ON A SIMPLIFIED METHOD OF THE ESTIMATION OF THE CORRELOGRAM FOR A STATIONARY GAUSSIAN PROCESS, III

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§ 1. Introduction.

In this paper we shall deal with a simplified method for the estimation of the correlogram for a stationary process.

Let $X(n)$ be a real-valued stationary process with discrete time parameter n. We assume $EX(n)=0$. We put

$$EX(n)^2=\sigma^2, \qquad EX(n)X(n+h)=\sigma^2\rho_h,$$

and we consider to estimate the correlogram ρ_h.

In the previous papers [4], [5], we discussed a simplified method for the estimation of the correlogram when σ^2 is known. But in the present paper, we discuss the case when σ^2 is unknown. For simplicity, let us assume the process $X(n)$ to be observed at $n=1, 2, \cdots, N, \cdots, N+h$.

Usually, in order to estimate the correlogram ρ_h, we use the estimate

$$\tilde{\Gamma}_h=\frac{\sum_{n=1}^{N} X(n)X(n+h)}{\sum_{n=1}^{N} X(n)^2}.$$

Now we shall modify the estimate $\tilde{\Gamma}_h$. The essential part of our modification is to replace $X(n)X(n+h)$ by $X(n)\operatorname{sgn}(X(n+h))$, where $\operatorname{sgn}(y)$ means $1, 0, -1$ correspondingly as $y>0$, $y=0$, $y<0$. The new estimate is

$$\Gamma_h=\frac{\sum_{n=1}^{N} X(n)\operatorname{sgn}(X(n+h))}{\sum_{n=1}^{N} |X(n)|}.$$

This new estimate Γ_h may be considered as follows. We make a nonlinear operation on the input $X(n)$ and assume that the output is $Y(n)=\operatorname{sgn}(X(n))$. Then, the estimate Γ_h consists of the cross-correlation of the input $X(n)$ and the output $Y(n)$.

We shall show below that when $X(n)$ is a Gaussian process satisfying some conditions, the estimate Γ_h is an asymptotically unbiased estimate of the correlogram

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ρ_h as $N\to\infty$. We evaluate the asymptotic variance of Γ_h. The estimate $\tilde{\Gamma}_h$ is also an asymptotically unbiased estimate of ρ_h. Further, Γ_h and $\tilde{\Gamma}_h$ are both consistent estimates of ρ_h. We compare, for the typical cases, the asymptotic variance of Γ_h with that of $\tilde{\Gamma}_h$.

§ 2. The estimate Γ_h.

Let $X(n)$ be a stationary Gaussian process having a finite moving average representation

$$(1)\qquad X(n)=G_0\xi(n)+G_1\xi(n-1)+\cdots+G_M\xi(n-M),$$

where $\xi(n)$ is the white noise with

$$E\xi(n)=0, \qquad E\xi(n)^2=1,$$
$$E\xi(n_1)\xi(n_2)=0 \quad \text{when } n_1\neq n_2,$$

M is some positive number and $\{G_k\text{'s}\}$ are constants.

Let $L_2(X;n)$ denote the closed linear manifold generated by $\{X(j); j\leqq n\}$ and $L_2(\xi;n)$ denote the closed linear manifold generated by $\{\xi(j); j\leqq n\}$.

LEMMA 1. *If $X(n)$ is a stationary Gaussian process which has the moving average representation* (1) *and if the condition*

$$(2)\qquad L_2(X;n)=L_2(\xi;n)$$

holds for an arbitrary integer n, $\xi(n)$ is a stationary Gaussion process.

In fact, we consider the joint distribution of $\xi(n_1),\cdots,\xi(n_k)$. As $\xi(n_\nu)\in L_2(X;n_\nu)$, there are constants $\{a_l; l=0,1,2,\cdots\}$ such that

$$\xi(n_\nu)=\underset{N\to\infty}{\mathrm{l.i.m.}}\sum_{l=0}^{N}a_lX(n_\nu-l).$$

Therefore for any real numbers $A_1, A_2, \cdots, A_k$,

$$A_1\xi(n_1)+A_2\xi(n_2)+\cdots+A_k\xi(n_k)$$
$$=\underset{N\to\infty}{\mathrm{l.i.m}}\left\{A_1\left(\sum_{l=0}^{N}a_lX(n_1-l)\right)+A_2\left(\sum_{l=0}^{N}a_lX(n_2-l)\right)+\cdots+A_k\left(\sum_{l=0}^{N}a_lX(n_k-l)\right)\right\}.$$

The distribution of

$$A_1\left(\sum_{l=0}^{N}a_lX(n_1-l)\right)+A_2\left(\sum_{l=0}^{N}a_lX(n_2-l)\right)+\cdots+A_k\left(\sum_{l=0}^{N}a_lX(n_k-l)\right)$$

is Guassian, so the distribution function of

$$A_1\xi(n_1)+A_2\xi(n_2)+\cdots+A_k\xi(n_k)$$

is Gaussian. This shows $\xi(n)$ is a Gaussian process.

As $\xi(n)$ is a white noise, $\xi(n_1)$ and $\xi(n_2)$ are orthogonal, for any $n_1\neq n_2$, so that, by the above lemma, $\xi(n_1)$ and $\xi(n_2)$ are mutually independent.

Now we determine the asymptotic distribution of the estimate Γ_h. Without loss of generality, we can assume that $h>0$. We have

$$\sqrt{N}(\Gamma_h-\rho_h)=\sqrt{N}\left(\frac{\sum_{n=1}^{N} X(n)\operatorname{sgn}(X(n+h))}{\sum_{n=1}^{N}|X(n)|}-\rho_h\right)$$

$$=\sqrt{N}\left(\frac{\frac{1}{N}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{n=1}^{N} X(n)\operatorname{sgn}(X(n+h))}{\frac{1}{N}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{n=1}^{N}|X(n)|}-\rho_h\right)$$

$$=\frac{\frac{1}{\sqrt{N}}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{n=1}^{N}\{X(n)\operatorname{sgn}(X(n+h))-\rho_h|X(n)|\}}{\frac{1}{N}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{n=1}^{N}|X(n)|}.$$

In the first place, we consider the statistic

$$\gamma_0=\frac{1}{N}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{n=1}^{N}|X(n)|.$$

Using the results in Huzii [4], we have

$$E(\gamma_0)=1$$

and

$$V(\gamma_0)=\text{the variance of }\gamma_0$$
$$=\frac{2}{N^2}\sum_{k=1}^{N-1}(N-k)(1-\rho_k^2)^{3/2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m)!}\Gamma(m+1)^2\right)+\frac{\pi}{2}\frac{1}{N}-1.$$

LEMMA 2. *If $X(n)$ is a process having the representation* (1), *then* $V(\gamma_0)\to 0$ *as* $N\to\infty$.

Proof. For our process $X(n)$, $\rho_k=0$ when $|k|>M$. So we have

$$V(\gamma_0)=\frac{2}{N^2}\sum_{k=1}^{M}(N-k)(1-\rho_k{}^2)^{3/2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m)!}\Gamma(m+1)^2\right)$$
$$+\frac{2}{N^2}\sum_{k=M+1}^{N}(N-k)+\frac{\pi}{2}\frac{1}{N}-1.$$

Now,

$$\frac{2}{N^2}\sum_{k=M+1}^{N}(N-k)=\frac{2}{N^2}\cdot\frac{(N-M-1)(N-M)}{2}=1-\frac{(2M+1)}{N}+\frac{M(M+1)}{N^2}.$$

Therefore we get

$$V(\gamma_0)=\frac{2}{N}\sum_{k=1}^{M}\left(1-\frac{k}{N}\right)(1-\rho_k^2)^{3/2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m)!}\Gamma(m+1)^2\right)$$
$$-\frac{(2M+1)}{N}+\frac{M(M+1)}{N^2}+\frac{\pi}{2}\frac{1}{N}.$$

This shows $V(\gamma_0)\to 0$ as $N\to\infty$.

From this Lemma 2, we can find the following result:

THEOREM 1. *γ_0 converges in probability to* 1 *as* $N\to\infty$.

In the next place, we consider the numerator of $\sqrt{N}(\Gamma_h-\rho_h)$, that is,

$$\frac{1}{\sqrt{N}}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{n=1}^{N}\{X(n)\operatorname{sgn}(X(n+h))-\rho_h|X(n)|\}.$$

Let us denote

$$Y(n)=\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\{X(n)\operatorname{sgn}(X(n+h))-\rho_h|X(n)|\}.$$

Since the process $X(n)$ has the representation (1) and the $\xi(n)$'s are mutually independent, $Y(n_1)$ and $Y(n_2)$ are mutually independent if $|n_1-n_2|>M+h$.

Here, we quote the result in Diananda [2].

DEFINITION 1 (Diananda). Let d_n be a function of n. Suppose $\{X_i\}$ $(i=1,2,\cdots)$ is a sequence of random variables such that the two sets of variables $(X_1, X_2, \cdots, X_r)$ and $(X_s, X_{s+1}, \cdots, X_n)$ are independent whenever $s-r>d_n$. Then we say that $\{X_i\}$ $(i=1,2,\cdots)$ is a sequence of *d_n-dependent variables* or is a *d_n-dependent process*.

LEMMA 3 (Diananda). *Let $\{X_i\}$ $(i=1,2,\cdots)$ be a sequence of stationary m-dependent scalar variables with the mean zero and $E(X_iX_j)=C_{i-j}$. Then the distribution function of the random variable $(X_1+X_2+\cdots+X_n)/\sqrt{n}\to$ the normal distribution function with the mean zero and the variance $\sum_{-m}^{m}C_p$ as $n\to\infty$.*

In our case, $Y(n)$ is a sequence of $(M+h)$-dependent variables and since $X(n)$ is a stationary Gaussian process, $Y(n)$ is a stationary process. It is clear that $EY(n)=0$. Let us denote $EY(n)Y(m)=C(n-m)$. From the above Lemma 3, the distribution function of the random variable

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{N}Y(n)=\frac{1}{\sqrt{N}}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{n=1}^{N}\{X(n)\operatorname{sgn}(X(n+h))-\rho_h|X(n)|\}$$

tends to the normal distribution function with the mean zero and the variance $\sum_{k=-(M+h)}^{M+h}C(k)$ *as* $N\to\infty$.

Now, we shall evaluate the value of $C(k)=EY(n)Y(n+k)$.

$$
\begin{aligned}
C(k)&=EY(n)Y(n+k)\\
&=\frac{\pi}{2}\frac{1}{\sigma^2}E\{(X(n)\operatorname{sgn}(X(n+h))-\rho_h|X(n)|)\\
&\qquad\cdot(X(n+k)\operatorname{sgn}(X(n+k+h))-\rho_h|X(n+k)|)\}\\
&=\frac{\pi}{2}\frac{1}{\sigma^2}EX(n)\operatorname{sgn}(X(n+h))X(n+k)\operatorname{sgn}(X(n+k+h))\\
&\quad-\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h EX(n)\operatorname{sgn}(X(n+h))|X(n+k)|\\
&\quad-\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h E|X(n)|X(n+k)\operatorname{sgn}(X(n+k+h))+\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h^2 E|X(n)||X(n+k)|.
\end{aligned}
$$

(i) When k is neither zero nor $\pm h$, we have, by using the results in the previous paper [5],

$$
\begin{aligned}
&\frac{\pi}{2}\frac{1}{\sigma^2}EX(n)\operatorname{sgn}(X(n+h))X(n+k)\operatorname{sgn}(X(n+k+h))\\
&=\frac{1}{2}\Big\{(AF^2+BF)D^{3/2}S_1(\rho_k)+(2AFG+BG+CF)D^{3/2}S_2(\rho_k)\\
&\qquad+(AG^2+CG)D^{3/2}S_1(\rho_k)+A\cdot\frac{\Delta}{2\sqrt{D}}S_3(\rho_k)\Big\}
\end{aligned}
$$

and

$$
\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h^2 E|X(n)|\,|X(n+k)|=\frac{1}{2}\rho_h^2 D^{3/2}S_2(\rho_k),
$$

where

$$
\Delta=\begin{vmatrix}1 & \rho_{k-h} & \rho_h\\ \rho_{k-h} & 1 & \rho_k\\ \rho_h & \rho_k & 1\end{vmatrix},\qquad
A=\frac{1}{\Delta}\begin{vmatrix}\rho_k & \rho_{k-h} & \rho_h\\ \rho_h & 1 & \rho_k\\ \rho_{k+h} & \rho_k & 1\end{vmatrix},
$$

$$
B=\frac{1}{\Delta}\begin{vmatrix}1 & \rho_k & \rho_h\\ \rho_{k-h} & \rho_h & \rho_k\\ \rho_h & \rho_{k+h} & 1\end{vmatrix},\qquad
C=\frac{1}{\Delta}\begin{vmatrix}1 & \rho_{k-h} & \rho_k\\ \rho_{k-h} & 1 & \rho_h\\ \rho_h & \rho_k & \rho_{k+h}\end{vmatrix},
$$

$$
D=\begin{vmatrix}1 & \rho_k\\ \rho_k & 1\end{vmatrix},\qquad
F=\frac{1}{D}\begin{vmatrix}\rho_{k-h} & \rho_k\\ \rho_h & 1\end{vmatrix},\qquad
G=\frac{1}{D}\begin{vmatrix}1 & \rho_{k-h}\\ \rho_k & \rho_h\end{vmatrix},
$$

$$
S_1(\rho_k)=2\Big(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m+1}}{(2m+1)!}\Gamma(m+2)\Gamma(m+1)\Big),
$$

$$
S_2(\rho_k)=2\Big(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m)!}\Gamma(m+1)^2\Big),
$$

$$
S_3(\rho_k)=2\Big(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m+1}}{(2m+1)!}\Gamma(m+1)^2\Big).
$$

Now, the value of

$$\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h EX(n)\,\mathrm{sgn}\,(X(n+h))|X(n+k)|$$

is as follows. Suppose that

$$X(n)=U_1X(n+k)+V_1X(n+h)+\nu_1(n),$$

where $\nu_1(n)$ is a Gaussian process with the mean zero and satisfies

$$E\nu_1(n)X(n+k)=0, \qquad E\nu_1(n)X(n+h)=0.$$

Then, U_1 and V_1 are determined by the following conditions:

$$E(X(n)-U_1X(n+k)-V_1X(n+h))X(n+k)=0,$$
$$E(X(n)-U_1X(n+k)-V_1X(n+h))X(n+h)=0.$$

From these, we get

$$U_1=\frac{1}{D_1}\begin{vmatrix}\rho_k & \rho_{h-k}\\ \rho_h & 1\end{vmatrix} \quad \text{and} \quad V_1=\frac{1}{D_1}\begin{vmatrix}1 & \rho_k\\ \rho_{h-k} & \rho_h\end{vmatrix},$$

where

$$D_1=\begin{vmatrix}1 & \rho_{h-k}\\ \rho_{h-k} & 1\end{vmatrix}.$$

The new random variable $\nu_1(n)$, determined in the above, is independent of $X(n+k)$, $X(n+h)$ and $(X(n+k), X(n+h))$. Using these results, we have

$$\begin{aligned}&EX(n)\,\mathrm{sgn}\,(X(n+h))|X(n+k)|\\ &=E(U_1X(n+k)+V_1X(n+h)+\nu_1(n))\,\mathrm{sgn}\,(X(n+h))|X(n+k)|\\ &=U_1EX(n+k)\,\mathrm{sgn}\,(X(n+h))|X(n+k)|+V_1E|X(n+h)||X(n+k)|\\ &=U_1\frac{\sigma^2}{\pi}D_1^{3/2}S_1(\rho_{h-k})+V_1\frac{\sigma^2}{\pi}D_1^{3/2}S_2(\rho_{h-k}).\end{aligned}$$

So we have

$$\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h EX(n)\,\mathrm{sgn}\,(X(n+h))|X(n+k)|=\frac{\rho_h}{2}\{U_1D_1^{3/2}S_1(\rho_{h-k})+V_1D_1^{3/2}S_2(\rho_{h-k})\}.$$

Similarly, we get

$$\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h E|X(n)|X(n+k)\,\mathrm{sgn}\,(X(n+k+h))=\frac{\rho_h}{2}\{U_2D_2^{3/2}S_1(\rho_{k+h})+V_2D_2^{3/2}S_2(\rho_{k+h})\},$$

where

$$D_2=\begin{vmatrix}1 & \rho_{h+k}\\ \rho_{h+k} & 1\end{vmatrix}, \qquad U_2=\frac{1}{D_2}\begin{vmatrix}\rho_k & \rho_{k+h}\\ \rho_h & 1\end{vmatrix} \quad \text{and} \quad V_2=\frac{1}{D_2}\begin{vmatrix}1 & \rho_k\\ \rho_{k+h} & \rho_h\end{vmatrix}.$$

Consequently, using the above results, we obtain

$$\begin{aligned}C(k)&=EY(n)Y(n+k)\\&=\frac{1}{2}\Big[\Big\{(AF^2+BF)D^{3/2}S_1(\rho_k)+(2AFG+BG+CF)D^{3/2}S_2(\rho_k)\\&\quad+(AG^2+CG)D^{3/2}S_1(\rho_k)+A\cdot\frac{\Delta}{2\sqrt{D}}S_3(\rho_k)\Big\}\\&\quad-\rho_h D_1^{3/2}\{U_1S_1(\rho_{h-k})+V_1S_2(\rho_{h-k})\}\\&\quad-\rho_h D_2^{3/2}\{U_2S_1(\rho_{h+k})+V_2S_2(\rho_{h+k})\}+\rho_h^2D^{3/2}S_2(\rho_k)\Big].\end{aligned}$$

(ii) Here we shall treat the case $|k|=h$. In the first place, let us consider the case $k=h$.

$$\begin{aligned}C(h)&=\frac{\pi}{2}\frac{1}{\sigma^2}EX(n)|X(n+h)|\operatorname{sgn}(X(n+2h))\\&\quad-\frac{\pi}{2}\frac{1}{\sigma^2}\rho_hEX(n)X(n+h)-\frac{\pi}{2}\frac{1}{\sigma^2}\rho_hE|X(n)|X(n+h)\operatorname{sgn}(X(n+2h))\\&\quad+\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h^2E|X(n)||X(n+h)|.\end{aligned}$$

In this expression,

$$\frac{\pi}{2}\frac{1}{\sigma^2}EX(n)|X(n+h)|\operatorname{sgn}(X(n+2h))=\frac{1}{2}D_h^{3/2}(H_1S_1(\rho_h)+K_1S_2(\rho_h)),$$

where

$$D_h=\begin{vmatrix}1&\rho_h\\\rho_h&1\end{vmatrix},\qquad H_1=\frac{1}{D_h}\begin{vmatrix}\rho_h&\rho_h\\\rho_{2h}&1\end{vmatrix}\qquad\text{and}\qquad K_1=\frac{1}{D_h}\begin{vmatrix}1&\rho_h\\\rho_h&\rho_{2h}\end{vmatrix}.$$

And

$$\frac{\pi}{2}\frac{1}{\sigma^2}\rho_hEX(n)X(n+h)=\frac{\pi}{2}\rho_h^2.$$

We treat the term

$$\frac{\pi}{2}\frac{1}{\sigma^2}\rho_hE|X(n)|X(n+h)\operatorname{sgn}(X(n+2h))$$

as the following. Let us put

$$X(n+h)=H_2X(n)+K_2X(n+2h)+\delta_2(n),$$

where $\delta_2(n)$ is independent of $X(n)$, $X(n+2h)$ and $(X(n), X(n+2h))$. The above condition is satisfied by determining the constants H_2 and K_2 from the following relations:

$$E\delta_2(n)X(n)=0\qquad\text{and}\qquad E\delta_2(n)X(n+2h)=0.$$

Then H_2 and K_2 are

$$H_2 = \frac{1}{D_{2h}} \begin{vmatrix} \rho_h & \rho_{2h} \\ \rho_h & 1 \end{vmatrix} \qquad \text{and} \qquad K_2 = \frac{1}{D_{2h}} \begin{vmatrix} 1 & \rho_h \\ \rho_{2h} & \rho_h \end{vmatrix},$$

where

$$D_{2h} = \begin{vmatrix} 1 & \rho_{2h} \\ \rho_{2h} & 1 \end{vmatrix}.$$

Hence we have

$$\begin{aligned} &\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h E|X(n)|X(n+h)\operatorname{sgn}(X(n+2h)) \\ &= \frac{\pi}{2}\frac{1}{\sigma^2}\rho_h H_2 EX(n)^2 \operatorname{sgn}(X(n))\operatorname{sgn}(X(n+2h)) \\ &\quad + \frac{\pi}{2}\frac{1}{\sigma^2}\rho_h K_2 E|X(n)||X(n+2h)| \\ &= \frac{1}{2}\rho_h D_{2h}^{3/2}(H_2 S_1(\rho_{2h}) + K_2 S_2(\rho_{2h})). \end{aligned}$$

Lastly, it is shown

$$\frac{\pi}{2}\frac{1}{\sigma^2}\rho_h^2 E|X(n)||X(n+h)| = \frac{1}{2}\rho_h^2 D_h^{3/2} S_2(\rho_h).$$

Consequently, we obtain

$$\begin{aligned} C(h) = \frac{1}{2}[&D_h^{3/2}(H_1 S_1(\rho_h) + K_1 S_2(\rho_h)) \\ &- \pi\rho_h^2 - \rho_h D_{2h}^{3/2}(H_2 S_1(\rho_{2h}) + K_2 S_2(\rho_{2h})) + \rho_h^2 D_h^{3/2} S_2(\rho_h)]. \end{aligned}$$

In the next place, when $k=-h$, we can consider

$$C(-h) = C(h).$$

(iii) When $k=0$,

$$\begin{aligned} C(0) &= \frac{\pi}{2}\frac{1}{\sigma^2} E(X(n)\operatorname{sgn}(X(n+h)) - \rho_h|X(n)|)^2 \\ &= \frac{\pi}{2}\frac{1}{\sigma^2}(EX(n)^2 - 2\rho_h EX(n)^2 \operatorname{sgn}(X(n))\operatorname{sgn}(X(n+h)) + \rho_h^2 EX(n)^2) \\ &= \frac{\pi}{2}\frac{1}{\sigma^2}\left(\sigma^2 - 2\rho_h\frac{\sigma^2}{\pi} D_h^{3/2} S_1(\rho_h) + \rho_h^2\sigma^2\right) \\ &= \frac{\pi}{2} - \rho_h D_h^{3/2} S_1(\rho_h) + \frac{\pi}{2}\rho_h^2. \end{aligned}$$

From the above results, we have

$$
\begin{aligned}
C_h &= \sum_{k=-(M+h)}^{M+h} C(k) = C(0) + 2\sum_{k=1}^{M+h} C(k) \\
&= \frac{\pi}{2} - \rho_h D_h^{3/2} S_1(\rho_h) - \frac{\pi}{2}\rho_h^2 + D_h^{3/2}(H_1 S_1(\rho_h) + K_1 S_2(\rho_h)) \\
&\quad - \rho_h D_{2h}^{3/2}(H_2 S_1(\rho_{2h}) + K_2 S_2(\rho_{2h})) + \rho_h^2 D_h^{3/2} S_2(\rho_h) \\
&\quad + \sum_{\substack{k=1 \\ (k\neq h)}}^{M+h} \Big[(AF^2+BF) D^{3/2} S_1(\rho_k) + (2AFG+BG+CF) D^{3/2} S_2(\rho_k) \\
&\quad + (AG^2+CG) D^{3/2} S_1(\rho_k) + A \cdot \frac{\Delta}{2\sqrt{D}} S_3(\rho_k) - \rho_h D_1^{3/2}(U_1 S_1(\rho_{h-k}) + V_1 S_2(\rho_{h-k})) \\
&\quad - \rho_h D_2^{3/2}(U_2 S_1(\rho_{h+k}) + V_2 S_2(\rho_{h+k})) + \rho_h^2 D^{3/2} S_2(\rho_k) \Big].
\end{aligned} \tag{3}
$$

Now we shall make the following assumptions:

(A, 1) The determinants Δ, D, D_1 and D_2 are not zero when $k \geqq 1$ and $k \neq h$.

(A, 2) $D_h \neq 0$ and $D_{2h} \neq 0$.

Here we rearrange the above results.

THEOREM 2. *If $X(n)$ is a stationary Gaussian process having a finite moving average representation* (1) *which satisfies the condition* (2) *and if the correlogram has the properties* (A, 1) *and* (A, 2), *the distribution function of $\sum_{n=1}^{N} Y(n)/\sqrt{N}$ tends to the normal distribution function with the mean zero and the variance C_h as $N \to \infty$.*

Now, we shall consider the distribution function of $\sqrt{N}(\Gamma_h - \rho_h)$. By Theorem 1,

$$\gamma_0 = \frac{1}{N}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{n=1}^{N} |X(n)|$$

converges in probability to 1 as $N \to \infty$. And by Theorem 2, the distribution function of

$$\frac{1}{\sqrt{N}}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma}\sum_{n=1}^{N}\{X(n)\,\mathrm{sgn}\,(X(n+h)) - \rho_h |X(n)|\}$$

tends to the normal distribution function with the mean zero and the variance C_h as $N \to \infty$. Therefore we have the following theorem.

THEOREM 3. *If $X(n)$ is a stationary Gaussian process having a finite moving average representation* (1) *which satisfies the condition* (2) *and if the correlogram has the properties* (A, 1) *and* (A, 2), *the distribution function of $\sqrt{N}(\Gamma_h - \rho_h)$ tends to the normal distribution function with the mean zero and the variance C_h as $N \to \infty$.*

§ 3. The estimate $\tilde{\Gamma}_h$.

In this section, we shall consider, with respect to the estimate $\tilde{\Gamma}_h$, the same as we did in § 2. Let the process $X(n)$ have the same properties as § 2.

Now we have

$$\sqrt{N}(\tilde{\Gamma}_h-\rho_h)=\sqrt{N}\left(\frac{\sum_{n=1}^{N}X(n)X(n+h)}{\sum_{n=1}^{N}X(n)^2}-\rho_h\right)=\sqrt{N}\left(\frac{\frac{1}{N}\frac{1}{\sigma^2}\sum_{n=1}^{N}X(n)X(n+h)}{\frac{1}{N}\frac{1}{\sigma^2}\sum_{n=1}^{N}X(n)^2}-\rho_h\right)$$

$$=\frac{\frac{1}{\sqrt{N}}\frac{1}{\sigma^2}\sum_{n=1}^{N}(X(n)X(n+h)-\rho_h X(n)^2)}{\frac{1}{N}\frac{1}{\sigma^2}\sum_{n=1}^{N}X(n)^2}.$$

We shall denote

$$\tilde{\gamma}_0=\frac{1}{N}\frac{1}{\sigma^2}\sum_{n=1}^{N}X(n)^2.$$

Then from the results in Huzii [4],

$$E(\tilde{\gamma}_0)=1$$

and

$$V(\tilde{\gamma}_0)=\frac{2}{N^2}\sum_{k=1}^{N-1}(N-k)(1+2\rho_k^2)+\frac{3}{N}-1$$

$$=\frac{1}{N}\left\{2+4\sum_{k=1}^{M}\left(1-\frac{k}{N}\right)\rho_k^2\right\}.$$

Hence, we have following lemma and theorem.

LEMMA 4. *If $X(n)$ is a stationary Gaussian process which has a finite moving average representation* (1), *$V(\tilde{\gamma}_0)$ tends to zero as $N\to\infty$.*

THEOREM 4. *If $X(n)$ is a stationary Gaussian process having a finite moving average representation* (1), *$\tilde{\gamma}_0$ converges in probability to* 1.

Now, we shall consider the statistic

$$\frac{1}{\sqrt{N}}\frac{1}{\sigma^2}\sum_{n=1}^{N}(X(n)X(n+h)-\rho_h X(n)^2).$$

Let us put

$$\tilde{Y}(n)=\frac{1}{\sigma^2}(X(n)X(n+h)-\rho_h X(n)^2).$$

As $X(n)$ is a stationary Gaussian process having a finite moving average representation (1) which satisfies the condition (2), $\tilde{Y}(n)$ is a $(M+h)$-dependent variable and $E\tilde{Y}(n)=0$. Clearly, $\tilde{Y}(n)$ is a stationary process. We shall denote

$$E\tilde{Y}(n)\tilde{Y}(m)=\tilde{C}(n-m).$$

By using the result of Lemma 3, the distribution function of the random variable

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\tilde{Y}(n)=\frac{1}{\sqrt{N}}\frac{1}{\sigma^2}\sum_{n=1}^{N}(X(n)X(n+h)-\rho_h X(n)^2)$$

tends to the normal distribution function with the mean zero and the variance $\sum_{k=-(M+h)}^{M+h}\tilde{C}(k)$ as $N\to\infty$.

Combining the above result with Theorem 4, we can say that the distribution function of $\sqrt{N}(\tilde{\Gamma}_h-\rho_h)$ tends to the normal distriburion function with the mean zero and the variance $\sum_{k=-(M+h)}^{M+h}\tilde{C}(k)$ as $N\to\infty$.

Let us now compute the value of $\sum_{k=-(M+h)}^{M+h}\tilde{C}(k)$.

$$\begin{aligned}\tilde{C}(k)&=E\tilde{Y}(n)\tilde{Y}(n+k)\\&=\frac{1}{\sigma^4}E(X(n)X(n+h)-\rho_h X(n)^2)(X(n+k)X(n+k+h)-\rho_h X(n+k)^2)\\&=\frac{1}{\sigma^4}\{EX(n)X(n+k)X(n+h)X(n+k+h)-\rho_h EX(n)^2X(n+k)X(n+k+h)\\&\qquad-\rho_h EX(n)X(n+k)^2X(n+h)+\rho_h^2EX(n)^2X(n+k)^2\}.\end{aligned}$$

(i) When k is neither zero nor $\pm h$,

$$\begin{aligned}\tilde{C}(k)&=(\rho_k^2+\rho_h^2+\rho_{h-k}\rho_{k+h})-\rho_h(\rho_h+2\rho_k\rho_{k+h})-\rho_h(\rho_h+2\rho_k\rho_{h-k})+\rho_h^2(1+2\rho_k^2)\\&=\rho_k^2+\rho_{h-k}\rho_{h+k}-2\rho_h\rho_k\rho_{k+h}-2\rho_h\rho_k\rho_{h-k}+2\rho_h^2\rho_k^2.\end{aligned}$$

(ii) When $k=h$,

$$\tilde{C}(h)=\rho_{2h}+2\rho_h^4-\rho_h^2-2\rho_h^2\rho_{2h}$$

and when $k=-h$,

$$\tilde{C}(-h)=\tilde{C}(h).$$

(iii) When $k=0$,

$$\tilde{C}(0)=1-\rho_h^2.$$

Putting

$$\tilde{C}_h=\sum_{k=-(M+h)}^{M+h}\tilde{C}(k),$$

we obtain, from the above results,

$$\begin{aligned}\tilde{C}_h=1-\rho_h^2&+2(\rho_{2h}+2\rho_h^4-\rho_h^2-2\rho_h^2\rho_{2h})\\&+2\sum_{\substack{k=1\\(k\neq h)}}^{M+h}(\rho_k^2+\rho_{h-k}\rho_{h+k}-2\rho_h\rho_k\rho_{h+k}-2\rho_h\rho_k\rho_{h-k}+2\rho_h^2\rho_k^2).\end{aligned}$$

Hence we have the following theorems:

THEOREM 5. *If* $X(n)$ *is a stationary Gaussian process having a finite moving average representation* (1) *which satisfies the condition* (2), *the distribution function* *of* $\sum_{n=1}^{N}\tilde{Y}(n)/\sqrt{N}$ *tends to the normal distribution function with the mean zero and the variance* $\tilde{C}_h$ *as* $N\to\infty$.

THEOREM 6. *If* $X(n)$ *is a stationary Gaussian process having a finite moving average representation* (1) *which satisfies the condition* (2), *the distribution function of* $\sqrt{N}(\tilde{\Gamma}_h-\rho_h)$ *tends to the normal distribution function with the mean zero and the variance* $\tilde{C}_h$ *as* $N\to\infty$.

§ 4. Comparison of the estimate Γ_h with the estimate $\tilde{\Gamma}_h$.

We shall compare the estimate Γ_h with the estimate $\tilde{\Gamma}_h$ on the viewpoint of the variance. Without loss of generality, we can assume $h>0$.

a) When $X(n)$ is a white noise, we have $\rho_k=0$ for any $k\neq 0$. So we have

$$C_h=\frac{\pi}{2} \qquad \text{and} \qquad \tilde{C}_h=1$$

for any $h\geqq 1$.

b) Let us assume

$$\rho_k=\begin{cases}\dfrac{1-\rho^{2(M-|k|+1)}}{1-\rho^{2(M+1)}}\cdot\rho^{|k|}\cos k\theta; & 0\leqq|k|\leqq M,\\ 0; & |k|\geqq M+1,\end{cases} \tag{4}$$

where ρ and θ are constants and $0\leqq\rho<1$. For simplicity, we write

$$\alpha_k=\frac{1-\rho^{2(M-|k|+1)}}{1-\rho^{2(M+1)}}\cos k\theta.$$

Then we have $|\alpha_k|<1$ and $\rho_k=\alpha_k\rho^{|k|}$.

In this case, we can say as follows:

THEOREM 7. *If* $|\rho_{h_0}|<\rho^{h_0}<\varepsilon$ *holds for sufficiently small positive number* ε, C_h *and* $\tilde{C}_h$ *are given approximately for any* $h\geqq h_0$ *as follows;*

$$C_h\sim\frac{\pi}{2}+2\sum_{k\geqq 1}\rho_k^2\sqrt{1-\rho_k^2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m+1)!}(m!)^2\right)$$

and

$$\tilde{C}_h\sim 1+2\sum_{k\geqq 1}\rho_k^2,$$

where the sign $\sim$ *is used to indicate that the left side and the right side are coinside by ignoring the magnitude of the order* ε.

Proof. Here, we shall prove this theorem only when $M\geqq h$. The stituation is the same when $h\geqq M+1$.

As $\rho_k=\alpha_k\rho^{|k|}$, we have for $h>k>0$, in the expression (3),

$$\Delta=1-\alpha_k^2\rho^{2k}-\alpha_{h-k}^2\rho^{2(h-k)}+O(\varepsilon),\qquad D=1-\alpha_k^2\rho^{2k},$$

$$D_1=1-\alpha_{h-k}^2\rho^{2(h-k)},\qquad D_2=1+O(\varepsilon^2),$$

$$D_h=1+O(\varepsilon^2),\qquad D_{2h}=1+O(\varepsilon^4),$$

$$A=\alpha_k\rho^k+O(\varepsilon),\qquad F=\alpha_{h-k}\rho^{h-k}/(1-\alpha_k^2\rho^{2k})+O(\varepsilon).$$

And each of B, C, G, H_1, K_1 is $O(\varepsilon)$. Further $AF^2D^{3/2}S_1(\rho_k)$ is $O(\varepsilon^2)$. Now we have

$$\begin{aligned}&A\cdot\frac{\Delta}{2\sqrt{D}}\cdot S_3(\rho_k)\\&=(\alpha_k\rho^k+O(\varepsilon))\cdot\frac{(1-\alpha_k^2\rho^{2k}-\alpha_{h-k}^2\rho^{2(h-k)}+O(\varepsilon))}{2\sqrt{1-\alpha_k^2\rho^{2k}}}\cdot 2\left(\sum_{m=0}^{\infty}\frac{(2\alpha_k\rho^k)^{2m+1}}{(2m+1)!}(m!)^2\right)\\&=2\alpha_k^2\rho^{2k}\sqrt{1-\alpha_k^2\rho^{2k}}\left(\sum_{m=0}^{\infty}\frac{(2\alpha_k\rho^k)^{2m}}{(2m+1)!}(m!)^2\right)+O(\varepsilon).\end{aligned}$$

Using the above results, we obtain

$$\begin{aligned}C_h&=\frac{\pi}{2}+\sum_{k\geqq 1}A\cdot\frac{\Delta}{2\sqrt{D}}S_3(\rho_k)+O(\varepsilon)\\&=\frac{\pi}{2}+2\sum_{k\geqq 1}\alpha_k^2\rho^{2k}\sqrt{1-\alpha_k^2\rho^{2k}}\left(\sum_{m=0}^{\infty}\frac{(2\alpha_k\rho^k)^{2m}}{(2m+1)!}(m!)^2\right)+O(\varepsilon)\\&=\frac{\pi}{2}+2\sum_{k\geqq 1}\rho_k^2\sqrt{1-\rho_k^2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m+1)!}(m!)^2\right)+O(\varepsilon).\end{aligned}$$

Similarly we have

$$\tilde{C}_h=1+2\sum_{k\geqq 1}\rho_k^2+O(\varepsilon).$$

Concerning the relation between C_h and $\tilde{C}_h$, we can obtain the following theorem:

THEOREM 8. *If the value of $|\rho_{h_0}|$ is sufficiently small, that is, $|\rho_{h_0}|<\rho^{h_0}<\varepsilon$ holds for sufficiently small positive number ε, it holds*

$$\frac{\pi}{2}\tilde{C}_h\geqq C_h>\tilde{C}_h$$

for any $h\geqq h_0$.

Proof. In the first place, we shall prove that $C_h>\tilde{C}_h$. By Theorem 7,

$$C_h\sim\frac{\pi}{2}+2\sum_{k\geqq 1}\rho_k^2\sqrt{1-\rho_k^2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m+1)!}(m!)^2\right)$$

and

$$\tilde{C}_h \sim 1+2\sum_{k\geqq 1}\rho_k^2.$$

We shall show

$$\sqrt{1-\rho_k^2}\left(\sum_{m=0}^{\infty}\frac{(2\rho_k)^{2m}}{(2m+1)!}(m!)^2\right)\geqq 1$$

for each k. For simplicity we put $\rho_k^2=X$, then the above relation is

$$\sqrt{1-X}\left(\sum_{m=0}^{\infty}\frac{2^{2m}(m!)^2}{(2m+1)!}X^m\right)\geqq 1.$$

We consider the function

$$f(X)=\left(\sum_{m=0}^{\infty}\frac{2^{2m}(m!)^2}{(2m+1)!}X^m\right)-\frac{1}{\sqrt{1-X}}$$

for $0\leqq X<1$. We have $f(0)=0$. Further

$$\begin{aligned}
f'(X)&=\sum_{m=1}^{\infty}\frac{2^{2m}(m!)^2 m}{(2m+1)!}X^{m-1}-\frac{1}{2}(1-X)^{-3/2}\\
&=\sum_{m=0}^{\infty}\frac{2^{2(m+1)}((m+1)!)^2(m+1)}{(2m+3)!}X^m-\sum_{m=0}^{\infty}\frac{(2m+1)(2m-1)\cdots 5\cdot 3\cdot 1}{m!\,2^{m+1}}X^m\\
&=\sum_{m=0}^{\infty}\left(\frac{2^{2(m+1)}((m+1)!)^2(m+1)}{(2m+3)!}-\frac{(2m+1)!!}{m!\,2^{m+1}}\right)X^m.\text{[1]}
\end{aligned}$$

Now we write

$$b_m=\frac{2^{2(m+1)}((m+1)!)^2(m+1)}{(2m+3)!},\qquad c_m=\frac{(2m+1)!!}{m!\,2^{m+1}}$$

and

$$a_m=b_m-c_m.$$

Then

$$f'(X)=\sum_{m=1}^{\infty}a_mX^m.$$

We have

$$b_0=\frac{2}{3}>c_0=\frac{1}{2}\qquad\text{and}\qquad a_0=\frac{2}{3}-\frac{1}{2}>0.$$

If $b_m>c_m$ holds, we find

$$b_{m+1}=b_m\cdot\frac{2^2(m+2)^3}{(2m+4)(2m+5)(m+1)}>c_{m+1}=c_m\cdot\frac{(2m+3)}{2(m+1)},$$

because

$$\frac{2^2(m+2)^3}{(2m+4)(2m+5)(m+1)}>\frac{(2m+3)}{2(m+1)}. \tag{5}$$

1) $(2m+1)!!=1\cdot 3\cdot 5\cdot\cdots\cdot(2m-1)\cdot(2m+1)$.

So we have $a_m>0$ for any positive integer m and this shows $f'(X)>0$ for $X\geqq 0$. This result shows $f(X)\geqq 0$ for $0\leqq X<1$ and we obtain

$$\sqrt{1-X}\left(\sum_{m=0}^{\infty}\frac{2^{2m}(m!)^2}{(2m+1)!}X^m\right)\geqq 1.$$

Consequently we have $C_h>\tilde{C}_h$.

In the next place we shall prove that $(\pi/2)\tilde{C}_h\geqq C_h$. For this purpose, we show

$$\frac{\pi}{2}>\sqrt{1-X}\left(\sum_{m=0}^{\infty}\frac{2^{2m}(m!)^2}{(2m+1)!}X^m\right),$$

by writing $\rho^{2k}=X$ as the above. Let us consider the function

$$g(X)=\frac{\pi}{2}\frac{1}{\sqrt{1-X}}-\sum_{m=0}^{\infty}\frac{2^{2m}(m!)^2}{(2m+1)!}X^m$$

for $0\leqq X<1$. We have $g(0)=\pi/2-1>0$ and

$$\begin{aligned} g'(X)&=\frac{\pi}{4}(1-X)^{-3/2}-\sum_{m=1}^{\infty}\frac{2^{2m}(m!)^2m}{(2m+1)!}X^{m-1}\\ &=\sum_{m=0}^{\infty}\left(\frac{\pi}{4}\frac{(2m+1)!!}{m!\,2^m}-\frac{2^{2(m+1)}((m+1)!)^2(m+1)}{(2m+3)!}\right)X^m. \end{aligned}$$

We shall write

$$e_m=\frac{\pi}{4}\frac{(2m+1)!!}{m!\,2^m},\qquad f_m=\frac{2^{2(m+1)}((m+1)!)^2(m+1)}{(2m+3)!}$$

and

$$g_m=e_m-f_m.$$

Then we have

$$e_0=\frac{\pi}{4}>f_0=\frac{2}{3}\qquad\text{and}\qquad g_0>0.$$

We show $g_m\geqq 0$ for any positive integer m. Let us assume that, for a certain integer m, $g_m<0$, that is, $e_m<f_m$. Then we find

$$e_{m+1}=e_m\cdot\frac{(2m+3)}{2(m+1)}<f_{m+1}=f_m\cdot\frac{2^2(m+2)^3}{(2m+4)(2m+5)(m+1)},$$

by using the relation (5). This shows $g_{m'}<0$ for any $m'\geqq m$ and we have

$$1>\frac{e_m}{f_m}>\frac{e_{m+1}}{f_{m+1}}>\frac{e_{m+2}}{f_{m+2}}>\cdots.$$

On the other hand,

$$\begin{aligned} \lim_{m\to\infty}\frac{e_m}{f_m}&=\lim_{m\to\infty}\frac{\pi}{4}\frac{(2m+1)!!}{m!\,2^m}\cdot\frac{(2m+3)!}{2^{2(m+1)}((m+1)!)^2(m+1)}\\ &=\lim_{m\to\infty}\frac{\pi}{4}\frac{(2m+1)!}{2^{2m}(m!)^2}\frac{(2m+3)!}{2^{2(m+1)}((m+1)!)^2(m+1)}. \end{aligned}$$

Using Stirling's formula

$$n!\sim(2\pi)^{1/2}n^{n+1/2}e^{-n},$$

we have

$$\lim_{m\to\infty}\frac{e_m}{f_m}=\lim_{m\to\infty}\frac{\pi}{4}\cdot\frac{(2\pi)^{1/2}(2m+1)^{2m+3/2}e^{-(2m+1)}}{2^{2m}(2\pi)m^{2m+1}e^{-2m}}\cdot\frac{(2\pi)^{1/2}(2m+3)^{2m+7/2}e^{-(2m+3)}}{2^{2(m+1)}(m+1)(2\pi)(m+1)^{2m+3}e^{-2(m+1)}}$$

$$=\frac{1}{e^2}\lim_{m\to\infty}\left(1+\frac{1}{2m}\right)^{2m}\left(1+\frac{1}{2m+2}\right)^{2m+2}\left(1+\frac{1}{2m}\right)^{3/2}\left(1+\frac{3}{2m}\right)^{3/2}\cdot\frac{1}{\left(1+\frac{1}{m}\right)^2}=1.$$

This is a contradiction. Consequently we have $g_m\geqq 0$ for all positive integer m.

From this result, we obtain $g'(X)>0$ for $0\leqq X<1$ and $g(X)>0$ for $0\leqq X<1$. This implies

Table 1.

h	ρ_h	C_h	$\tilde{C}_h$
1	0.4322	0.484	0.279
2	−0.2663	1.244	0.829
3	−0.5069	2.000	1.423
4	−0.2677	2.630	1.948
5	0.0929	3.101	2.360
6	0.2517	3.430	2.661
7	0.1581	3.650	2.870
8	−0.0244	3.793	3.010
9	−0.1223	3.887	3.103
10	−0.0901	3.950	3.165
11	0.0004	3.992	3.207
12	0.0580	4.022	3.236
13	0.0499	4.044	3.256
14	0.0060	4.060	3.271
15	−0.0267	4.073	3.284
16	−0.0269	4.084	3.293
17	−0.0062	4.091	3.300
18	0.0119	4.096	3.305
19	0.0142	4.098	3.307
20	0.0047	4.098	3.307
21	−0.0050	4.099	3.308
22	−0.0072	4.099	3.308
23	−0.0031	4.099	3.308
24	0.0019	4.099	3.308
25	0.0035	4.099	3.308
30	0.0001	4.100	3.309

$$\frac{\pi}{2} > \sqrt{1-X}\left(\sum_{m=0}^{\infty}\frac{2^{2m}(m!)^2}{(2m+1)!}X^m\right)$$

and we obtain $(\pi/2)\tilde{C}_h \geqq C_h$.

c) As it is difficult to compare C_h with $\tilde{C}_h$ generally, we make a comparison numerically.

For this purpose, we treat the case when the correlogram ρ_k is defined by (4).

Considering the case

$$\rho=0.8, \qquad \theta=0.25 \quad \text{and} \quad M=30,$$

we obtain the result of numerical comparison as Table 1. This result is also shown as Figure 1.

The situation of the other cases, assuming each of the parameters ρ, θ and M to have various values, will be similar to that of the above case. Generally, C_h will be greater than $\tilde{C}_h$.

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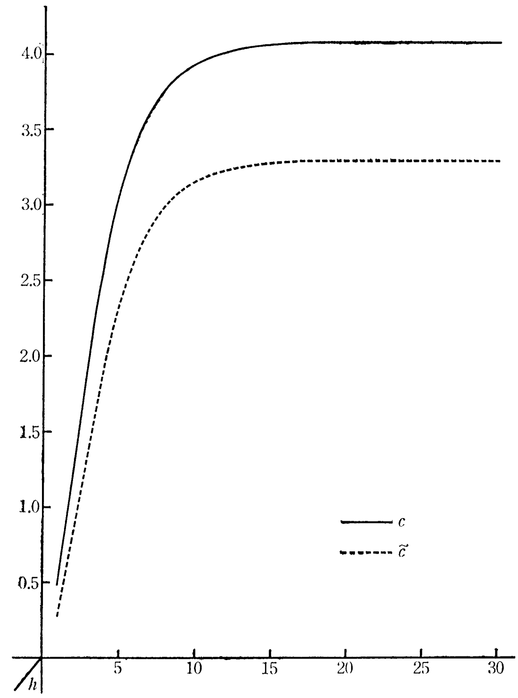

Figure 1.

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