CODING THEOREMS FOR THE COMPOUND SEMI-CONTINUOUS MEMORYLESS CHANNELS

BY KEN-ICHI YOSHIHARA

1. Summary.

The idea of "compound channels" was introduced by Blackwell, Breiman, and Thomasian [1] and Wolfowitz [4], independently, and coding theorems and their strong converses for the compound discrete memoryless channels were proved by Wolfowitz, mainly. In [3], Kesten considered coding theorems and their weak converses for the compound semi-continuous memoryless channels in the case where the output alphabet is the set of integers.

In this paper, under some assumptions, we shall consider coding theorems and their strong converses for the compound semi-continuous memoryless channels in the case where the output space is the real line. In Section 2, we shall make assumptions and in Sections 3 and 4, we shall prove some lemmas by which coding theorems will be proved in the following sections. The results in this paper contain, as a special case, Wolfowitz's results in [6].

2. Assumptions.

In this paper, we shall consider the semi-continuous compound channels under the following assumptions:

A 1: Let S be defined by

$$S = \mathop{\times}_{j=1}^{m} [\gamma_1^{(j)}, \gamma_2^{(j)}] \tag{1}$$

where $[\gamma_1^{(j)}, \gamma_2^{(j)}]$ denotes a (bounded) closed interval of the real line, and $\mathop{\times}\limits_{j=1}^{m}$ denotes the Cartesian products of these intervals.

A 2: (a) For each $s \in S$, the channel $(D'_n, D''_n, h(\cdot|\cdot|s))$ is the semi-continuous memoryless channel where

(i) $D'_n = \mathop{\times}\limits_{j=1}^{n} \{1, \cdots, a\}$,

(ii) $D''_n = \mathop{\times}\limits_{j=1}^{n} D''_1$, D''_1 being the real line,

Received September 4, 1964.

(iii) for $u=(x_1, \cdots, x_n)\in D'_n$ and $v=(y_1, \cdots, y_n)\in D''_n$,

$$h(v|u|s)=\prod_{j=1}^{n} w(y_j|x_j|s) \tag{2}$$

where $w(\cdot|x|s)$ denotes the probability density for each $x\in\{1, \cdots, a\}$ and for each $s\in S$.

(b) If s_0 and s_1 are in S and $s_1 \neq s_0$, then

$$\int_{\{y:\ w(y|x|s_1)\neq w(y|x|s_0)\}} w(y|x|s_0)dy>0 \qquad \text{for some } x\in\{1, \cdots, a\}. \tag{3}$$

A 3: (a) For all x and for almost all y, the partial derivatives

$$\frac{\partial \log_2 w(y|x|s)}{\partial s^{(i)}} \qquad (i=1, \cdots, m)$$

exist for every $s=(s^{(1)}, \cdots, s^{(m)})\in S$.

(b) Let s_0 be an arbitrary point in S. There exists a positive number ρ, independent of s_0, such that, for any $s\in\{s': |s'-s_0|<\rho\}\cap S$, the following two conditions are satisfied:

(i) There exists a bounded and integrable function $F(y)$ for which

$$\left|\frac{\partial w(y|x|s)}{\partial s^{(i)}}\right|\leqq F(y) \qquad \text{for almost all } y \tag{4}$$

holds for all i $(i=1, \cdots, m)$ and for all x $(x=1, \cdots, a)$.

(ii) There exists a measurable function $H(y)$ for which

$$\left|\frac{\partial \log_2 w(y|x|s)}{\partial s^{(i)}}\right|\leqq H(y) \qquad \text{for almost all } y \tag{5}$$

holds for all i $(i=1, \cdots, m)$ and for all x $(x=1, \cdots, a)$, and

$$\int_{-\infty}^{\infty} w(y|x|s)H(y)dy<\infty. \tag{6}$$

A 4: There exists a positive number β such that, for all t $(0<t\leqq\beta)$, for all $s\in S$ and for all probability a-vector π,

$$E^{(s)}\left[\exp_e t\log_2 \frac{w(Y_1|X_1|s)}{q^{(s)}(Y_1|\pi)}\middle|\pi\right]<\infty \tag{7}$$

where

$$q^{(s)}(y|\pi)=\sum_{x=1}^{a}\pi(x)w(y|x|s). \tag{8}$$

A 5: For any probability a-vector π,

$$E^{(s)}[\{\log_2 w(Y_1|x|s)\}^6|\pi]\leqq M_0<\infty \qquad \text{for all } x\in\{1, \cdots, a\} \tag{9}$$

3. Some Lemmas.

In this section, we shall prove some lemmas which are needed in later.

LEMMA 1. *For any* $s \in S$ *and* $x \in \{1, \cdots, a\}$, *let*

$$G_{s,x}(T)=\{y\colon |\log_2 w(y|x|s)| \geqq T\}. \tag{10}$$

Then

$$\int_{G_{s,x}(T)} w(y|x|s) \cdot |\log_2 w(y|x|s)| dy = O\left(\frac{1}{T^4}\right) \qquad \textit{for all } x. \tag{11}$$

Similarly, for any $s \in S$ *and* $x \in \{1, \cdots, a\}$, *let*

$$G_s'(T)=\{y\colon |\log_2 q^{(s)}(y|\pi)| \geqq T\}. \tag{12}$$

Then

$$\int_{G_s'(T)} q^{(s)}(y|\pi)|\log_2 q^{(s)}(y|\pi)| dy = O\left(\frac{1}{T^4}\right). \tag{13}$$

Proof. By A 5, for any probability a-vector π,

$$E^{(s)}[\{\log_2 w(Y_1|x|s)\}^6|\pi] \leqq M_0 \qquad \text{for all } x.$$

Thus, for all x

$$P^{(s)}\{G_{s,x}(k)|x\} \leqq \frac{M_0}{k^6}$$

and

$$\int_{G_{s,x}(T)} w(y|x|s) \cdot |\log_2 w(y|x|s)| dy$$

$$\leqq \sum_{k=[T]}^{\infty} \int_{G_{s,x}(k)-G_{s,x}(k+1)} w(y|x|s)|\log_2 w(y|x|s)| dy$$

$$\leqq \sum_{k=[T]}^{\infty} (k+1) P^{(s)}\{G_{s,x}(k)-G_{s,x}(k+1)|x\}$$

$$\leqq \sum_{k=[T]}^{\infty} (k+1) P^{(s)}\{G_{s,x}(k)|x\} \leqq \sum_{k=[T]}^{\infty} \frac{(k+1)\cdot M_0}{k^6} \leqq \frac{M_0'}{T^4}.$$

Since by A 5,

$$E^{(s)}[\{\log_2 q^{(s)}(Y_1|\pi)\}^6|\pi] \leqq M_1,$$

so, similarly, we have (13).

LEMMA 2. *Let $s_0 \in S$ be arbitrary. Let n be a large number such that $K2^{-\sqrt[3]{n}} < \rho$, where K is a constant. For any $s \in \{s' : |s'-s_0| < K2^{-\sqrt[3]{n}}\} \frown S$ and for any probability a-vector π*

$$|H(q^{(s)}|s) - H(q^{(s_0)}|s_0)| = O(n^{-4/3}) \tag{14}$$

and

$$\left|\sum_{x=1}^{a} \pi(x) H(w(\cdot|x|s|)) - \sum_{x=1}^{a} \pi(x) H(w(\cdot|x|s_0))\right| = O(n^{-4/3}) \tag{15}$$

under assumptions from A1 *to* A5.

Proof. We shall prove (15), at first.

$$\begin{aligned} \Delta_x &= |H(w(\cdot|x|s)) - H(w(\cdot|x|s_0))| \\ &= \left|\int_{-\infty}^{\infty} w(y|x|s) \log_2 w(y|x|s) dy - \int_{-\infty}^{\infty} w(y|x|s_0) \log_2 w(y|x|s_0) dy\right| \\ &\le \left|\int_{\overline{G_{s_0,x}(\sqrt[3]{n}/2)}} w(y|x|s) \log_2 w(y|x|s) dy - \int_{\overline{G_{s_0,x}(\sqrt[3]{n}/2)}} w(y|x|s_0) \log_2 w(y|x|s_0) dy\right| \\ &\quad + \int_{G_{s_0,x}(\sqrt[3]{n}/2)} w(y|x|s) |\log_2 w(y|x|s)| dy + \int_{G_{s_0,x}(\sqrt[3]{n}/2)} w(y|x|s_0) |\log_2 w(y|x|s_0)| dy \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{16}$$

By A3(b), we have, for n sufficiently large,

$$\begin{aligned} I_1 &\le \int_{\overline{G_{s_0,x}(\sqrt[3]{n}/2)}} w(y|x|s) |\log_2 w(y|x|s) - \log_2 w(y|x|s_0)| dy \\ &\quad + \int_{\overline{G_{s_0,x}(\sqrt[3]{n}/2)}} |w(y|x|s) - w(y|x|s_0)| \cdot |\log_2 w(y|x|s_0)| dy \\ &\le K2^{-\sqrt[3]{n}} \int_{\overline{G_{s_0,x}(\sqrt[3]{n}/2)}} w(y|x|s) \left\{\sum_{i=1}^{m} \left|\frac{\partial \log_2 w(y|x|s')}{\partial s^{(i)}}\right|\right\} dy \\ &\quad + \frac{\sqrt[3]{n}}{2} K2^{-\sqrt[3]{n}} \int_{\overline{G_{s_0,x}(\sqrt[3]{n}/2)}} \left\{\sum_{i=1}^{m} \left|\frac{\partial w(y|x|s'')}{\partial s^{(i)}}\right|\right\} dy \\ &\le K2^{-\sqrt[3]{n}} \int_{-\infty}^{\infty} m w(y|x|s) H(y) dy + \frac{\sqrt[3]{n}}{2} K2^{-\sqrt[3]{n}} \int_{-\infty}^{\infty} m F(y) dy \end{aligned} \tag{17}$$

$$=O(2^{-\sqrt[3]{n}}).$$

By Lemma 1,

$$I_3=O(n^{-4/3}). \tag{18}$$

On the other hand, by A 3(b), we can find a positive number K_1 such that $F(y)\leqq K_1$ for all y, and thus we have

$$w(y|x|s_0)-mK_1K2^{-\sqrt[3]{n}}$$
$$\leqq w(y|x|s)\leqq w(y|x|s_0)+mK_1K2^{-\sqrt[3]{n}}.$$

Accordingly, for $y\in G_{s_0,x}(\sqrt[3]{n}/2)$

$$|\log_2 w(y|x|s)|\geqq\frac{\sqrt[3]{n}}{3} \tag{19}$$

for sufficiently large n. By (19) and Lemma 1, we have

$$I_2=O(n^{-4/3}). \tag{20}$$

Combining (17), (18) and (20), we conclude that

$$\Delta_x=O(n^{-4/3}). \tag{21}$$

Using (21), we have

$$\left|\sum_{x=1}^{a}\pi(x)H(w(\cdot|x|s))-\sum_{x=1}^{a}\pi(x)H(w(\cdot|x|s_0))\right|$$

$$\leqq\sum_{x=1}^{a}\pi(x)\Delta_x=O(n^{-4/3}).$$

We can prove (14) by the same method as we proved (15), using Lemma 1 and the following two inequalities:

(i)
$$\left|\frac{\partial q^{(s)}(y)}{\partial s^{(i)}}\right|=\left|\frac{\partial}{\partial s^{(i)}}\left\{\sum_{x=1}^{a}\pi(x)w(y|x|s)\right\}\right|$$

$$\leqq\sum_{x=1}^{a}\pi(x)\left|\frac{\partial w(y|x|s)}{\partial s^{(i)}}\right|\leqq F(y)\qquad\text{for all } i\ (i=1,\cdots,m);$$

(ii) since, for any $p_k(\geqq 0)$ and $q_k(\geqq 0)$,

$$\frac{p_1+p_2+\cdots+p_n}{q_1+q_2+\cdots+q_n}\leqq\frac{p_1}{q_1}+\frac{p_2}{q_2}+\cdots+\frac{p_n}{q_n}$$

where $0/0=0$, so

$$\left|\frac{\partial \log_2 q^{(s)}(y)}{\partial s^{(i)}}\right|=\left|\frac{\partial}{\partial s^{(i)}}\log_2\left\{\sum_{x=1}^{a}\pi(x)w(y|x|s)\right\}\right|$$

$$\leqq \sum_{x=1}^{a}\pi(x)\left|\frac{\partial w(y|x|s)}{\partial s^{(i)}}\right|\Big/\sum_{x=1}^{a}\pi(x)w(y|x|s)\leqq \sum_{x=1}^{a}\left|\frac{\partial w(y|x|s)}{\partial s^{(i)}}\right|\Big/w(y|x|s)$$

$$=\sum_{x=1}^{a}\left|\frac{\partial \log_2 w(y|x|s)}{\partial s^{(i)}}\right|\leqq aH(y) \qquad \text{for all } i\ (i=1,\cdots,m).$$

LEMMA 3. *Let $s_0\in S$ be arbitrary. Let n be a large number such that $K2^{-\sqrt[3]{n}}<\rho$ where K is a constant. Furthermore, let u_0 be any n-sequence in D_n'. Then, for any $s\in\{s'\colon |s'-s_0|<K2^{-\sqrt[3]{n}}\}\frown S$ and for any $B\in\mathfrak{B}(D_n'')$*

$$|P^{(s)}\{B|u_0\}-P^{(s_0)}\{B|u_0\}|=O(n^{-1/3}) \tag{22}$$

under the assumptions from A 1 *to* A 5.

Proof. Let $u_0=(x_1,\cdots,x_n)$ be fixed. Let

$$F_k=\{y\colon w(y|x_k|s_0)\geqq 2^{-\sqrt[3]{n}/2}\}.$$

Define F as the n-dimensional Cartesian product of F_k $(k=1,\cdots,n)$, i.e., $F=F_1\times\cdots\times F_n$. Then

$$\begin{aligned}|P^{(s)}\{B|u_0\}-P^{(s_0)}\{B|u_0\}|&=\left|\int_B h(v|u_0|s)dv-\int_B h(v|u_0|s_0)dv\right|\\ &\leqq\int_{B\frown\bar F} h(v|u_0|s)dv+\int_{B\frown\bar F} h(v|u_0|s_0)dv+\int_{B\frown F}|h(v|u_0|s)-h(v|u_0|s_0)|dv.\end{aligned} \tag{23}$$

Now, if $v=(y_1,\cdots,y_n)$ is an n-sequence in F,

$$w(y_k|x_k|s_0)\geqq 2^{-\sqrt[3]{n}/2} \qquad \text{for all } k\ (k=1,\cdots,n) \tag{24}$$

and thus, using A 3(b),

$$\int_{B\frown F}|h(v|u_0|s)-h(v|u_0|s_0)|dv$$

$$=\int_{B\frown F}h(v|u_0|s_0)\left|\frac{h(v|u_0|s)}{h(v|u_0|s_0)}-1\right|dv$$

$$\leqq\int_{B\frown F}h(v|u_0|s_0)\left|\frac{\prod_{k=1}^{n}\{w(y_k|x_k|s_0)+mKK_1 2^{-\sqrt[3]{n}}\}}{\prod_{k=1}^{n}w(y_k|x_k|s_0)}-1\right|dv \tag{25}$$

$$\leqq \{(1+mKK_1 2^{-\sqrt[3]{n}/2})^n - 1\} \int_{B \frown F} h(v|u|s_0)dv$$

$$\leqq (1+mKK_1 2^{-\sqrt[3]{n}/2})^n - 1 = O(2^{-\sqrt[3]{n}})$$

for sufficiently large n, where K_1 is a constant such that $F(y) \leqq K_1$ for all y.

On the other hand, if $v \in \overline{F}$, then there is at least one k for which (24) does not hold. Thus, using Lemma 1,

$$\int_{B \frown \overline{F}} h(v|u_0|s_0)dv \leqq \int_{\overline{F}} h(v|u_0|s_0)dv$$

$$\leqq \sum_{k=1}^{n} \int_{\{y_k:\ w(y_k|x_k|s_0) < 2^{-\sqrt[3]{n}/2}\}} w(y_k|x_k|s_0)dy_k \tag{26}$$

$$\leqq \sum_{k=1}^{n} \int_{\{y_k:\ |\log_2 w(y_k|x_k|s_0)| > -\sqrt[3]{n}/2\}} w(y_k|x_k|s_0)dy_k = nO(n^{-4/3}) = O(n^{-1/3}).$$

Similarly, using A 3 and Lemma 1, we have

$$\int_{B \frown \overline{F}} h(v|u_0|s)dv = O(n^{-1/3}). \tag{27}$$

Combining (25), (26) and (27), we have (22). Thus the proof is completed.

4. Capacities and their approximations.

Let $S_n{}^*$ be the set of all points $s^* \in S$ whose j-th coordinate is of the form

$$\left(k_j + \frac{1}{2}\right)(\gamma_2{}^{(j)} - \gamma_1{}^{(j)})2^{-\sqrt[3]{n^2}} + \gamma_1{}^{(j)} \qquad (k_j = 0, \cdots, [2^{\sqrt[3]{n^2}}];\ j = 1, \cdots, m). \tag{28}$$

Define $\bar{C}$ and $\bar{C}_n{}^*$ as follows:

$$\bar{C} = \max_{\pi} \min_{s \in S} R(\pi|s) \tag{29}$$

and

$$\bar{C}_n{}^* = \max_{\pi} \min_{s^* \in S_n{}^*} R(\pi|s^*) \tag{30}$$

where

$$R(\pi|s) = E^{(s)}\left[\log_2 \frac{w(Y_1|X_1|s)}{q^{(s)}(Y_1|\pi)} \middle| \pi \right].$$

Then, by Lemma 2, we have

(31) $$\bar{C}_n{}^* - \frac{M_1}{\sqrt[3]{n}} \leqq \bar{C} \leqq \bar{C}_n{}^* + \frac{M_1}{\sqrt[3]{n}}$$

for sufficiently large n. In Section 5, we shall show that $\bar{C}$ is the capacity for the compound semi-continuous memoryless channel whose channel probability function (c.p.f.) is not known by the sender.

Similarly, define $\bar{\bar{C}}$ and $\bar{\bar{C}}_n{}^*$ as follows:

(32) $$\bar{\bar{C}} = \min_{s \in S} \max_{\pi} R(\pi | s)$$

and

(33) $$\bar{\bar{C}}_n{}^* = \min_{s^* \in S_n{}^*} \max_{\pi} R(\pi | s^*).$$

Then, also by Lemma 2, we have

(34) $$\bar{\bar{C}}_n{}^* - \frac{M_2}{\sqrt[3]{n}} \leqq \bar{\bar{C}} \leqq \bar{\bar{C}}_n{}^* + \frac{M_2}{\sqrt[3]{n}}$$

for sufficiently large n. In Section 7, we shall show that $\bar{\bar{C}}$ is the capacity for the compound semi-continuous memoryless channel whose c.p.f. is known by the sender.

5. A coding theorem and its strong converse for the compound semi-continuous memoryless channel with c.p.f. unknown to both sender aud receiver.

DEFINITION 1. A code (N, α) for the compound semi-continuous memoryless channel with c.p.f. unknown to both sender and receiver is a system

$$\{(u_1, A_1), \cdots, (u_N, A_N)\}$$

which satisfies the following conditions:

(i) the u_i are n-sequences and the A_i are disjoint sets in $\mathfrak{B}(D_n'')$, and

(ii) $\alpha_s(u_i) \leqq \alpha$, $i=1, \cdots, N$; $s \in S$,

where

$$P^{(s)}\{A_i | u_i\} = 1 - \alpha_s(u_i).$$

In this section we shall prove a coding theorem and its strong converse for the compound semi-continuous memoryless channel with c.p.f. unknown to both sender and receiver.

(a) A coding theorem.

To prove Theorem 1, we use the following two theorems.

THEOREM A. (Blackwell-Breiman-Thomasian) (cf. [1] and [7]) *Let θ, θ' and $\alpha<1$ be arbitrary positive numbers. For the compound channel with T c.p.f.'s whose c.p.f. is unknown to both sender and receiver, there exists an (N, α) code such that*

$$N>\frac{2^{\theta}}{T}\left[\alpha-T^{2}2^{-\theta'}-\sum_{s=1}^{T}P^{(s)}(A^{(s)}(\theta+\theta'))\right]$$

where $P^{(s)}$ is the probability distribution corresponding to $p(u)h(v|u|s)$ and

$$A^{(s)}(\theta+\theta')=\left\{(u, v)\colon \log_{2}\frac{h(v|u|s)}{q^{(s)}(v)}\leqq\theta+\theta'\right\}.$$

THEOREM B. (Cramér) (cf. [2]) *Let $Z_1, Z_2, \cdots$ be independent, identically distributed random variables with mean 0 and finite variance σ^2. Let $\nu(x)$ be the distribution function of Z_n's, and*

$$F_{n}(x)=P\left\{\frac{Z_{1}+\cdots+Z_{n}}{\sqrt{n}}\leqq x\right\}.$$

We assume that there exists a positive constant A such that

$$R=\int_{-\infty}^{\infty}e^{ty}\nu(dy)\qquad for\quad |t|<A$$

always exists. If $x>1$ and $x=O(n^{1/6})$, then

$$F_{n}(-x)=\Phi(-x)\left\{e^{-c_{0}x^{3}/\sqrt{n}}+O\left(\frac{x\log n}{\sqrt{n}}\right)\right\},\ c_{0}=\frac{\gamma_{3}}{6\sigma^{3}}$$

where

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-x^{2}/2}dx.$$

THEOREM 1. *Let α, $0<\alpha\leqq1$, be arbitrary. Under the assumptions from A1 to A5, there exists a positive number K_0 such that, for sufficiently large n, there exists a code (n, N, α) for S with c.p.f. unknown to both sender and receiver, with*

$$N>2^{n\bar{C}-K_{0}\sqrt[3]{n^{2}}}.$$

Proof. Let L_n be the number of elements of $S_n{}^*$. Then, by construction,

$$L_{n}\leqq([2^{\sqrt[3]{n^{2}}}]+1)^{m}\leqq2^{(m+1)\sqrt[3]{n^{2}}}.$$

Let $\bar{\pi}$ be a probability a-vector such that

$$\bar{C}_n^* = \max_{\pi} \min_{s \in S_n^*} R(\pi|s) = \min_{s \in S_n^*} R(\bar{\pi}|s).$$

Since, by A 4 and by the fact that, for any π, and for any t $(-1 \leqq t \leqq 0)$

$$E^{(s)}\left[\exp_e\left\{t \log_2 \frac{w(Y_1|X_1|s)}{q^{(s)}(Y_1|\pi)}\right\}\bigg|\pi\right] < \infty,$$

we can conclude that, the above relation also holds, for any t $(|t| < \min(1, \beta))$, and since

$$\bar{C}_n^* \leqq R(\bar{\pi}|s) = E^{(s)}\left[\log_2 \frac{w(Y_1|X_1|s)}{q^{(s)}(Y_1|\bar{\pi})}\bigg|\bar{\pi}\right]$$

for all $s \in S_n^*$, so, from Lemma 3 and Theorem B, we have that, for any $K>0$ and n sufficiently large

$$P^{(s)}\{A^{(s)}(n\bar{C}_n^* - K\sqrt[3]{n^2})\} \leqq P^{(s)}\{A^{(s)}(nR(\bar{\pi}|S) - K\sqrt[3]{n^2})\}$$

(35)

$$\leqq \frac{1}{\sqrt{2\pi}K\sqrt[6]{n}} \cdot e^{-K^2\sqrt[3]{n}/2}\left\{e^{-c \cdot K^3} + O\left(\frac{\log n}{\sqrt[3]{n}}\right)\right\} \leqq Me^{-K^2\sqrt[3]{n}/2}$$

for all $s \in S_n^*$, where $P^{(s)}(\cdot)$ denotes the probability distribution corresponding to $\Pi_{j=1}^n \bar{\pi}(x_j) w(y_j|x_j|s)$.

Let $K_1 > 4(m+1)$ and put $\theta = n\bar{C}_n^* - K_1\sqrt[3]{n^2}$ and $\theta' = K_1\sqrt[3]{n^2}/2$. Since, by (35)

$$\sum_{s \in S_n^*} P^{(s)}\{A^{(s)}(\theta+\theta')\} \leqq L_n M e^{-K_1^2\sqrt[3]{n^2}/2} \leqq Me^{-(K_1/2-(m+1))\sqrt[3]{n^2}}$$

and

$$L_n 2^{-\theta'} \leqq 2^{-(K_1/2-2(m+1))\sqrt[3]{n^2}}$$

for sufficiently large n, so

$$L_n^2 2^{-\theta'} + \sum_{s \in S_n^*} P^{(s)}\{A^{(s)}(\theta+\theta')\} \leqq \frac{\alpha}{2}$$

for sufficiently large n. Applying Theorem A to this case, we conclude that, for n sufficiently large, there exists a code $(n, N, 3\alpha/4)$ for the compound channel S_n^* such that

$$N > \frac{\alpha}{4} 2^{n\bar{C}_n^* - K_1\sqrt[3]{n^2}}.$$

Now, we choose $K_0 > K_1 + M_1$. Then, combining the above result and Lemma 3, and using (31), we have the desired result.

(b) Strong converse of the coding theorem.

Corresponding to Theorem 1, we shall show the strong converse. In order to do this, we use the following theorem proved by Kemperman (cf. [7]):

THEOREM C (Kemperman) *Let* α, $0\leqq\alpha<1$, *be arbitrary. For any semi-continuous memoryless channel, there exists a constant* $K_0>0$ *such that, for every* n, *a code* (n, N, α) *satisfies*

$$N<2^{nC+K_0\sqrt{n}}.$$

Now, we state and prove the strong converse.

THEOREM 2. *Let* $\varepsilon>0$ *and* α, $0\leqq\alpha<1$, *be arbitrary. Under the assumptions from* A1 *to* A5, *there exists a positive constant* K_0 *such that, for every* n, *any* (n, N, α) *code for the compound semi-continuous memoryless channel* S *with c.p.f. unknown to both sender and receiver satisfies*

$$N<2^{n\bar{C}+K_0\sqrt{n}}.$$

Proof. It is sufficient to prove the theorem for $\alpha>0$. Let π be any probability a-vector and $\bar{s}\in S$ such that

$$R(\pi|\bar{s})<\bar{C}+\frac{1}{\sqrt{n}}. \tag{36}$$

Since an (n, N, α) code for S is surely an (n, N, α) code for the semi-continuous memoryless channel with (single) c.p.f. $w(\cdot|\cdot|\bar{s})$, so, by Theorem C and (36), we have the desired result.

6. A coding theorem and its strong converse for the compound semi-continuous memoryless channel with c.p.f. known to the receiver but not to the sender

DEFINITION 2. For each s in S, let T_s be a code (n, N, α) for the semi-continuous memoryless channel with (single) c.p.f. $w(\cdot|\cdot|s)$, thus,

$$T_s=\{(u_1, A_1(s)), \cdots, (u_N, A_N(s))\}$$

where u_i, $i=1, \cdots, N$, is the same for all s, but $A_i(s)$ is a function of s and $A_i(s)\frown A_j(s)=\phi$ for each s in S. A set of codes T_s, $s\in S$, is called a code (n, N, α) for the compound semi-continuous memoryless channel with c.p.f. known to the receiver but not to the sender.

Now, we shall consider a coding theorem and its strong converse for the compound semi-continuous memoryless channel with c.p.f. known to the receiver but not to the sender. Following the same method in [7], we have

THEOREM 3. *Let $\varepsilon>0$ and α, $0<\alpha\leqq 1$, be arbitrary. Under the assumptions from* A1 *to* A5, *there exists a positive constant K_0 such that, for sufficiently large n, there exists a code (n, N, α) for the compound semi-continuous memoryless channel with c.p.f. unknown to the receiver but not to the sender with*

$$N>2^{n\bar{C}-K_0\sqrt[3]{n^2}}$$

THEOREM 4. *Let α, $0\leqq\alpha<1$, be arbitrary. There exists a positive constant K_0' such that, for every n, any code (n, N, α) for the compound semi-continuous memoryless channel with c.p.f. known to the receiver but not to the sender satisfies*

$$N<2^{n\bar{C}+K_0'\sqrt{n}}.$$

7. A coding theorem and its strong converse for the compound semi-continuous memoryless channel with c.p.f. known to the sender but not to the receiver.

DEFINITION 3. For each s in S, let T_s' be a code (n, N, α) for the semi-continuous channel with (single) c.p.f. $w(\cdot|\cdot|s)$, thus:

$$T_s'=\{(u_1(s), A_1), \cdots, (u_N(s), A_N)\},$$

here $u_i(s)$ depends on s but A_i is the same for all s. A set of codes T_s', $s\in S$, is called a code (n, N, α) for the compound semi-continuous memoryless channel with c.p.f. known to the sender but not to the receiver.

In this case, to prove that $\bar{\bar{C}}$ is the capacity, we need the following theorem proved by Wolfowitz.

THEOREM D (Wolfowitz) (cf. [5] and [7]) *Let θ, θ', and $\alpha<1$ be arbitrary positive number. For the compound channel with T c.p.f.'s whose c.p.f. is known only to the sender, there exists an (N, α) code such that*

$$N>\frac{2^\theta}{T}\left[\alpha-T^2\cdot 2^{-\theta'}-\sum_{s=1}^{T}P^{(s)}\{B^{(s)}(\theta+\theta')\}\right]$$

where $P^{(s)}(\cdot)$ is the probability distribution corresponding to $p(u|s)h(v|u|s)$, and

$$B^{(s)}(\theta+\theta')=\left\{(u, v)\colon \log_2\frac{h(v|u|s)}{\int p(u|s)h(v|u|s)\lambda(du)}<\theta+\theta'\right\}.$$

THEOREM 5. *Let $\varepsilon>0$ and α, $0<\alpha\leqq 1$, be arbitrary. Under the assumptions from* A1 *to* A5, *there exists a constant K_0 such that for sufficiently large n there exists an $(n, 2^{n-K\bar{\bar{C}}_0\sqrt[3]{n^2}}, \alpha)$ for the compound semi-continuous memoryless channel with c.p.f. known to the sender but not to the receiver.*

Proof. We shall lean heavily on the proof of Theorem 1. For each $s\in S$, define $\pi(s)$ to be a probability a-vector for which

$$\max_{\pi} R(\pi|s)=R(\pi(s)|s)=C(s).$$

Since

$$\bar{\bar{C}}_n{}^*\leqq C(s)=E^{(s)}\left[\log_2\frac{w(Y_1|X_1|s)}{q^{(s)}(Y_1|\pi(s)}\,\middle|\,\pi(s)\right]\qquad\text{for all}\quad s\in S_n{}^*,$$

so, from Lemma 3 and Theorem B, we have that, for any $K>0$ and n sufficiently large,

$$P^{(s)}\{A^{(s)}(n\bar{\bar{C}}_n-K\sqrt[3]{n^2})\}\leqq P^{(s)}\{A^{(s)}(nC(s)-K\sqrt[3]{n^2})\}$$

$$\leqq\frac{1}{\sqrt{2\pi}\cdot K\cdot\sqrt[6]{n}}\cdot e^{-K^2\sqrt[3]{n}/2}\left\{e^{-c_0K^3}+O\left(\frac{\log n}{\sqrt[3]{n}}\right)\right\}\leqq Me^{-K^2\sqrt[3]{n}/2}$$

for all $s\in S_n{}^*$, where $P^{(s)}(\cdot)$ denotes the probability distribution corresponding $\Pi_{j=1}^n\pi(x_j|s)w(y_j|x_j|s)$.

Thus, putting $K_0>4(m+1)$, $\theta=n\bar{\bar{C}}_n{}^*-K_0\sqrt[3]{n^2}$ and $\theta'=K_0\sqrt[3]{n^2}/2$, and using Theorem D, we have the theorem.

THEOREM 6. *Let $\varepsilon>0$ and α, $0\leqq\alpha<1$, be arbitrary. Under the assumptions from* A 1 *to* A 5, *there exists a positive constant K_0' such that, for every n, any (n, N, α) code for the compound semi-continuous memoryless channel with c.p.f. known to the sender but not to the receiver satisfies*

$$N<2^{n\bar{\bar{C}}+K_0'\sqrt{n}}.$$

Proof. Apply Theorem C to the individual code T_s', each of which has the capacity $C(s)$ and we have the theorem using the fact that $\bar{\bar{C}}$ is the infinimum of the capacities $C(s)$, $s\in S$.

8. Applications.

As a special case, we shall consider the next compound semi-continuous memoryless channel which was considered by Wolfowitz in [6].

Let the input alphabet consist of the k (real) numbers $a_1, \cdots, a_k$. Let $u_0=(x_1, \cdots, x_n)$ be any n-sequence sent, each x_j being one of $a_1, \cdots, a_k$, and let

$$v(u_0)=(Y_1(u_0), \cdots, Y_n(u_0))$$

be a sequence of independent random variables received, where $Y_j(u_0)-x_j$ has a Gaussian distribution with mean μ and variance σ^2. Let J_1 and J_2 be bounded closed intervals of the real line; and J_2 is to contain positive numbers only. The parameters μ and σ^2 lie, respectively, in J_1 and J_2.

In this case all the assumptions from A 1 to A 5 are satisfied. Thus, if we put

$$\bar{C}_1 = \max_{\pi} \min_{(\mu,\sigma^2)\in J_1\times J_2} R(\pi(\mu, \sigma^2)) = \max_{\pi} \min_{\sigma^2\in J_2} R(\pi|(0, \sigma^2)), \tag{37}$$

and

$$\bar{\bar{C}}_1 = \min_{(\mu,\sigma^2)\in J_1\times J_2} \max_{\pi} R(\pi|(\mu, \sigma^2)) = \min_{\sigma^2\in J_2} \max_{\pi} R(\pi|(0, \sigma^2)), \tag{38}$$

the same results of Theorems 1 to 6 are true, and they are somewhat stronger than Wolfowitz's ones.

ACKNOWLEDGEMENT. The author is indebted to Prof. M. Udagawa, Tokyo University of Education, for suggesting the problem and for his encouragement and guidance.

REFERENCES

[1] BLACKWELL, D., L. BREIMAN, AND A. J. THOMASIAN, The capacity of a class of channels. Ann. Math. Stat. **30** (1959), 1229–1241.

[2] CRAMÉR, H., Sur un nouveau théorème-limite de la théorie des probabilités, Actualités Sci., Hermann, Paris (1938).

[3] KESTEN, H., Some remarks on the capacity of compound channels, in the semi-continuous case. Information and Control **4** (1961), 69–184.

[4] WOLFOWITZ, J., Simultaneous channels. Arch, Rational Mech. Anal. **4** (1960), 371–386.

[5] WOLFOWITZ, J., On coding theorems for simultaneous channels. Trans. IRE. Prof. Group on circuit theory, CT–**7** (1960), 513–516.

[6] WOLFOWITZ, J., On channels in which the distributions of error is known only to the receivcr or only to the sender. Information and decision processes 178–182 McGraw-hill Book Co., Inc., New York (1960).

[7] WOLFOWITZ, J., Coding theorems of information theory, Springer, Berlin (1961).

FACULTY OF ART EDUCATION, YOKOHAMA NATIONAL UNIVERSITY.