(r,e^H)$. If one of $L_1(z)$ and $L_2(z)$ is not a polynomial, then we get

$$\varliminf_{r\to\infty}\frac{N(r,\tilde{R})}{T(r,f_0)}=0,$$

because $H(z)$ is a polynomial, which is the assumption of our theorem. This fact yields $P(f_0)\leqq 2$ by virtue of the Nevanlinna-Selberg theory [11], which contradicts $P(f_0)=3$. Therefore both of $L_1(z)$ and $L_2(z)$ are polynomials.

Since $H(z)$, $L_1(z)$ and $L_2(z)$ are polynomials, the method by which Ozawa [9] proved his theorem 3 implies that the identity (6.2) offers (i) $L_1=L_2$, (ii) $L_1=2L_2$, $\beta_2^2=16\beta_1$, (iii) $2L_1=L_2$, $\beta_1^2=16\beta_2$, (iv) $L_1=-L_2$, $16\beta_1\beta_2=1$. In these cases the ultrahyperelliptic surface R_1 defined by the equation $y^2=\tilde{g}(z)$ has $P(R_1)=4$, while the ultrahyperelliptic surface R_2 defined by the equation $y^2=f^2Q(e^H-\alpha)(e^H-\beta)$ has $P(R_2)\leqq 3$ by means of theorem 1. This is a contradiction, because we have $P(R_1)=P(R_2)$ from the identity (6.2). Therefore we have $P(R)\neq 3$. Thus theorem 2 has been proved.

7. Analytic mappings. Now we shall consider the problem (B) in the case $P(R)=P(S)=4$.

Let R and S be two ultrahyperelliptic surfaces with $P(R)=P(S)=4$ and defined by two equations $y^2=G(z)$ and $u^2=g(w)$, where G and g are two entire functions having no zero other than an infinite number of simple zeros respectively. Then we may assume that $G(z)$ satisfies the equation (4.1) and $g(w)$ satisfies the following equation:

$$
\begin{aligned}
(7.1) \qquad & f(w)^2g(w)=(e^{L(w)}-\gamma)(e^{L(w)}-\delta), \\
& L(w)\not\equiv\text{const.}, \qquad L(0)=0, \qquad \gamma\delta(\gamma-\delta)\neq 0,
\end{aligned}
$$

with two suitable entire functions f and L and two constants γ and δ.

And let $\tilde{R}$ and $\tilde{S}$ be finite modifications of R and S, respectively, defined by two equations $y^2=Q(z)G(z)$ and $u^2=q(w)g(w)$, with $Q(z)$ given by (4.2) and

$$(7.2) \qquad q(w)=\prod_{j=1}^{\sigma}(w-d_j)\prod_{j=1}^{\tau}\frac{1}{w-e_j}, \qquad \sigma+\tau\geqq 1,$$

where d_j and e_j are mutually distinct constants and their moduli are less than r_0.
Then we shall prove the following theorem:

THEOREM 3. *There exists an analytic mapping φ of R into $\tilde{S}$ if and only if there exist an entire functions $h(z)$ and a meromorphic function $f^*(z)$ satisfying one of the following conditions*

(a) $$H(z)=L\circ h(z)-L\circ h(0), \qquad f^*(z)^2=q\circ h(z),$$
$$\frac{\gamma}{\alpha}=\frac{\delta}{\beta}=e^{L\circ h(0)} \quad \textit{or} \quad \frac{\delta}{\alpha}=\frac{\gamma}{\beta}=e^{L\circ h(0)},$$

(b) $$H(z)=-L\circ h(z)+L\circ h(0), \qquad f^*(z)^2=q\circ h(z),$$
$$\alpha\gamma=\beta\delta=e^{L\circ h(0)} \quad \textit{or} \quad \alpha\delta=\beta\gamma=e^{L\circ h(0)}$$

and $\sigma+\tau=1$ *or* 2 *in* (7.2) *when* $h(z)$ *is transcendental, or* $\sigma+\tau=1$ *when* $h(z)$ *a polynomial.*

By virtue of this theorem, Ozawa [10] and Hiromi and Ozawa [3], we easily get

COROLLARY 1. *There is no analytic mapping of R into $\tilde{R}$.*

COROLLARY 2. *If there is an analytic mapping of R into $\tilde{S}$, then there exists an analytic mapping of R into S whose projection is the same $h(z)$.*

From the proof of theorem 3 given in 8 we can deduce a perfect condition for the existence of an analytic mapping of $\tilde{R}$ into $\tilde{S}$. Therefore we state, without proof,

THEOREM 4. *There exists an analytic mapping φ of $\tilde{R}$ into $\tilde{S}$ if and only if there exist an entire function $h(z)$ and a meromorphic function $f^*(z)$ satisfying one of the following conditions*

(a) $$H(z)=L\circ h(z)-L\circ h(0), \qquad f^*(z)^2=q\circ h(z)/Q(z),$$
$$\frac{\gamma}{\alpha}=\frac{\delta}{\beta}=e^{L\circ h(0)} \quad \textit{or} \quad \frac{\delta}{\alpha}=\frac{\gamma}{\beta}=e^{L\circ h(0)},$$

(b) $$H(z)=-L\circ h(z)+L\circ h(0), \qquad f^*(z)^2=q\circ h(z)/Q(z),$$
$$\alpha\gamma=\beta\delta=e^{L\circ h(0)} \quad \textit{or} \quad \alpha\delta=\beta\gamma=e^{L\circ h(0)}$$

and $\sigma+\tau=1$ *or* 2 *in* (7.2) *when* $h(z)$ *is transcendental.*

Similarly we easily get

COROLLARY 3. *If there is an analytic mapping of $\tilde{R}$ into $\tilde{S}$, then there exists an analytic mapping R into S whose projection is the same $h(z)$.*

8. *Proof of Theorem* 3. Assume that there exists an analytic mapping of R into $\tilde{S}$. Then by means of Ozawa's theorem [8] there exist an entire function $h(z)$ and a meromorphic function $f(z)$ such that

$$f(z)^2(e^{H(z)}-\alpha)(e^{H(z)}-\beta)=q\circ h(z)(e^{L\circ h(z)}-\gamma)(e^{L\circ h(z)}-\delta), \tag{8.1}$$

where $f(z)$ has zeros and poles possibly at the multiple zeros of $q\circ h(e^{L\circ h}-\gamma)(e^{L\circ h}-\delta)$ and $(e^H-\alpha)(e^H-\beta)$. Therefore we see that

$$\begin{aligned} &m(r, e^H)\sim m(r, e^{L\circ H}), \\ &T(r, f'/f)=o(m(r, e^H))=o(m(r, e^{L\circ h})) \end{aligned} \tag{8.2}$$

outside a set of finite measure. Compare (8.1) and (8.2) with (5.1) and (5.2), and this case can be similarly treated as in the process of our proof of theorem 1. Hence according to the reasoning of **5** it is sufficient to consider the following two cases (I) and (II):

(I) $m(r, e^{H+L\circ h})=o(m(r, e^H))$ outside a set of finite measure, and

$$f(z)^2=dq\circ h(z)e^{-2H(z)}, \tag{8.3}$$

where d is a non-zero constant.

(II) $f(z)^2=dq\circ h(z)e^{-2H(z)+2L\circ h(z)}$ with a non-zero constant d.

First we shall prove $\sigma+\tau=1$ or 2 in (7.2). We assume that $\sigma+\tau\geqq 3$ and $h(z)$ is a transcendental entire function. Then the relations (8.3) and (II) imply that every root of equations $h(z)=d_j$ or e_j must be of even multiplicity whenever it esists. Hence $\Theta(d_j, h)\geqq 1/2$ and $\Theta(e_j, h)\geqq 1/2$ hold. Therefore by assumption $\sigma+\tau\geqq 3$, we have $\sum^{\sigma}\Theta(d_j, h)+\sum^{\tau}\Theta(e_j, h)\geqq(3/2)$, which contradicts the Nevanlinna ramification relation for h: $\sum\Theta(a, h)\leqq 1$ [1, 4].

Next we assume that $\sigma+\tau\geqq 2$ and $h(z)$ is a polynomial and d and e are two distinct elements of the set $\{d_j, e_j\}$ in (7.2). Then the relations (8.3) and (II) offer that there exist two polynomial $h_1(z)$ and $h_2(z)$ such that $h(z)-d=h_1(z)^2$ and $h(z)-e=h_2(z)^2$. Eliminating $h(z)$ we obtain $(h_1(z)-h_2(z))(h_1(z)+h_2(z))=e-d\neq 0$, which implies a contradictory fact that $h_1(z)$ and $h_2(z)$ are constants.

Now we shall get the former conditions in our theorem.

Case (I). Putting $\zeta_1=-(\alpha+\beta)$, $\zeta_2=\alpha\beta$, $\eta_1=-(\gamma+\delta)$, $\eta_2=\gamma\delta$, the identity (8.1) reduces to

$$(d-\eta_2)e^{2H(z)}+(\zeta_1 d-\eta_1 e^{H(z)+L\circ H(z)})e^{H(z)}+\zeta_2 d-e^{2H(z)+2L\circ h(z)}=0.$$

Hence the impossibility of Borel's identity implies

$$d=\eta_2, \quad \zeta_1 d-\eta_1 e^{H+L\circ h}=0 \quad \text{and} \quad \zeta_2 d-e^{2H+2L\circ h}=0.$$

Accordingly the function $H(z)+L\circ h(z)$ must be the constant $L\circ h(0)$. Then we have $(\alpha+\beta)\gamma\delta=(\gamma+\delta)e^{L\circ h(0)}$ and $\alpha\beta\gamma\delta=e^{2L\circ h(0)}$. These relations yield $\alpha\gamma=\beta\delta=e^{L\circ h(0)}$ or $\alpha\delta=\beta\gamma=e^{L\circ h(0)}$. Thus we attain to the case (b) in our theorem.

Case (II). Substituting $f(z)^2$ into (8.1), the identity (8.1) reduces to

$$\begin{aligned} &(1-d)e^{2H(z)+2L\circ h(z)}+\eta_1 e^{2H(z)+L\circ h(z)} \\ &\quad -d\zeta_1 e^{H(z)+2L\circ h(z)}+\eta_2 e^{2H(z)}-d\zeta_2 e^{2L\circ h(z)}=0. \end{aligned} \tag{8.4}$$

Since $\eta_2\neq 0$, lemma A gives

$$\eta_2 e^{2H}+c_1(1-d)e^{2H+2L\circ h}-c_2 d\zeta_2 e^{2L\circ h}=0, \tag{8.5}$$

where c_1 and c_2 are constants. If $c_1c_2(1-d)\neq 0$, then writing the identity (8.5) in the form $\eta_2 e^{2H-2L\circ h}+c_1(1-d)e^{2H}=c_2d\zeta_2$, and using lemma 1 in [3], we have

$$c_1'\eta_2 e^{2H-2L\circ h}+c_2'c_1(1-d)e^{2H}=0, \text{ that is, } c_1'\eta_2 e^{-2L\circ h}+c_2'c_1(1-d)=0,$$

where c_1' and c_2' are two constants which do not vanish simultaneously. This contradicts $L\circ h(z)\not\equiv$const. If $c_1=c_2=0$, then the identity (8.5) is clearly impossible because of $\eta_2\neq 0$. If $c_2=0$ and $c_1(1-d)\neq 0$, then the identity (8.5) reduces to $\eta_2+c_1(1-d)e^{2L\circ h}=0$, which is untenable. If $c_2\neq 0$ and $c_1(1-d)=0$, then the function $H(z)-L\circ h(z)$ must be constant $-L\circ h(0)$. Then the identity (8.4) reduces to

$$(1-d)e^{4L\circ h(0)}e^{4H(z)}+(\eta_1 e^{L\circ h(0)}-d\zeta_1 e^{2L\circ h(0)})e^{3H(z)}+(\eta_2-d\zeta_2 e^{2L\circ h(0)})e^{2H(z)}=0.$$

By virtue of the impossibility of Borel's identity we obtain

$$d=1, \quad \eta_1=\zeta_1 e^{L\circ h(0)} \quad \text{and} \quad \eta_2=\zeta_2 e^{2L\circ h(0)}.$$

Thus we attain the case (a) in our theorem.

The sufficiency part is evident by Ozawa's theorem [8]. Thus we have proved theorem 3.

9. *Remark.* Our theorem 3 is best possible. This fact is derived from the sharpness of the Nevanlinna ramification relation. Let R, S and $\tilde{S}$ be three ultrahyperelliptic surfaces defined by the three equations $y^2=(e^{\sin z}-\alpha)(e^{\sin z}-\beta)$, $u^2=(e^w-\alpha)(e^w-\beta)$ and $u^2=(w-1)(w+1)(e^w-\alpha)(e^w-\beta)$, with non-zero distinct constants α and β, respectively. Then it is clear that $P(R)=P(S)=4$ and $P(\tilde{S})=2$. And we have $(i\cos z)^2=(\sin z-1)(\sin z+1)$. Therefore we claim that there exists an analytic mapping of R into $\tilde{S}$ whose projection is $h(z)=\sin z$ and that simultaneously there exists an analytic mapping of R into S whose projection is also $h(z)=\sin z$.

Further the sharpness of theorem 3 is clear when $h(z)$ is a polynomial.

§ 4. Regularly branched three-sheeted covering Riemann surfaces.

Now we shall discuss the similar problems as (A), (B) in the case of regularly branched three-sheeted covering Riemann surfaces.

10. Picard's constant. Let R be a regularly branched three-sheeted covering Riemann surface with $P(R)=6$ defined by the equation $y^3=G(z)$, where $G(z)$ is an entire function whose zeros are infinite in number and each is of multiplicity $\leqq 2$. Then by a characterization of R with $P(R)=6$ [2], we can put

$$\begin{aligned} &F(z)^3G(z)=(e^{H(z)}-\alpha)(e^{H(z)}-\beta)^2, \\ &H(z)\not\equiv\text{const.}, \quad H(0)=0, \quad \alpha\beta(\alpha-\beta)\neq 0, \end{aligned} \tag{10.1}$$

with two entire functions $F(z)$ and $H(z)$ and two constants α and β. Let $\tilde{R}$ be a finite modification of R defined by the equation $y^3=\tilde{G}(z)$ with $\tilde{G}(z)=Q(z)G(z)$, where $Q(z)$ has the following form:

$$\prod_{j=1}^{\lambda}(z-a_j)^{\kappa_j}\prod_{j=1}^{\mu}\frac{1}{(z-b_j)^{\nu_j}}, \qquad \kappa_j, \nu_j=1 \text{ or } 2, \qquad \lambda+\mu\geqq 1, \tag{10.2}$$

where a_j and b_j are mutually distinct constants and their moduli are less than r_0.

Now we shall prove the following theorem:

THEOREM 5. *Let R be a regularly branched three-sheeted covering Riemann surface and $\tilde{R}$ be its finite modification. Then $P(R)=6$ implies $P(\tilde{R})\leqq 4$.*

11. *Proof of Theorem* 5. Let R and $\tilde{R}$ be two regularly branched three-sheeted covering Riemann surfaces defined by two equations $y^3=G(z)$ and $y^3=Q(z)G(z)$ with $G(z)$ satisfying (10.1) and $Q(z)$ given by (10.2). In order to prove $P(R)\leqq 4$, by virtue of theorem 1 and theorem 2 in [2], it is sufficient to show the impossibility of an identity of the form

$$\begin{aligned}&f(z)^3(e^{H(z)}-\alpha)(e^{H(z)}-\beta)^2=P(z)(e^{L(z)}-\gamma)(e^{L(z)}-\delta)^2,\\ &P(z)=1/Q(z),\ L(z)\not\equiv\text{const.},\ L(0)=0, \quad \gamma\delta(\gamma-\delta)\neq 0,\end{aligned} \tag{11.1}$$

where γ and δ are two constants, $L(z)$ is an entire function and $f(z)$ is a meromorphic function which has zeros and poles possibly at the zeros of multiplicity $\geqq 3$ of $(e^L-\gamma)(e^L-\delta)^2$ and $Q(e^H-\alpha)(e^H-\beta)^2$. We can easily see from the reasoning in [5] that the identity (11.1) offers

$$\begin{aligned}&m(r, e^H)\sim m(r, e^L),\\ &T(r, f'/f)=o(m(r, e^H))=o(m(r, e^L))\end{aligned} \tag{11.2}$$

outside a set of finite measure. By differentiating the both side of (11.1) and setting $\zeta_1=-(\alpha+2\beta)$, $\zeta_2=2\alpha\beta+\beta^2$, $\zeta_3=-\alpha\beta^2$, $\eta_1=-(\gamma+2\delta)$, $\eta_2=2\gamma\delta+\delta^2$, $\eta_3=-\gamma\delta^2$, and eliminating f^3, we obtain

$$\begin{aligned}&(3f'/f+3H'-3L'-P'/P)e^{3H+3L}+\zeta_1\eta_1(3f'/f+2H'-2L'-P'/P)e^{2H+2L}\\ &+\zeta_2\eta_2(3f'/f+H'-L'-P'/P)e^{H+L}+\eta_1(3f'/f+3H'-2L'-P'/P)e^{3H+2L}\\ &+\zeta_1(3f'/f+2H'-3L'-P'/P)e^{2H+3L}+\eta_2(3f'/f+3H'-L'-P'/P)e^{3H+L}\\ &+\zeta_2(3f'/f+H'-3L'-P'/P)e^{H+3L}+\zeta_1\eta_2(3f'/f+2H'-L'-P'/P)e^{2H+L}\\ &+\zeta_2\eta_1(3f'/f+H'-2L'-P'/P)e^{H+2L}+\eta_3(3f'/f+3H'-P'/P)e^{3H}\\ &+\zeta_1\eta_3(3f'/f+2H'-P'/P)e^{2H}+\zeta_2\eta_3(3f'/f+H'-P'/P)e^{H}\\ &+\zeta_3(3f'/f-3L'-P'/P)e^{3L}+\zeta_3\eta_1(3f'/f-2L'-P'/P)e^{2L}\\ &+\zeta_3\eta_2(3f'/f-L'-P'/P)e^{L}+\zeta_3\eta_3(3f'/f-P'/P)=0.\end{aligned} \tag{11.3}$$

According to the reasoning in [5, p. 243–246] (cf. 5 in this paper), we can derive $3f'(z)/f(z)+3H'(z)-3L'(z)-P'(z)/P(z)\equiv 0$ from the coefficient of e^{3H+3L} in (11.3), or $3f'(z)/f(z)+3H'(z)-P'(z)/P(z)\equiv 0$ from the coefficient of e^{3H} in (11.3).

Hence we attain a contradictory fact that $f(z)^3=dP(z)e^{-3H(z)+3L(z)}$ or $f(z)^3=dP(z)e^{-3H(z)}$ with a non-zero constant d. Q.E.D.

12. Analytic mappings. We shall consider the similar problem (B) in the case $P(R)=P(S)=6$.

Let R and S be two regularly branched three-sheeted covering Riemann surfaces with $P(R)=P(S)=6$ defined by two equations $y^3=G(z)$ and $u^3=g(w)$, respectively, where G and g are two entire functions whose zeros are infinite in number and each is of multiplicity $\leqq 2$. Then we may assume that $G(z)$ satisfies the equation (10.1) and $g(w)$ satisfies the following equation

$$f(w)^3 g(w)=(e^{L(w)}-\gamma)(e^{L(w)}-\delta)^2, \qquad L(w)\not\equiv\text{const.}, \quad L(0)=0, \quad \gamma\delta(\gamma-\delta)\neq 0, \tag{12.1}$$

with two suitable entire functions f and L and two constants γ and δ.

And let $\tilde{R}$ and $\tilde{S}$ be two finite modification of R and S, respectively, defined by two equations $y^2=Q(z)G(z)$ and $u^2=q(w)g(w)$, with $Q(z)$ given by (10.2) and

$$q(w)=\prod_{j=1}^{\sigma}(w-d_j)^{\kappa_j}\prod_{j=1}^{\tau}\frac{1}{(w-e_j)^{\nu_j}}, \qquad \kappa_j,\ \nu_j=1 \text{ or } 2, \qquad \sigma+\tau\geqq 1, \tag{12.2}$$

where d_j and e_j are mutually distinct constants and their moduli are less than r_0.

Then we similarly have the following theorems:

THEOREM 3′. *There exists an analytic mapping of R into $\tilde{S}$ if and only if $\sigma+\tau=1$ in (12.2) there exist an entire function $h(z)$ and a meromorphic function $f^*(z)$ satisfying one of the following conditions*

(a) $$H(z)=L\circ h(z)-L\circ h(0), \qquad f^*(z)^3=q\circ h(z),$$
$$\frac{\gamma}{\alpha}=\frac{\delta}{\beta}=e^{L\circ h(0)} \quad \text{or} \quad \frac{\delta}{\alpha}=\frac{\gamma}{\beta}=e^{L\circ h(0)},$$

(b) $$H(z)=-L\circ h(z)+L\circ h(0), \qquad f^*(z)^3=q\circ h(z),$$
$$\alpha\gamma=\beta\delta=e^{L\circ h(0)} \quad \text{or} \quad \alpha\delta=\beta\gamma=e^{L\circ h(0)}.$$

COROLLARY 1′. *There is no analytic mapping of R into $\tilde{R}$.*

COROLLARY 2′. *If there is an analytic mapping of R into $\tilde{S}$, then there exists an analytic mapping of R into S whose projection is the same $h(z)$.*

It is clear that our theorem 3′ is best possible.

These theorem and corollaries can be deduced from the process of 4–8 and of our previous paper [5, p. 247–249]. Therefore we refrain the proofs of theorem 3′ and corollaries 1′ and 2′.

Similarly we can state the theorem 4′ and corollary 3′ corresponding to theorem 4 and corollary 3, respectively.

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