ON THE EXISTENCE OF ANALYTIC MAPPINGS, II

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1. Let $G(z)$ and $g(z)$ be two entire functions having no zero other than an infinite number of simple zeros, respectively. Let R and S be two ultrahyperelliptic surfaces defined by two equations $y^2=G(x)$ and $y^2=g(x)$, respectively. In our previous paper [3] we offered a conjectural problem: *Is the order ρ_G of G an integral multiple of the order ρ_g of g, when there is an analytic mapping φ from R into S?* As we remarked there, in this problem we should assume that $\rho_G<\infty$ and $0<\rho_g<\infty$ and further suitable normalizations on G and g are done. Let G_c and g_c be two canonical products having the same zeros with the same multiplicities as those of G and g, respectively. In this paper an analytic mapping means a non-trivial one.

Theorem 1. *Assume that $\rho_{G_c}<\infty$ and $0<\rho_{g_c}<\infty$ and that there exists an analytic mapping φ from R into S. Then ρ_{G_c} is an integral multiple of ρ_{g_c}.*

This is somewhat effective criterion for the non-existence of an analytic mapping from R into S. Theorem 1 can be stated in the following form:

Assume that $\rho_{N(r;0,G)}<\infty$ and $0<\rho_{N(r;0,g)}<\infty$ and that there exists an analytic mapping φ from R into S. Then the former one is an integral multiple of the latter one.

2. To prove theorem 1 we need an elegant theorem due to Valiron [7]. We can state his result in the following manner.

Let $h(z)$ be an entire function satisfying one of the following conditions:
(a) *$h(z)$ has a finite order;*
(b) *There is a suitable number $\lambda>1$ satisfying*

$$\lim_{r\to\infty}\frac{\log V(r^\lambda)}{V(r)}=0,\quad V(r)=\log M(r),\quad M(r)=\max_{|z|\leqq r}|h(z)|.$$

Then the equation $h(z)=w$ has at least one solution z in the annulus

$$M^{-1}(|w|)\leqq|z|\leqq M^{-1}(|w|)^{1+\alpha}$$

for an arbitrary small positive number α, if $|w|$ is sufficiently large, $|w|>A(\alpha)$.

As Valiron remarked, (b) implies (a) and (b) is satisfied by a quite wide class of entire functions, which contains some entire functions of infinite order. He also gave another theorem which is more precise and applicable than the above.

Received March 23, 1965.

3. Next we shall prove some estimations on the value-distribution of a composite entire function $g\circ h(z)$, where g and h are two entire functions. Let $g(z)$ be as in § 1. Since g has no zero other than an infinite number of simple zeros, the N-function $N_2(r; 0, g\circ h)$ of simple zeros of $g\circ h$ satisfies

$$N(r; 0, g\circ h)=N_2(r; 0, g\circ h)+N_1(r; 0, g\circ h)+\bar{N}_1(r; 0, g\circ h)$$

and

$$\begin{aligned}\bar{N}_1(r; 0, g\circ h)&\leqq N_1(r; 0, g\circ h)\leqq N(r; 0, h')\leqq m(r, h')\\ &\leqq m(r, h)+m(r, h'/h)\leqq(1+\varepsilon)m(r, h), \qquad \lim_{r\to\infty}\varepsilon=0\end{aligned}$$

with some negligible exceptional intervals. Further

$$N(r; 0, g\circ h)\geqq \sum_1^p N(r; w_\mu, h)$$

for an arbitrary but fixed number p of zeros $\{w_\mu\}$ of $g(z)$ and for all sufficiently large r. Assume that h is transcendental. By the second fundamental theorem for h

$$\begin{aligned}\sum_1^p N(r; w_\mu, h)&\geqq(p-1)m(r, h)-O(\log rm(r, h))\\ &\geqq(p-1-\varepsilon')m(r, h), \qquad \lim_{r\to\infty}\varepsilon'=0\end{aligned}$$

with some negligible exceptional intervals. Thus we have

$$N(r; 0, g\circ h)\geqq Km(r, h), \qquad N_2(r; 0, g\circ h)\geqq Km(r, h)$$

for an arbitrary but fixed positive number K and for all sufficiently large r with some negligible exceptional intervals. These imply that

$$\text{(A)} \qquad \varlimsup_{r\to\infty}\frac{N(r; 0, g\circ h)}{N_2(r; 0, g\circ h)}=1, \qquad \varlimsup_{r\to\infty}\frac{N_2(r; 0, g\circ h)}{N(r; 0, g\circ h)}=1$$

and

$$\text{(A}'\text{)} \qquad \varlimsup_{r\to\infty}\frac{N(r; 0, g\circ h)}{m(r, h)}=\infty, \qquad \varlimsup_{r\to\infty}\frac{m(r, h)}{N(r; 0, g\circ h)}=0.$$

Further here we assume that h is of finite order. Then

$$n(r; 0, g\circ h)=\sum{}^* n(r; w_\mu, h),$$

where $\sum^*$ indicates the summation over all w_μ for which there exists at least one root of $h(z)=w_\mu$ in $|z|<r$. By Valiron's theorem and by the second fundamental theorem for h we have

$$\begin{aligned}\sum{}^* n(r; w_\mu, h)&\geqq \sum{}^* N(r; w_\mu, h)/\log(r/r_0)\\ &\geqq(n(M(r^{1/(1+\alpha)}); 0, g)-2)\frac{m(r, h)}{\log r-\log r_0}.\end{aligned}$$

Since $m(r, h)\geqq P(\log r-\log r_0)$ for all sufficiently large r and for an arbitrary but fixed positive number P, we have

$$n(r;0,g\circ h)\geqq P(n(M(r^{1/(1+\alpha)});0,g)-2).$$

Since $M(r^{1/(1+\alpha)})>r$ for all sufficiently large r and $N(r)$, $n(r)$ are monotone for r, we have by dividing by r and by integrating

$$\text{(B)}\qquad N(r;0,g\circ h)\geqq P(N(r;0,g)-2\log r+O(1)).$$

We construct another estimation for $N(r;0,g\circ h)$ under the same assumptions. By the well-known inequalities

$$n(r/2)\log 2\leqq N(r)=\int_{r_0}^{r}\frac{n(r)}{r}dr\leqq n(r)(\log r-\log r_0),$$

we have

$$\text{(B}'\text{)}\qquad \begin{aligned} N(r;0,g\circ h)&\geqq n\left(\frac{r}{2};0,g\circ h\right)\log 2\\ &\geqq\left(\frac{N(M((r/2)^{1/(1+\alpha)});0,g)}{\log M((r/2)^{1/(1+\alpha)})-\log c_0}-2\right)P\log 2\end{aligned}$$

for all sufficiently large r and for an arbitrary but fixed positive number P.

If h is a polynomial of degree ν and has a form $a_0z^\nu+a_1z^{\nu-1}+\cdots+a_\nu$, then we have

$$\begin{aligned} n(r;0,g\circ h)&\geqq\sum{}^{*}n(r;w_\mu,h)\geqq\sum{}^{*}\nu\\ &=\nu n(|a_0|r^\nu(1-\varepsilon);0,g)-O(1),\qquad \varepsilon>0,\end{aligned}$$

and hence

$$\text{(C)}\qquad N(r;0,g\circ h)\geqq N(|a_0|r^\nu(1-\varepsilon);0,g)-O(\log r).$$

4. We shall now enter into our proof of theorem 1. In our previous papers [3], [4] we proved the following theorem.

If there exists an analytic mapping φ from R into S, then there exist two entire functions h and f satisfying an equation of the form

$$f(z)^2G_c(z)=g_c\circ h(z)$$

and vice-versa.

By this theorem we have

$$\text{(D)}\qquad \begin{aligned} N_2(r;0,g_c\circ h)&\leqq N(r;0,G_c)=N(r;0,g_c\circ h)-2N(r;0,f)\\ &\leqq N_2(r;0,g_c\circ h)+2m(r,h)-2N(r;0,f).\end{aligned}$$

This shows that by (A), (A′)

$$\text{(D}'\text{)}\qquad N(r;0,f)\leqq m(r,h)=o(N(r;0,g_c\circ h))=o(N_2(r;0,g_c\circ h)).$$

Thus we have

$$N_2(r;0,g_c\circ h)\leqq N(r;0,G_c)\leqq(1+\varepsilon)N_2(r;0,g_c\circ h)\leqq(1+\varepsilon')N(r;0,g_c\circ h).$$

In the first place we assume that

$$m(r, G_c)=O(m(r, h)).$$

Then we have

$$Km(r, h)\leqq N_2(r; 0, g_c\circ h)\leqq N(r; 0, G_c)\leqq m(r, G_c)=O(m(r, h)),$$

which shows that K is bounded above. This contradicts the arbitrariness of K. If h is of infinite order, then by the order finiteness of G_c we have

$$m(r, G_c)=o(m(r, h)).$$

Next we assume that h is of finite non-zero order. Then by (B′) and by the non-zero property of ρ_{g_c} there exists an infinite sequence $\{r_n\}$ for which

$$\frac{N(r_n; 0, g_c\circ h)}{N(r_n; 0, G_c)}\geqq\frac{P\log 2}{r_n^{\rho_{G_c}+\varepsilon}}\left(\frac{\exp((r_n/2)^{1/(1+\alpha)(\rho_h-\varepsilon)}(\rho_{g_c}-\varepsilon))}{(r_n/2)^{(\rho_h+\varepsilon)/(1+\alpha)}-c_1}-2\right).$$

The right hahd side term tends to ∞ if n tends to ∞. This implies that

$$\varlimsup_{r\to\infty}\frac{N(r; 0, g_c\circ h)}{N(r; 0, G_c)}=\infty,$$

which is untenable by (D) and (D′).

If h is transcendental but of order zero, then we have

$$M((r/2)^{1/(1+\alpha)})\geqq(r/2)^{p/(1+\alpha)}$$

for an arbitrary positive number p and for all sufficiently large r. Then by (B′) we have

$$\frac{N(r; 0, g_c\circ h)}{N(r; 0, G_c)}\geqq\frac{P\log 2}{r^{\rho_{G_c}+\varepsilon}}\left(\frac{(r/2)^{p(\rho_{g_c}-\varepsilon)/(1+\alpha)}}{(r/2)^{\varepsilon/(1+\alpha)}-c_1}-2\right)$$

and hence by the non-zero property of ρ_{g_c} and by the arbitrariness of p we can say that the right hand side term tends to ∞ when r tends to ∞ along a suitable sequence $\{r_n\}$. This implies that

$$\varlimsup_{r\to\infty}\frac{N(r; 0, g_c\circ h)}{N(r; 0, G_c)}=\infty,$$

which is again untenable by (D) and (D′).

If h is a polynomial of degree ν, then by (C)

$$\log m(r, g_c\circ h)\geqq\log N(r; 0, g_c\circ h)\geqq\log N(|a_0|r^\nu(1-\varepsilon); 0, g_c)-O(\log\log r)$$

and hence

$$\rho_{g_c\circ h}\geqq\rho_{N(r;0,g_c\circ h)}\geqq\varlimsup_{r_n\to\infty}\frac{\log N(|a_0|r_n^\nu(1-\varepsilon); 0, g_c)}{\log r_n}$$

$$\geqq\varlimsup_{r_n\to\infty}\frac{(\rho_{g_c}-\varepsilon)\nu\log r_n-c}{\log r_n}=(\rho_{g_c}-\varepsilon)\nu,$$

for a suitable sequence $\{r_n\}$, $r_n \to \infty$ and for an arbitrary positive number ε. Thus we have

$$\rho_{g_c \circ h} \geqq \nu \rho_{g_c}.$$

Further we have

$$\nu \rho_{g_c} \leqq \rho_{N(r;0,g_c \circ h)} = \rho_{G_c} = \overline{\lim_{r\to\infty}} \frac{\log N(r; 0, G_c)}{\log r}$$

$$= \rho_{N(r;0,G_c)} = \rho_{N(r;0,g_c \circ h)} \leqq \rho_{g_c \circ h}.$$

Evidently we have by Pólya's method [5]

$$\rho_{g_c \circ h} \leqq \nu \rho_{g_c}.$$

Thus we have the desired result and its corollary.

5. We shall prove the following theorem:

THEOREM 2. *Let g be an entire function of finite order having no zero other than an infinite number of simple zeros. Let R be an ultrahyperelliptic surface defined by $y^2 = g(x)$. If there exists an analytic mapping φ from R into itself, then φ is a univalent conformal mapping from R onto itself and the corresponding entire function $h(z)$ is a linear function of the form $e^{2\pi i q/p} z + b$ with a suitable rational number q/p.*

Proof. If $\rho_{g_c} > 0$, then by theorem 1 we have $\rho_{g_c} = \nu \rho_{g_c}$. This implies $\nu = 1$ in this case. Thus h must be a linear function $az + b$.

If $\rho_{g_c} = 0$ and $0 < \rho_h$, then $m(r, g_c) = o(m(r, h))$. On the other hand by the equation $f(z)^2 g(z) = g \circ h(z)$ we have

$$N_2(r; 0, g_c \circ h) \leqq N(r; 0, g_c) = N(r; 0, g_c \circ h) - 2N(r; 0, f)$$

$$\leqq N_2(r; 0, g_c \circ h) + 2m(r, h) - 2N(r; 0, f)$$

and by (A) and (A′)

$$N(r; 0, g_c \circ h) \sim N_2(r; 0, g_c \circ h) \sim N(r; 0, g_c).$$

This implies that $N(r; 0, g_c \circ h) = o(m(r, h))$, which contradicts (A′).

If $\rho_{g_c} = 0$ and $\rho_h = 0$, then by (B)

$$N(r; 0, g_c \circ h) \geqq P(N(r; 0, g_c) - 2 \log r).$$

Therefore by $\log r = o(N(r; 0, g_c))$

$$\overline{\lim_{r\to\infty}} \frac{N(r; 0, g_c \circ h)}{N(r; 0, g_c)} \geqq P.$$

Since P is arbitrary, we finally have

$$\overline{\lim_{r\to\infty}} \frac{N(r; 0, g_c \circ h)}{N(r; 0, g_c)} = \infty.$$

This contradicts

$$\overline{\lim_{r\to\infty}}\frac{N(r;0,g_c\circ h)}{N(r;0,g_c)}=1.$$

If $\rho_{g_c}=0$ and h is a polynomial of degree ν, then $\rho_{g_c}\circ h=0$ and by (C) or more direct enumeration

$$\overline{\lim_{r\to\infty}}\frac{N(r;0,g_c\circ h)}{N(r;0,g_c)}\geqq\nu.$$

On the other hand the left hand side term is equal to 1. Hence $\nu=1$. Thus h must be a linear function $az+b$.

In all cases we have that h must be a linear function $az+b$. This implies the first part of theorem 2, that is, φ is a univalent conformal mapping from R onto itself. Then its iteration $\varphi_n=\varphi\circ\varphi_{n-1}$ is also of the same nature. These mappings carry every branch point to a branch point and vice versa. If $|\alpha|>1$, then the set E of zero points of $g(z)$ satisfies $E=(E-b)/a$, and hence E has a finite cluster point $b/(1-a)$. This is a contradiction. If $|\alpha|<1$, then the set E of zero points of $g(z)$ has a finite cluster point $b/(1-a)$. This is also untenable. Let a be $e^{2\pi i\alpha}$, α: real. If α is an irrational number, then E has at least one cluster point on a circumference with a suitable radius and the center $b/(1-a)$. This is also untenable. Thus we have the desired result.

As a corollary we have the following fact:

COROLLARY 1. *If there exist two into analytic mappings φ: $R\to S$ and ψ: $S\to R$ and if R is an ultrahyperelliptic surface defined by an equation $y^2=g(x)$ having no zero other then an infinite number of simple zeros and satisfying $\rho_g<\infty$, then R and S are conformally equivalent with each other.*

6. Remarks. We should here remark that Shimizu [6] solved the following equation

$$g(z)=g\circ h(z).$$

To solve the Shimizu equation is somewhat easier than ours. In fact, as he did, it is sufficient to compare two Nevanlinna characteristic functions $m(r,g)$ and $m(r,g\circ h)$ and then he obtained the solution $h(z)=e^{2\pi iq/p}z+b$ with a suitable rational number q/p. However the appearance of an unknown function $f(z)$ makes our problem difficult and we cannot prove theorem 2 by comparison of two characteristic functions $m(r,g)$ and $m(r,g\circ h)$. It would be very plausible to conjecture that theorem 2 holds without any additional condition on the order of g.

In some special cases the equation

$$f(z)^2G(z)=g\circ h(z)$$

was perfectly solved [1], [3]. In these cases f had a quite few zeros. This is indeed a general property. We shall prove this. By the equation we have

$$N(r;0,f)\leqq m(r,h)=o(N(r;0,g\circ h)).$$

This shows that the number of zeros of f is quite few in relation to that of $g\circ h$. Further we can conclude that the defect

$$\delta(0, f)=1-\overline{\lim_{r\to\infty}}\frac{N(r; 0, f)}{m(r, f)}$$

is equal to 1, when $m(r, f)\geqq O(N(r; 0, g\circ h))$. If $m(r, f)=o(N(r; 0, g\circ h))$, then the above fact does not hold in general. This is shown by several examples.

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