ON ULTRAHYPERELLIPTIC SURFACES

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§ **1.** Let R be an open Riemann surfaces. Let $\mathfrak{M}(R)$ be a family of non-constant meromorphic functions on R. Let f be a member of $\mathfrak{M}(R)$. Let $P(f)$ be the number of Picard's exceptional values of f, where we say α a Picard's value of f when α is not taken by f on R. Let $P(R)$ be a quantity defined by

$$\sup_{f\in\mathfrak{M}(R)} P(f).$$

In general $P(R)\geqq 2$. In [4] we showed that this was an important quantity belonging to R for a criterion of non-existence of analytic mapping.

Now let R be an ultrahyperelliptic surface, which is a proper existence domain of a two-valued algebroid function $\sqrt{g(z)}$ with an entire function $g(z)$ of z whose zeros are all simple and are infinite in number. Then by Selberg's generalization of Nevanlinna's theory we have $P(R)\leqq 4$. Further we showed that $P(R)$ was equal to 2 in almost all cases of ultrahyperelliptic surfaces, that is, we had the following result: If $g(z)$ is of non-integral finite order, then $P(R)=2$. In the present paper we shall establish the existence of an ultrahyperelliptic surface R with $P(R)=3$. The existence of the surfaces with $P(R)=4$ is evident, however we need a characterization of these surfaces with $P(R)=4$ for our purpose. We do not give any characterization of the ultrahyperelliptic surfaces with $P(R)=3$.

§ **2. A lemma on the number of simple zeros of the function** $e^{h(z)}-\nu$. In the sequel we need a property of the function $e^h-\nu$ on the number of simple zeros several times. Let T, m, N, N_1, $\bar{N}$ and S be the quantities defined in Nevanlinna's theory [3]. Let $N_2(r;\, a, f)$ and $\bar{N}_1(r;\, a, f)$ be the N-functions with respect to the simple a-points and to the multiple a-points of the indicated function f, which is counted only once, respectively.

LEMMA. *Let h be an arbitrary given entire function of z. Then we have*

$$\varlimsup_{r\to\infty}\frac{N_2(r;\,\nu,e^h)}{T(r;\,e^h)}=1$$

for every non-zero constant ν.

Proof. By Nevanlinna's second fundamental theorem we have

$$T(r, e^h)<N(r;\,0,e^h)+N(r;\,\infty,e^h)+N(r;\,\nu,e^h)-N_1(r;\,e^h)+S(r),$$

$$S(r)<O(\log rT(r,e^h))$$

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with some suitable exceptional intervals. In this case $N(r;\ 0,\ e^h)=N(r;\ \infty,\ e^h)=0$. On the other hand we have

$$m(r,\ h')=m(r,\ h'e^h/e^h)=O(\log rT(r,\ e^h))$$

with some exceptional intervals. Since h' is an entire function, we have

$$T(r,\ h')=m(r,\ h').$$

Thus we have

$$N(r;\ 0,\ h')\leqq T(r,\ h')=O(\log rT(r,\ e^h)).$$

However we have

$$N_1(r;\ e^h)=N(r;\ 0,\ h').$$

Since there are relations

$$N(r;\ \nu,\ e^h)-\bar{N}(r;\ \nu,\ e^h)=N_1(r;\ \nu,\ e^h)\leqq N_1(r;\ e^h),$$

we have

$$\bar{N}_1(r;\ \nu,\ e^h)\leqq N_1(r;\ \nu,\ e^h)=O(\log rT(r,\ e^h)).$$

Therefore by the second fundamental theorem we have

$$\varlimsup_{r\to\infty}\frac{N(r;\ \nu,\ e^h)-N_1(r;\ \nu,\ e^h)-\bar{N}_1(r:\ \nu,\ e^h)}{T(r,\ e^h)}=1.$$

Thus we have

$$\varlimsup_{r\to\infty}\frac{N_2(r;\ \nu,\ e^h)}{T(r,\ e^h)}=1,$$

which is the desired result: If h is a polynomial, then our result is evident.

§ **3.** We shall here give a characterization of R with $P(R)=4$ by the form of defining function $g(z)$. Suppose that $P(R)=4$. Then there is a two-valued entire algebroid function f of z which is regular on R and whose defining equation is

$$F(z,f)\equiv f^2-2f_1(z)f+f_1(z)^2-f_2(z)^2g(z)=0$$

with two single-valued entire functions $f_1(z)$ and $f_2(z)$ of z. Further we may assume that 0, 1, a and ∞ are four Picard's values of f. Then, by Rémoundos' reasoning of his celebrated generalization of Picard's theorem [6] pp. 25–27, we have three possibilities:

$$\begin{pmatrix} F(z,\ 0) \\ F(z,\ 1) \\ F(z,\ a) \end{pmatrix}=\begin{pmatrix} c \\ \beta_1e^{H_1} \\ \beta_2e^{H_2} \end{pmatrix} \text{ or } \begin{pmatrix} \beta_1e^{H_1} \\ c \\ \beta_2e^{H_2} \end{pmatrix} \text{ or } \begin{pmatrix} \beta_1e^{H_1} \\ \beta_2e^{H_2} \\ c \end{pmatrix},$$

where β_1 and β_2 are non-zero constants and H_1 and H_2 are two entire functions of z satisfying $H_1(0)=H_2(0)=0$ and being non-constant functions.

In the first case we have

$$\begin{cases} f_1^2-f_2^2g=c \\ 1-2f_1+f_1^2-f_2^2g=\beta_1e^{H_1} \\ a^2-2af_1+f_1^2-f_2^2g=\beta_2e^{H_2}, \end{cases}$$

Then we have

$$(a-c)(1-a)=a\beta_1e^{H_1}-\beta_2e^{H_2}.$$

On the other hand the impossiblity of an identity of the form

$$A_1e^{H_1}+A_2e^{H_2}=A_3,$$

where A_1, A_2 and A_3 are constants, when $A_3\neq 0$, is equivalent to Picard's theorem. This is nothing but Borel's formulation of Picard's theorem [1], [2]. Thus we have $a=c$, since $a\neq 1$. Simultaneously we have $a\beta_1=\beta_2$ and $H_1\equiv H_2$. Then we have

$$(1-a)^2-2(1+a)\beta_1e^{H_1}+\beta_1^2e^{2H_1}=4f_2^2g.$$

Let b be a zero of $g(z)$, then we have

$$(1-a)^2-2(1+a)\beta_1e^{H_1(b)}+\beta_1^2e^{2H_1(b)}=0,$$

that is,

$$\beta_1e^{H_1(b)}=1+a\pm\sqrt{a}\,.$$

Thus we have

$$\begin{aligned} 4f_2^2g&=(\beta_1e^{H_1}-\beta_1e^{H_1(b)})(\beta_1e^{H_1}+\beta_1e^{H_1(b)}-2-2a) \\ &=\beta_1^2(e^{H_1}-\gamma)(e^{H_1}-\delta), \\ \gamma&=(1+\sqrt{a})^2/\beta_1, \qquad \delta=(1-\sqrt{a})^2/\beta_1. \end{aligned}$$

Since $a\neq 0, 1$, we have $\gamma\delta\neq 0$ and $\gamma\neq\delta$. Then $g(z)$ is equal to an expression of the following form:

$$\frac{(e^{H_1}-\gamma)(e^{H_1}-\delta)}{U^2},$$

where $U(z)$ is an entire function of z which is defined in the following manner: If the function $e^{H_1}-\gamma$ has a point z as its zero of multiplicity $2n$ or $2n+1$, then the function V has the point z as a zero of multiplicity n. Similarly we shall define a function W for $e^{H_1}-\delta$. Then we put $U=VW$. Thus we have

$$f_2=\pm\frac{\beta_1}{2}U, \qquad f_1=\frac{1+a}{2}-\frac{\beta_1}{2}e^{H_1}.$$

Therefore we finally have

$$f=\frac{1+a}{2}-\frac{\beta_1}{2}e^{H_1}\pm\frac{\beta_1}{2}\sqrt{(e^{H_1}-\gamma)(e^{H_1}-\delta)},$$

Hence the surface R is defined by an equation of the form

$$y^2=(e^{H_1(x)}-\gamma)(e^{H_1(x)}-\delta)/U^2(x),$$

$$\gamma\delta\neq 0, \qquad \gamma\neq\delta.$$

In the second case we have similarly a representation

$$f=\frac{a}{2}+\frac{\beta_1}{2}e^{H_1}\pm\frac{\beta_1}{2}\sqrt{(e^{H_1}-\gamma')(e^{H_1}-\delta')},$$

$$\gamma'=(1+\sqrt{1-a})^2/\beta_1, \quad \delta'=(1-\sqrt{1-a})^2/\beta_1, \quad \gamma'\delta'\neq 0, \quad \gamma'\neq\delta'$$

and a defining equation of R with an entire function U defined quite similarly

$$y^2=(e^{H_1(x)}-\gamma')(e^{H_1(x)}-\delta')/U^2(x).$$

In the third case we have a representation

$$f=\frac{1}{2}+\frac{\beta_1}{2a}e^{H_1}\pm\frac{\beta_1}{2\sqrt{a}}\sqrt{(e^{H_1}-\gamma'')(e^{H_1}-\delta'')},$$

$$\gamma''=-a(1-2a+\sqrt{(1-3a)(1-a)}), \delta''=-a(1-2a-\sqrt{(1-3a)(1-a)}).$$

Since $a\neq 0$, we have $\gamma''\delta''\neq 0$. If $\gamma''=\delta''$, then $a=1/3$, since $a\neq 1$. If $a=1/3$, then f is reduced to a single-valued entire function and hence $P(f)=2$, which may be omitted. Thus we have $\gamma''\neq\delta''$. Hence we have the defining equation of R

$$y^2=(e^{H_1(x)}-\gamma'')(e^{H_1(x)}-\delta'')/U(x)^2.$$

Here we should remark that the function $e^H-\gamma$, $\gamma\neq 0$, has an infinite number of simple zeros. This is due to the Lemma in § **2**, although we can prove this qualitatively by the ramification relation in Nevanlinna theory [2].

In every case we have a defining equation of R in the following form

$$y^2=(e^{H(x)}-\gamma)(e^{H(x)}-\delta),$$

$$\gamma\delta\neq 0, \quad \gamma\neq\delta, \quad H(0)=0.$$

Here U may be omitted. This is a characterization of R with $P(R)=4$. To construct a function f with $P(f)=4$ is easy now. In fact it is sufficient to consider a meromorphic function

$$\sqrt{\frac{e^H-\gamma}{e^H-\delta}},$$

which omits evidently four values $1, -1, \sqrt{\gamma/\delta}, -\sqrt{\gamma/\delta}$.

§ **4.** We shall here prove the existence of an ultrahyperelliptic surface R with $P(R)=3$. Let R be an ultrahyperelliptic surface defined by an equation

$$y^2=81e^{4x}-72e^{3x}-2e^{2x}-8e^x+1.$$

Let f be an entire algebroid function

$$\frac{1}{2}(1+4e^z-9e^{2z})+\frac{1}{2}\sqrt{81e^{4z}-72e^{3z}-2e^{2z}-8e^z+1}$$

of z, which is an entire function on R, then f does not take three values 0, 1 and ∞ on R. To this end we examine this by Rémoundos' reasoning. In fact we have that

$$F(z,f)\equiv f^2-(1+4e^z-9e^{2z})f+\frac{1}{4}(1+4e^z-9e^{2z})^2$$
$$-\frac{1}{4}(81e^{4z}-72e^{3z}-2e^{2z}-8e^z+1)$$

satisfies $F(z,0)=4e^z$ and $F(z,1)=9e^{2z}$. Thus $f\neq 0$, 1 and ∞ on R.

Since we have

$$g(z)=(e^z-1)(81e^{3z}+9e^{2z}+7e^z-1)$$
$$=81(e^z-1)(e^z-\varepsilon_1)(e^z-\varepsilon_2)(e^z-\varepsilon_3),$$
$$|\varepsilon_j|\neq 1,\ j=1,2,3;\ (\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)\neq 0,$$

$g(z)$ has no double zeros. Next we should prove that $g(z)$ does not satisfy the equation

$$h^2g=f^2\frac{(e^L-\gamma)(e^L-\delta)}{U^2},\ \gamma\neq\delta,\ \gamma\delta\neq 0,$$

where L, f and h are three entire functions of z satisfying $L(0)=0$ and $f\not\equiv 0$, $h\not\equiv 0$ and U is the entire function determined as in § 3. If it is not so, then the equation holds. Let both side terms be denoted by $X(z)$ and $Y(z)$ for simplicity's sake. If L is a transcendental entire function, then e^L has infinite order by Pólya's theorem [5]. Let $N_3(r;\ 0,\ Y)$ be the N-function with respect to zeros of odd multiplicity of the indicated function Y, which are all counted only once. Then $N_3(r;\ 0,\ Y) \geqq N_2(r;\ \gamma,\ e^L)+N_2(r;\ \delta,\ e^L)$ and hence it has infinite order by the Lemma. On the other hand $N_3(r;\ 0,\ X)=N_2(r;\ 0,\ g)$ has order one, which is absurd. If L is a polynomial of degree p, then $N_3(r;\ 0,\ Y)$ has order p and hence p must be equal to one. Therefore our equation reduces to the following form

$$X(z)\equiv h(z)^2g(z)=f(z)^2(e^{\beta z}-\gamma)(e^{\beta z}-\delta)\equiv Y(z),$$

since $e^{\beta z}-\gamma$ and $e^{\beta z}-\delta$ have only simple zeros and U is constructed from the multiple zeros of $e^L-\gamma$ and $e^L-\delta$. Since $g(z)$ has the form

$$81(e^z-1)(e^z-\varepsilon_1)(e^z-\varepsilon_2)(e^z-\varepsilon_3),$$
$$|\varepsilon_j|\neq 1,\ j=1,2,3;\ (\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)\neq 0,$$

$2n\pi i$ is a simple zero of g and hence

$$(u^n-\gamma)(u^n-\delta)=0,\qquad u=e^{2\beta\pi i}.$$

Then the modulus of u is equal to 1. If $u\neq\pm 1$, then $u^n\neq\gamma$ and $u^n\neq\delta$ for some integer n, which is absurd. If $u=1$, then β is a non-zero integer p and $\gamma=1$ or $\delta=1$. Therefore we have

$$X(z)=f(z)^2(e^{pz}-1)(e^{pz}-\zeta),\ \zeta\neq 0,\ \zeta\neq 1.$$

If $p\neq\pm 1$, then

$$\frac{h(z)^2 g(z)}{e^z-1}=f(z)^2\frac{e^{pz}-1}{e^z-1}(e^{pz}-\zeta)$$

has at least one zero with odd multiplicity which is due to the function $(e^{pz}-1)$ $\div(e^z-1)$. For this zero z_0 we may assume that $e^{z_0}=e^{2\pi i/p}$. However the left hand side term has it as a zero of even multiplicity. This is a contradiction. If $p=1$, then

$$X(z)=f(z)^2(e^z-1)(e^z-\zeta).$$

If $p=-1$, then we have

$$X(z)=f(z)^2(e^z-1)(e^z-1/\zeta)\zeta e^{-2z}.$$

Both cases are absurd by the form of $g(z)$. If $u=-1$, then β is a non-zero half integer q and $\gamma=1$ and $\delta=-1$. Therefore we have

$$X(z)=f(z)^2(e^{qz}-1)(e^{qz}+1)=f(z)^2(e^{2qz}-1).$$

If $q\neq\pm 1/2$, then

$$\frac{X(z)}{e^z-1}=f(z)^2\frac{e^{2qz}-1}{e^z-1}$$

has the zero z_0 satisfying $e^{z_0}=e^{2\pi i/2q}$, which is of odd multiplicity. However it has at most even multiplicity in the left hand side term, which is absurd. If $q=\pm 1/2$, then

$$X(z)=f(z)^2(e^{\pm z}-1),$$

and hence

$$\frac{X(z)}{e^z-1}=f(z)^2 \quad \text{or} \quad -f(z)^2e^{-z}.$$

These are also absurd. Therefore we have the desired fact.

This shows that the ultrahyperelliptic surface R defined by $y^2=g(x)$ satisfies $P(R)=3$. Thus the existence of the surface with $P(R)=3$ is established. Some characterizations of such surfaces would be possible, though it would be very troublesome to settle. This is an open problem.

References

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