A FUNCTIONAL METHOD ON AMOUNT OF ENTROPY

BY HISAHARU UMEGAKI

1. Introduction.

The theory of information, originated by Shannon, was applied in the new subject to investigate the theory of transformation with invariant measure by Kolmogorov and his school, cf. Rokhlin [12]. Recently, Halmos [7] gave a very clarified note relative to their investigations. While, in order to achieving the channel capacity in stationary finite memory channels, cf. Feinstein [6], some important properties of the entropy (the average amount of information) of information sources in these channels were studied by Khinchin [8], Takano [13], Traregradsky [14], Breiman [2], Parthasarathy [11] and others. The basic space of information sources of the channels is the doubly infinite product set A^I (the messages space) of the alphabet A, which becomes a compact metric space relative to the weak product topology and in which the shift transformation is a homeomorphism on A^I (so-called the Bernoulli automorphism).

The purpose of this paper is to study a functional property of the entropy of information sources of general type in the abstract analysistic formula. From a general point of view, throuthout this paper, the basic space of the information sources will be taken as a measurable space with an invertible measurable transformation (automorphism, say). The description for such general space is not only used to discuss the functional properties of the amounts of entropies of various information sources, but also will be given a preliminary method of the concept of information for von Neumann algebras, some of which were partly discussed in the papers, cf. Nakamura-Umegaki [9], [10], Daivs [3], Echigo-Nakamura [5] and Umegaki [15].

In § 2, a functional form of the amount of entropies for finite partition and automorphism will be defined as functional of invariant measures which will be called the entropy functional, and, in § 3, it is represented as non-negative linear functional over the Banach space of the invariant bounded signed measures. The method will be done by using a technique of Breiman [2]. In § § 4 and 5, as the basic space take a measurable space with denombrably generated Borel field and apply to it the result in the previous section. This space contains various basic spaces of information sources, e.g. the message space A^I, the phase spaces in dynamical systems, the compact metric space or the spectrum of commutative von Neumann

Received April 4, 1963.

algebra on separable Hilbert space. In § 4, the extension of the entropy functional to stationary functional over Banach space of signed measures is proved and this fact is applied to prove the integral representations of this functionals for finite partitions and automorphism. This is a generalization of Parthasarathy's theorem [11]. In § 5, the entropy of automorphism is defined, and Rokhlin approximation theorem will be proved and it is applied, with the theorem given in § 4, to the integral representations of the entropy of automorphism. In appendix, without assuming the denombrability of the basic space, the entropy functional for finite partition $\mathfrak{A}$ and automorphism S is defined over the space of bounded signed measures which are bounded measures defined on the subfield generated by $\mathfrak{A}$ and S.

The concept of the entropy functional may be used for a generalization of theorems in finite memory channels. This will be discussed in the forthcoming paper. The abstract of this paper was published in [16].

2. Definition of entropy functional.

Let $(X, \mathfrak{X})$ be a measurable space with an automorphism S(=invertible measurable transformation from X onto X). A subfamily $\mathfrak{A}$ of $\mathfrak{X}$ is called a *partition*, if it covers X and any pair of different sets $U, V \in \mathfrak{A}$ is disjoint. Denote $\boldsymbol{F}(X)$ the collection of all finite partitions $\{\mathfrak{A}, \mathfrak{B}, \cdots\}$. For $\mathfrak{A}, \mathfrak{A}^{(1)}, \mathfrak{A}^{(2)}, \cdots, \mathfrak{A}^{(n)}$ and $\mathfrak{B} \in \boldsymbol{F}(X)$, denote

$$\mathfrak{A} \vee \mathfrak{B} \qquad \text{or} \qquad \bigvee_{k=1}^{n} \mathfrak{A}^{(k)}$$

the partitions generated by $\mathfrak{A}$ and $\mathfrak{B}$ or, by $\{\mathfrak{A}^{(k)}\}_{k=1}^{n}$, respectively, that is, e.g., $\mathfrak{A} \vee \mathfrak{B}$ consists of $U \cap V$ $(U \in \mathfrak{A},\ V \in \mathfrak{B})$, and also denote

$$\text{(1)} \qquad \mathfrak{A}_n = \bigvee_{k=1}^{n} S^{-k}\mathfrak{A}, \qquad S^{-k}\mathfrak{A} = \{S^{-k}U;\ U \in \mathfrak{A}\}.$$

Then $\mathfrak{A} \vee \mathfrak{B}$, $\vee_{k=1}^{n} \mathfrak{A}^{(k)}$, $S^{-k}\mathfrak{A}$ and $\mathfrak{A}_n$ are finite partitions. Furthermore, for $\mathfrak{A} \in \boldsymbol{F}(X)$, denote

$$\text{(2)} \qquad \bar{\mathfrak{A}}_n = \bigcup_{k=1}^{n} S^{-k}\mathfrak{A} \qquad \text{or} \qquad \bar{\mathfrak{A}}_\infty = \bigcup_{n=1}^{\infty} \mathfrak{A}_n$$

the σ-subfields of $\mathfrak{X}$ generated by $\{S^{-k}\mathfrak{A};\ k=1, 2, \cdots, n\}$ or $\{\mathfrak{A}_n;\ n=1, 2, \cdots\}$ respectively. For $\mathfrak{A}, \mathfrak{B} \in \boldsymbol{F}(X)$, if any $U \in \mathfrak{A}$ contains some $V \in \mathfrak{B}$ then $\mathfrak{B}$ is finer than $\mathfrak{A}$ and this relation is denoted by

$$\mathfrak{A} \leqslant \mathfrak{B}.$$

Let $P(X, S)$ be the set of all S-invariant probability measures over the measurable space $(X, \mathfrak{X})$. *Assume that* $P(X, S)$ *is non-empty.* Let $L(X, S)$ be the linear space of all S-invariant bounded (complex) signed measures $\xi, \eta, \cdots$. Then the space $L(X, S)$ is a complex Banach space with the norm:

$$\text{(3)} \qquad \|\xi\|_1 = \text{total-variation of } \xi.$$

Denote $L^+(X, S)$ the non-negative elements in $L(X, S)$.

For any probability measure p and for any fixed σ-subfield $\mathfrak{B}$ of $\mathfrak{X}$, denote $P_p(U|\mathfrak{B})$ the conditional probability, in the probability space $(X, \mathfrak{X}, p)$, of a set $U\in\mathfrak{X}$ conditioned by $\mathfrak{B}$. For any non-trivial $\xi\in L^+(X, S)$, $\xi_1=\xi/||\xi||_1$ belongs to $P(X, S)$, and put

$$P_\xi(U|\mathfrak{B}) = P_{\xi_1}(U|\mathfrak{B}).$$

For a fixed $\mathfrak{F}\in\boldsymbol{F}(X)$, define a functional over $L^+(X, S)$

$$H(\xi, \mathfrak{F}, S)=-\sum_{U\in\mathfrak{F}}\int_X P_\xi(U\mid\bar{\mathfrak{F}}_\infty)\log P_\xi(U\mid\bar{\mathfrak{F}}_\infty)\,d\xi(x) \tag{4}$$

for every $\xi\in L^+(X, S)$, where for $\xi=0$ put $H(\xi, \mathfrak{F}, S)=0$.[1] Then this satisfies the following equalities:

$$H(\xi, \mathfrak{F}, S)=-\sum_{U\in\mathfrak{F}}\lim_{n\to\infty}\int_X P_\xi(U\mid\bar{\mathfrak{F}}_n)\log P_\xi(U\mid\bar{\mathfrak{F}}_n)\,d\xi(x) \tag{5}$$

$$=-\lim\frac{1}{n}\sum_{U\in\mathfrak{F}\vee\mathfrak{F}_{n-1}}\xi(U)\log\xi(U)\leqq-\sum_{U\in\mathfrak{F}}\xi(U)\log\xi(U)+\xi(X)\log\xi(X).\text{[2]} \tag{6}$$

Since $\{P_\xi(U\mid\bar{\mathfrak{F}}_n)\log P_\xi(U\mid\bar{\mathfrak{F}}_n);\ n=1, 2, \cdots\}$ is semi-martingale (in the sense of Doob [4]) in the probability space $(X, \mathfrak{X}, \xi_1)$, the equality (5) is proved by semi-martingale's convergence theorem and the equality (6) is proved by McMillan's convergence theorem (e.g. cf. Halmos [7], p. 34).

Beasides, any measure $\xi\in L(X, S)$ is uniquely expressed such as

$$\xi=\xi^{(1)}-\xi^{(2)}+i\xi^{(3)}-i\xi^{(4)}\qquad(\imath=\sqrt{-1})$$

for $\xi^{(k)}\in L^+(X, S)$ $(k=1, 2, 3, 4)$, and the domain of the functional $H(\cdot,\mathfrak{F}, S)$ is extended over the space $L(X, S)$ by

$$H(\xi, \mathfrak{F}, S)=H(\xi^{(1)}, \mathfrak{F}, S)-H(\xi^{(2)}, \mathfrak{F}, S)+iH(\xi^{(3)}, \mathfrak{F}, S)-iH(\xi^{(4)}, \mathfrak{F}, S).$$

The functional $H(\cdot, \mathfrak{F}, S)$ will be called *entropy functional for the partition* $\mathfrak{F}$ *and for the automorphism* S over the measurable space $(X, \mathfrak{X})$. The following properties are equivalent each other: (1°) $H(p, \mathfrak{F}, S)=0$ for a fixed $p\in P(X, S)$, (2°) $\mathfrak{F}\subset\bar{\mathfrak{F}}_\infty$ (mod p) and (3°) $\bar{\mathfrak{F}}_\infty=\bigcup_{-\infty}^{\infty}S^{-k}\mathfrak{F}$ (mod p), where S^0 is the identity transformation. The well-definedness of the functional $H(\cdot, \mathfrak{F}, S)$ will be proved in the next section.

3. Linearity of entropy functional.

The entropy functionals $H(\xi, \mathfrak{F}, S)$, which were defined in the preceeding section, are three variable functions of measures ξ, partition $\mathfrak{F}$ and automorphisms S. If

1) The function $\lambda\log\lambda$ $(\lambda\geqq 0)$ is defined such as $\lambda\cdot\log\lambda$ $(\lambda>0)$, and 0 $(\lambda=0)$. The logarithm has the base 2.

2) For the partition $\mathfrak{F}$ not necessarily finite, the entropy functional $H(\xi, \mathfrak{F}, s)$ is defined, and when $\{\sum\xi(U)\log\xi(U);\ U\in\mathfrak{F}\}$ is finite, it is always finite and (5), (6) are satisfied for such $\xi\in L^+(X, s)$.

the measure ξ is stationary probability measure, then the amount $H(\xi, \mathfrak{F}, S)$ conicides with the amount of the entropy of the automorphisim S, relative to the partition $\mathfrak{F}$, in the usual sense, cf. e.g. Halmos [7], Rokhlin [12]. The entropy functionals have the property as linear functional and this property will be essentially used for the channel's theory.

THEOREM 1. *For any fixed finite partition $\mathfrak{F}$, the entropy functional $H(\cdot, \mathfrak{F}, S)$ is a non-negative bounded linear functional over the Banach space $L(X, S)$.*

To show the theorem, we shall prove five lemmas.[3)]

LEMMA 1.1. *The entropy functional $H(\cdot, \mathfrak{F}, S)$ is linear over $L^+(X, S)$ with respect to non-negative cofficients.*

Proof. The proof will be done by applying the Breiman's technique [2]. For any pair $\xi, \eta \in L^+(X, S)$ and any pair of positive numbers $\alpha, \beta > 0$,

$$\begin{aligned}
&-\frac{1}{n}\sum(\alpha\xi+\beta\eta)(U)\log(\alpha\xi+\beta\eta)(U)\\
&=-\frac{1}{n}\sum(\alpha\xi(U)+\beta\eta(U))\log(\alpha\xi(U)+\beta\eta(U))\\
&=-\frac{1}{n}\sum(\alpha\xi(U)\log\xi(U)-\frac{1}{n}\sum\beta\eta(U)\log\eta(U)\\
&\quad-\frac{1}{n}\sum\alpha\xi(U)\log\left(\alpha+\beta\frac{\eta(U)}{\xi(U)}\right)\\
&\quad-\frac{1}{n}\sum\beta\eta(U)\log\left(\beta+\alpha\frac{\xi(U)}{\eta(U)}\right)
\end{aligned}\tag{7}$$

where $\sum$ is taken as the summation for U running over the sets in $\mathfrak{F}\vee\mathfrak{F}_n$, and where in the first and third terms of the last side of these equalities in (7) if $\xi(U_0)=0$ for some U_0 in $\mathfrak{F}\vee\mathfrak{F}_n$ then let the summands containing $\xi(U_o)$ be zero and same for the other terms if $\eta(U_o)=0$. Since for $\xi(U)\neq 0$,

$$\log\alpha\leqq\log\left(\alpha+\beta\frac{\eta(U)}{\xi(U)}\right)\leqq\log\alpha+\left(\frac{\beta}{\alpha}\right)\frac{\eta(U)}{\xi(U)},$$

it holds

$$\begin{aligned}
\frac{1}{n}\xi(X)\,\alpha\log\alpha&\leqq\frac{1}{n}\sum\alpha\xi(U)\log\left(\alpha+\beta\frac{\eta(U)}{\xi(U)}\right)\\
&\leqq\frac{1}{n}\xi(X)\,\alpha\log\alpha+\frac{1}{n}\beta\eta(X),
\end{aligned}$$

where $\sum$ is taken in the same meaning as in (7). Therefore as $n\to\infty$ the third term in the last side of (7) tends to zero and similarily the final term of its side tends to zero. Consequently

3) The Lemmas 1.1~1.3 are satisfied for denombrable partition $\mathfrak{F}$ within the measure $\xi\in L^+(X, s)$ such as $\{\sum|\xi(U)\log\xi(U)|;\ U\in\mathfrak{F}\}<\infty$.

$$-\lim_{n\to\infty}\frac{1}{n}\sum(\alpha\xi+\beta\eta)(U)\log(\alpha\xi+\beta\eta)(U)$$

$$=-\alpha\lim_{n\to\infty}\frac{1}{n}\sum\xi(U)\log\xi(U)-\beta\lim_{n\to\infty}\frac{1}{n}\sum\eta(U)\log\eta(U)$$

and hence

$$H(\alpha\xi+\beta\eta,\mathfrak{F},S)=\alpha H(\xi,\mathfrak{F},S)+\beta H(\eta,\mathfrak{F},S). \tag{8}$$

When $\alpha\cdot\beta=0$, (8) is obvious from the definition of $H(\cdot,\mathfrak{F},S)$ and $H(0,\mathfrak{F},S)=0$.

LEMMA 1.2. *For any two pairs $\xi,\eta\in L^+(X,S)$ and $\xi',\eta'\in L^+(X,S)$ with $\xi-\eta=\xi'-\eta'$, the following equality holds*:

$$H(\xi,\mathfrak{F},S)-H(\eta,\mathfrak{F},S)=H(\xi',\mathfrak{F},S)-H(\eta'\ \mathfrak{F},S). \tag{9}$$

Proof. Since $\xi-\eta=\xi'-\eta'$ implies $\xi+\eta'=\xi'+\eta$, by Lemma 1.1

$$\begin{aligned}H(\xi,\mathfrak{F},S)+H(\eta',\mathfrak{F},S)&=H(\xi+\eta',\mathfrak{F},S)\\&=H(\xi'+\eta,\mathfrak{F},S)\\&=H(\xi'\ \mathfrak{F},S)+H(\eta,\mathfrak{F},S)\end{aligned}$$

and (9) holds.

LEMMA 1.3. *$H(\cdot,\mathfrak{F},S)$ is well-defined and real linear over the real part $L^{(r)}(X,S)$ of $L(X,S)$.*

Proof. By Lemmas 1.1, 1.2, and by the definition of $H(\cdot,\mathfrak{F},S)$, the fact that

$$H(\alpha\xi,\mathfrak{F},S)=\alpha H(\xi,\mathfrak{F},S) \qquad \text{for all real } \alpha \text{ and all } \xi\in L^+(X,S)$$

and the well-definedness are obvious. Taking $\xi=\sum_{k=1}^n\xi_k$, $\xi_k=\xi'_k-\xi''_k$ with $\xi'_k,\xi''_k\in L^+(X,S)$, since $\xi=\sum\xi'_k-\sum\xi''_k$ and $\sum\xi'_k,\sum\xi''_k\in L^+(X,S)$.

$$H(\xi,\mathfrak{F},S)=H(\sum_{k=1}^n\xi'_k,\mathfrak{F},S)-H(\sum_{k=1}^n\xi''_k,\mathfrak{F},S)$$

by Lemma 1.2,

$$=\sum_{k=1}^n\{H(\xi'_k,\mathfrak{F},S)-H(\xi''_k,\mathfrak{F},S)\}$$

by Lemma 1.1,

$$=\sum_{k=1}^n H(\xi'_k-\xi''_k,\mathfrak{F},S)$$

$$=\sum_{k=1}^n H(\xi_k,\mathfrak{F},S).$$

Also for $\xi=\xi'-\xi''$ with $\xi',\xi''\in L^+(X,S)$

$$\begin{aligned}H(-\xi,\mathfrak{F},S)&=H(\xi''-\xi',\mathfrak{F},S)\\&=H(\xi'',\mathfrak{F},S)-H(\xi'\ \mathfrak{F},S)\\&=-H(\xi,\mathfrak{F},S),\end{aligned}$$

hence by Lemma 1.1, for any real α

$$H(\alpha\xi, \mathfrak{F}, S)=\alpha H(\xi', \mathfrak{F}, S)-\alpha H(\xi'', \mathfrak{F}, S)$$
$$=\alpha H(\xi'-\xi'', \mathfrak{F}, S)=\alpha H(\xi, \mathfrak{F}, S).$$

LEMMA 1.4. *$H(\cdot, \mathfrak{F}, S)$ is strongly continous over $L^{(r)}(X, S)$, more precisely*

$$|H(\xi, \mathfrak{F}, S)| \leqq |\mathfrak{F}|\cdot||\xi||_1/2 \tag{10}$$

for every $\xi \in L^{(r)}(X, S)$, where $|\mathfrak{F}|$ is the number of the disjoint sets in $\mathfrak{F}$.

Proof. For $\xi \in L^{+}(X, S)$

$$0 \leqq -\int P_\xi(U|\bar{\mathfrak{F}}_\infty) \log P_\xi(U|\bar{\mathfrak{F}}_\infty)\, d\xi(x)$$
$$\leqq \xi\text{-ess.}\sup_{x\in X} |P_\xi(U|\bar{\mathfrak{F}}_\infty)(x) \log P_\xi(U|\bar{\mathfrak{F}}_\infty)(x)|\cdot\xi(X)$$
$$\leqq ||\xi||_1/2.$$

Hence

$$H(\xi, \mathfrak{F}, S) \leqq |\mathfrak{F}|\cdot||\xi||_1/2.$$

For general $\xi \in L^{(r)}(X, S)$, putting $|\xi|=\xi^{+}+\xi^{-}$ where ξ^{+} and ξ^{-} are the positive and the negative parts of ξ which belong to $L^{+}(X, S)$, respectively, then

$$-|\xi| \leqq \xi \leqq |\xi| \quad \text{and} \quad ||\,|\xi|\,||_1=||\xi||_1.$$

Therefore by the linearity and the non-negative definiteness of $H(\cdot, \mathfrak{F}, S)$

$$|H(\xi, \mathfrak{F}, S)| \leqq H(|\xi|, \mathfrak{F}, S) \leqq |\mathfrak{F}|\cdot||\xi||_1/2$$

and (10) is obtained.

LEMMA 1.5. *$H(\cdot, \mathfrak{F}, S)$ is well-defined linear and bounded over the space $L(X, S)$:*

$$|H(\xi, \mathfrak{F}, S)| \leqq |\mathfrak{F}|\cdot||\xi||_1 \quad \textit{for every } \xi \in L(X, S). \tag{11}$$

Proof. By the definition of $H(\zeta, \mathfrak{F}, S)$ for the complex measure $\zeta \in L(X, S)$, $H(i\xi, \mathfrak{F}, S)=iH(\xi, \mathfrak{F}, S)$ for $\xi \in L^{(r)}(X, S)$ holds, and hence the lemma is obvious from Lemmas 1.1~1.4 excepting the boundedness. Since for every $\xi, \eta \in L^{(r)}(X, S)$

$$\max(||\xi||_1, ||\eta||_1) \leqq ||\xi+i\eta||_1 \leqq \sqrt{||\xi||_1^2+||\eta||_1^2} \leqq 2\cdot||\xi+i\eta||_1,$$

it hlods

$$|H(\xi+i\eta, \mathfrak{F}, S)|=\sqrt{|H(\xi, \mathfrak{F}, S)|^2+|H(\eta, \mathfrak{F}, S)|^2}$$
$$=|\mathfrak{F}|\cdot\sqrt{||\xi||_1^2+||\eta||_1^2}/2$$
$$\leqq |\mathfrak{F}|\cdot||\xi+i\eta||_1,$$

that is, (11) is obtained.

By Lemmas 1.1~1.5, the proof of Theorem 1 is complete.

4. Extension of entropy functional.

Assume the measurable space $(X, \mathfrak{X})$ has denombrable generator, that is, there exists a denombrable subfamily of $\mathfrak{X}$ which generates $\mathfrak{X}$. Then $P(X, S)$ is necessarily non-empty and there exists a measure $\mu \in P(X, S)$ which dominates all $p \in P(X, S)$, that is, every $p \in P(X, S)$ is absolutely continuous with respect to μ, $p \ll \mu$ say. Let $L(X)$ be linear space of all bounded signed measures ξ, $\ll \mu$, with the norm $\|\xi\|_1$ defined by its total variation, cf. the formula (3), and $L^+(X)$ be the set of non-negative $\xi \in L(X)$. Then $L(X)$ is a Banach space which contains $L(X, S)$ as subspace.

Since a set in X is μ-null set if and only if it is p-null set for all $p \in P(X, S)$, the μ-null sets are determined by $P(X, S)$ and hence it is dependently determined only on the automorphism S.[4)]

For a pair f, g of measurable functions, the equality $f(x)=g(x)$ μ-a.e. $x \in X$ will be merely written by $f(x)=g(x)$, a.e. $x \in X$.

For any function f and measure ξ denote $(Sf)(x)=f(Sx)$ and $(S\xi)(U)=\xi(S^{-1}U)$. A linear functional $F(\cdot)$ on $L(X)$ is *S-stationary* if $F(S\xi)=F(\xi)$ for every $\xi \in L(X)$. Then the following is proved:

THEOREM 2. *The entropy functional $H(\cdot, \mathfrak{F}, S)$ for the partition $\mathfrak{F} \in \boldsymbol{F}(X)$ and for the automorphism S is uniquely extended to non-negative bounded S-stationary linear functional over the space $L(X)$.*

Proof. Let $\mathfrak{S}$ be the σ-subfield of $\mathfrak{X}$ consisting of all S-invariant (mod μ) measurable sets. Denote $E[\cdot \mid \mathfrak{S}]$ the conditional expectation conditioned by $\mathfrak{S}$ in the probability space $(X, \mathfrak{X}, \mu)$, cf. Doob [4]. Since $\xi \ll \mu$ for every $\xi \in L(X)$, the Radon-Nikodym derivative $d\xi/d\mu$ exists. Whence

$$E\left[\frac{d\xi}{d\mu}\middle|\mathfrak{S}\right]d\mu \qquad \text{for every } \xi \in L(X)$$

is a signed measure and belongs to $L(X, S)$, and the amount

$$H\left(E\left[\frac{d\xi}{d\mu}\middle|\mathfrak{S}\right]d\mu, \mathfrak{F}, S\right), \qquad \xi \in L(X),$$

is defined. Denote it by $\bar{H}(\cdot, \mathfrak{F}, S)$. Then $\bar{H}(\cdot, \mathfrak{F}, S)$ is an extension of $H(\cdot, \mathfrak{F}, S)$, and is well-defined and linear over the full space $L(X)$, because the transformation

$$d\xi \to E\left[\frac{d\xi}{d\mu}\middle|\mathfrak{S}\right]d\mu$$

is linear. Furthermore it is bounded. Indeed, the boundedness follows from

$$\begin{aligned}|\bar{H}(\xi, \mathfrak{F}, S)| &= \left|H\left(E\left[\frac{d\xi}{d\mu}\middle|\mathfrak{S}\right]d\mu, \mathfrak{F}, S\right)\right| \\ &\leqq |\mathfrak{F}| \cdot \left\|E\left[\frac{d\xi}{d\mu}\middle|\mathfrak{S}\right]d\mu\right\|_1\end{aligned}$$

4) By this reason, the domain $L(X)$ of the variable ξ of $H(\cdot, \cdot, S)$ is automatically restricted only by S. It can be also assumed that the symbols $\subset$, $\cup$ etc. are defined within mod μ.

$$\leqq |\mathfrak{F}| \cdot ||\xi||_1$$

Show that $\overline{H}(\cdot, \mathfrak{F}, S)$ is S-stationary. For every $\xi \in L(X)$

$$\overline{H}(S\xi, \mathfrak{F}, S) = H\left(E\left[\frac{d(S\xi)}{d\mu} \middle| \mathfrak{S} \right] d\mu, \mathfrak{F}, S \right)$$

$$= H\left(E\left[\frac{d\xi}{d\mu} \middle| \mathfrak{S} \right] d\mu, \mathfrak{F}, S \right) = \overline{H}(\xi, \mathfrak{F}, S),$$

because $(d(S\xi)/d\mu)(x) = (d\xi/d\mu)(Sx)$ and $E[Sf | \mathfrak{S}] = E[f | \mathfrak{S}]$ for every measurable function f.

Finally, it is shown the uniquness of the extension. Let $M(X)$ be Banach space of all essentially bounded measurable functions with the norm

$$||f||_\infty = \text{ess. sup}\{|f(x)|;\ x \in X\}.$$

Then $M(X)$ is the conjugate Banach space of $L(X)$ and therefore there exists uniquely (within a.e. $x \in X$) a function $h(x) \in M(X)$ such that

$$\overline{H}(\xi, \mathfrak{F}, S) = \int h(x) d\xi(x) \qquad \text{for every } \xi \in L(X).$$

By the S-stationarity of $\overline{H}(\cdot, \mathfrak{F}, S)$

$$\int h(x) d\xi(x) = \overline{H}(S\xi, \mathfrak{F}, S) = \int h(x) d\xi(S^{-1}x) = \int h(Sx) d\xi(x)$$

for every $\xi \in L(X)$ and hence $h(x) = h(Sx)$ a.e. $x \in X$, i.e. h is S-invariant. If $H'(\cdot)$ is another S-stationary extension of $H(\xi, \mathfrak{F}, S)$ $(\xi \in L(X, S))$ onto $L(X)$, then $H'(\cdot)$ is uniquely expressed by S-invariant $h'(x) \in M(x)$ such as

$$H'(\xi) = \int h'(x) d\xi(x) \qquad \text{for every } \xi \in L(X).$$

Therefore for every $\xi \in L(X)$

$$\int h'(x) d\xi(x) = \int h'(x) \frac{d\xi}{d\mu} d\mu(x)$$

$$= \int h'(x) E\left[\frac{d\xi}{d\mu} \middle| \mathfrak{S} \right] d\mu(x)$$

$$= \overline{H}(\xi, \mathfrak{F}, S)$$

by $E[d\xi/d\mu | \mathfrak{S}] d\mu(x) \in L(X, S)$,

$$= \int h(x) d\xi(x)$$

and $h(x) = h'(x)$ a.e. $x \in X$, and $\overline{H}(\xi, \mathfrak{F}, S) = H'(\xi)$ for all $\xi \in L(X)$. The proof is complete. q.e.d.

The extended functional obtained in Theorem 2 will be denoted by the same symbol $H(\cdot, \mathfrak{F}, S)$ and also called by *entropy functional* for the partition $\mathfrak{F}$ and the automorphism S, which depends only of the $\mathfrak{F}$ and the S, but is independendently determined of the choice of the S-stationary probability measure μ. Therefore the function $h(x)$, obtained in the final part of the proof Theorem 2, is dependently determined by the finite partition $\mathfrak{F}$ and the autmorphism S. Denote this function $h(\cdot)$ by $h(\cdot, \mathfrak{F}, S)$, which is essentially bounded, non-negative and S-invariant. The following theorem is an immediate consequence of the above fact.

THEOREM 3. *There exists uniquely, within a.e., a bounded non-negative S-invariant measurable function $h(\cdot, \mathfrak{F}, S)$ such that*

$$H(\xi, \mathfrak{F}, S)=\int h(x, \mathfrak{F}, S)d\xi(x) \quad \textit{for every } \xi\in L(X). \tag{12}$$

The kernel function $h(\cdot, \mathfrak{F}, S)$ of the entropy functional $H(\cdot, \mathfrak{F}, S)$ will be called the *entropy fuuction* for the partition $\mathfrak{F}$ over the measurable space $(X, \mathfrak{X})$. Theorem 3 implies

$$H(p, \mathfrak{F}, S)=\int h(x, \mathfrak{F}, S)dp(x) \tag{13}$$

for every S-invariant probability measure p. This is a generalization of the integral representation theorem of Parthasarathy [11].

There are several known properties of the functional $H(\cdot, \mathfrak{F}, S)$ for the fields $\mathfrak{F}$ and the automorphisms S, cf. Halmos [7], such that for every $\xi\in L^{+}(X)$

$$H(\xi, \mathfrak{F}, S)\leqq H(\xi, \mathfrak{G}, S) \quad \text{if} \quad \mathfrak{F}\subset \bigvee_{k=-\infty}^{\infty} S^{-k}\mathfrak{G} \tag{14}$$

especially

$$H(\xi, \mathfrak{F}, S)\leqq H(\xi, \mathfrak{G}, S) \quad \text{if} \quad \mathfrak{F}\subset\mathfrak{G} \tag{15}$$

and

$$H(\xi, T^{-1}\mathfrak{F}, S)=H(\xi, \mathfrak{F}, S) \text{ for automorphism } T \text{ with } ST=TS \tag{16}$$

where ξ is invariant with respect to both automorphisms S ann T, especially

$$H(\xi, S^{-1}\mathfrak{F}, S)=H(\xi, \mathfrak{F}, S). \tag{17}$$

The corresponding properties of the entropy function $h(\cdot, \mathfrak{F}: S)$ also hold in the a.e. sense, i.e. (14)$\sim$(17) hold for $h(\cdot, \cdot, \cdot)$ in the place of $H(\cdot, \cdot, \cdot)$. Furthermore, for every pair $\mathfrak{F}, \mathfrak{G}\in \boldsymbol{F}(X)$,

$$H(\xi, \mathfrak{F}\vee\mathfrak{G}, S)\leqq H(\xi, \mathfrak{F}, S)+H(\xi, \mathfrak{G}, S)\leqq 2H(\xi, \mathfrak{F}\vee\mathfrak{G}, S) \tag{18}$$

for every $\xi\in L^{+}(X)$ and

$$h(x, \mathfrak{F}\vee\mathfrak{G}, S)\leqq h(x, \mathfrak{F}, S)+h(x, \mathfrak{G}, S)\leqq 2h(x, \mathfrak{F}\vee\mathfrak{G}, S) \tag{19}$$

for a.e. $x\in X$, hold.

5. Entropy of automorphisms.

As in §4, assume the denombrability of $(X, \mathfrak{X})$ The amount $H(\xi, \mathfrak{F}, S)$ or

$h(x, \mathfrak{F}, S)$ is determined by three variables $\xi \in L(X)$, $\mathfrak{F} \in \boldsymbol{F}(X)$ and automorphisms S, or a.e. $x \in X$, $\mathfrak{F} \in \boldsymbol{F}(X)$ and S, respectively. Putting

$$H[\xi, S] = \sup_{\mathfrak{F} \in \boldsymbol{F}(X)} H(\xi, \mathfrak{F}, S), \ \xi \in L^{+}(X, S),\text{[5]} \tag{20}$$

this amount depends only on $\xi \in L^{+}(X)$ and automorphism S, and will be called *the entropy of the automorphism* S with respect to the measure ξ. If for a fixed S there exists a finite partition $\mathfrak{F} \in \boldsymbol{F}(X)$ such that

$$\bigcup_{n=-\infty}^{\infty} S^{-n}\mathfrak{F} = \mathfrak{X}, \tag{21}$$

then

$$H[\xi, S] = H(\xi, \mathfrak{F}, S), \ \xi \in L^{+}(X). \tag{22}$$

This follows immediately from (14). The message space A^{I} with its shift transformation has always this propecty. In general, the following holds:

THEOREM 4. *For fixed automorphism* S *and for any monotone increasing sequence of partitions* $\mathfrak{F}(m) \in \boldsymbol{F}(X)$:

$$\mathfrak{F}(1) \leqslant \mathfrak{F}(2) \leqslant \cdots, \quad \bigcup_{m=1}^{\infty} \mathfrak{F}(m) = \mathfrak{X}, \tag{23}$$

the equality

$$H[\xi, S] = \lim_{m \to \infty} H(\xi, \mathfrak{F}(m), S) \tag{24}$$

holds for every $\xi \in L^{+}(X)$.[6]

Proof. By Halmos ([7], p. 36), for any fixed $\mathfrak{F} \in \boldsymbol{F}(X)$,

$$\begin{aligned}
&-\frac{1}{k} \sum \{\xi(U) \log \xi(U);\ U \in \bigvee_{j=0}^{k-1} S^{-j}\mathfrak{F}\} \\
&\leqq -\frac{2n+k}{k} \frac{1}{2n+k} \sum \{\xi(U) \log \xi(U);\ U \in \bigvee_{j=0}^{2n+k-1} S^{-j}\mathfrak{F}(m)\} \\
&\quad - \sum_{U \in \mathfrak{F}} \int_X C_U(x) \cdot \log P_{\xi}\Big(U \Big| \bigcup_{j=-n}^{n} S^{j}\mathfrak{F}(m)\Big) d\xi(x) \\
&= I(k, m, n) + J(m, n), \text{ say.}
\end{aligned}$$

Since for any $\varepsilon > 0$ and for suitable m_0

$$0 \leqq J(m, n) \leqq J(m, 0) < \varepsilon \quad (m \geqq m_0),$$

the inequality

5) This amount equals also to the case when the 'sup' is taken for the denombrable partitions $\mathfrak{F}$ with finite $\{\sum \xi(U) \log \xi(U);\ U \in \mathfrak{F}\}$.

6) This is Rokhlin's theorem [12], which was stated for denombrable partitions in the place of $\mathfrak{F}(m)$, but the proof for its case can be done by the similar way of this proof.

$$-\frac{1}{k}\sum\{\xi(U)\log\xi(U);\ U\in\bigvee_{j=0}^{k-1}S^{-j}\mathfrak{A}\}<I(k, m, n)+\varepsilon \quad (m\geqq m_0) \tag{25}$$

holds. Furthermore, as $k\to\infty$, $I(k, m, n)\to H(\xi, \mathfrak{A}(m), S)$ and also the left side of (25) tends to $H(\xi, \mathfrak{A}, S)$. Therefore

$$H(\xi, \mathfrak{A}, S)\leqq H(\xi, \mathfrak{A}(m), S)+\varepsilon \quad (m\geqq m_0). \tag{26}$$

Since by (15) $H(\xi, \mathfrak{A}(m), S)$ is monotone-increasing with respect to m, there exists

$$H_\xi' = \lim H(\xi, \mathfrak{A}(m), S),$$

this limit is uniquely determined as non-negative finite or $+\infty$. Hence (26) implies

$$H[\xi, S]\leqq H_\xi'\leqq H[\xi, S]$$

and (24) is obtained. q.e.d.

By a simply modified proof of Theorem 4,

COROLLARY 4.1. *If $\mathfrak{A}(m)\in \boldsymbol{F}(X)$ is a sequence of partitions such that*

$$\mathfrak{S}(1)\subset\mathfrak{S}(2)\subset\cdots,\quad \bigcup_{m=1}^{\infty}\mathfrak{S}(m)=\mathfrak{X}, \tag{27}$$

where $\mathfrak{S}(m)=\bigcup_{k=1}^{\infty}S^{-k}\mathfrak{A}(m)$ is the subfields of $\mathfrak{X}$, then the equality (24) *holds.*

Theorem 4 is applied to the integral representation of entropy of automorphism:

THEOREM 5. *For any automorphism S, there exists uniquely, within a.e., an S-invariant non-negative measurable function $h[\cdot, S]$ over $(X, \mathfrak{X})$ such that*

$$H[\xi, S]=\int h[x, S]d\xi(x) \tag{28}$$

for every $\xi\in L^+(X)$, where the equality (28) *is meant by that if either side is finite then another side is also finite and they are equal.*

Proof. Taking $\mathfrak{A}(m)\in \boldsymbol{F}(X)$ given in Theorem 4, the sequence $\{H(\xi, \mathfrak{A}(m), S)\}$ is monotone increasing and convergences to $H[\xi, S]$, and the corresponding sequence $\{h(\cdot, \mathfrak{A}(m), S)\}$ of the entropy functions is also monotone increasing. Since by Theorem 3

$$H(\xi, \mathfrak{A}(m), S)=\int h(x, \mathfrak{A}(m), S)d\xi(x),\quad m=1, 2, \cdots,$$

taking the limit function of $h(\cdot, \mathfrak{A}(m), S)$, say $h[\cdot, S]$, then by the monotone convergence theorem it satisfies (28) for every $\xi\in L^+(X)$ and it is the required function.

If $h'(\cdot)$ is an function satisfying (28) for every $\xi\in L^+(X)$, then

$$\int h'(x)d\xi(x)=\int h[x, S]d\xi(x)$$

for every $\xi\in L^+(X)$. Hence

$$\{x;\ h'(x)<+\infty\}=\{x;\ h[x, S]<+\infty\}$$

(mod null set) and $h'(x)=h[x, S]$ a.e. $x\in X$, admitting $+\infty$, and the uniqueness of $h[x, S]$ is proved. q.e.d.

By this proof, it is easily proved that

COROLLARY 5.1. *Let $\mathfrak{F}(m)\in \boldsymbol{F}(X)$ be monotone iucreasing sequence of partitions with $\bigcup_{m=1}^{\infty}\mathfrak{F}(m)=\mathfrak{X}$ or more generally satisfying* (27), *then the sequence of functions $h(\cdot, \mathfrak{F}(m), S)$ is monotone increasing and converging to the function $h[\cdot, S]$ a.e. $x\in X$.*

REMARK. From Theorems 4 and 5, the integral representation theorem of the entropy functional for denombrable partition and automorphism follows in the form of (12), in which the corresponding integrand (the entropy function) in not necessarily bounded.

Now we shall describe for the functional $H[\cdot, \cdot]$ and the function $h[\cdot, \cdot]$ by a known interesting result relative to the entropy of flow. A one-parameter family $\{S_t\}$ of automorphisms is called *flow* if $S_{s+t}x=S_s(S_tx)$ for every pair of real numbers s, t and for all $x\in X$. A flow $\{S_t\}$ is called measurable, if the two variable function $\psi(t, x)=S_tx$ is measurable on the product measurable space $(-\infty, \infty)\times X$. Let $\bar{P}(X, S)$ be the set of all probability measures which are invariant with respect to all atomorphisms $\{S_t;\ -\infty<t<\infty\}$ and $L(X, S)$ be the closed linear subspace of $L(X)$ generated by $\bar{P}(X, S)$ (the space of all invariant bounded signed measures with respect to $\{S_t\}$). It is well-known, by the fixed-point theorem of Markoff-Tychonoff, that $\bar{P}(X, S)$ is non-empty.

The entropy functional $H(\cdot, \mathfrak{F}, S_t)$ over $L(X)$ is defined for $\mathfrak{F}\in \boldsymbol{F}(X)$ and for every $S_t\ (-\infty<t<\infty)$ and the amounts of entropies $H[\xi, S_t]$ of the automorphisms S_t are also defined for every $\xi\in L^+(X)$. Then by the theorem of Abramov [1], it holds that

$$H[\xi, S_t]=|t|\cdot H[\xi, S_1] \qquad (-\infty<t<\infty)$$

for $\xi\in L^+(X)$. This implies

$$h(x, S_t)=|t|\cdot h(x, S_1) \qquad (-\infty<t<\infty)$$

a.e. $x\in X$.

7. Appendix.

In this appendix, it is not necessarily assumed the denombrability of the measurable space $(X, \mathfrak{X})$. For any finite partition $\mathfrak{F}\in \boldsymbol{F}(X)$ and for a fixed automorphism S, putting

$$\tilde{\mathfrak{F}}_\infty = \bigcup_{-\infty}^{\infty} S^{-k}\mathfrak{F},$$

$(X, \tilde{\mathfrak{F}}_\infty)$ is denombrably generated measurable space on which S is an automorphism. Putting $P_{\mathfrak{F}}(X, S)$ the set of all S-invariant probability measures on $(X, \tilde{\mathfrak{F}}_\infty)$, it is non-empty and there exists $\mu_{\mathfrak{F}}\in P_{\mathfrak{F}}(X, S)$ which dominates $P_{\mathfrak{F}}(X, S)$. Let $L_{\mathfrak{F}}(X, S)$ or $L_{\mathfrak{F}}(X)$ be Banach spaces of all S-invariant or $\mu_{\mathfrak{F}}$-absolutely continuous bounded signed measures over $(X, \tilde{\mathfrak{F}}_\infty)$, respectively. As in the space $(X, \mathfrak{X})$, the entropy functional, denote $H_0(\cdot, \mathfrak{F}, S)$ for $\mathfrak{F}$ and S, in the space $(X, \tilde{\mathfrak{F}}_\infty)$ is defined, which is non-negative linear and S-stationary over $L_{\mathfrak{F}}(X)$. Then by Theorem 3, it is immediate that.

THEOREM 6. *For any partitian $\mathfrak{F} \in \boldsymbol{F}(X)$ and automorphism S, there exists uniquely, within $\mu_{\mathfrak{F}}$-a.e., an $\tilde{\mathfrak{F}}_\infty$ -measurable function $h_0(\cdot, \mathfrak{F}, S)$ such that it is S-iuvariant and*

$$H_0(\xi, \mathfrak{F}, S) = \int h_0(x, \mathfrak{F}, S) d\xi \quad \textit{for every } \xi \in L_{\mathfrak{F}}(X).$$

The equalities or inequalities (14)～(19) are also holds for the entropy functionals $H_0(\cdot, \cdot, \cdot)$ and the functions $h_0(\cdot, \cdot, \cdot)$, here ξ should be taken in $L_{\mathfrak{G}}{}^+(X)$ or $L^+{}_{\mathfrak{F}\vee\mathfrak{G}}(X)$ and the a.e. terms be taken as $\mu_{\mathfrak{G}}$- or $\mu_{\mathfrak{F}\vee\mathfrak{G}}$-a.e. terms which depend only on S and on $\mathfrak{G}$ or $\mathfrak{F} \vee \mathfrak{G}$ respectively.

If $(X, \mathfrak{X})$ is denombrably generated, then the measure $\mu_{\mathfrak{F}}$ over $(X, \tilde{\mathfrak{F}}_\infty)$ can be taken as the $[\mu/\tilde{\mathfrak{F}}_\infty]$ restriction of μ onto $(X, \tilde{\mathfrak{F}}_\infty)$. Let $E[d\xi/d\mu \mid \tilde{\mathfrak{F}}_\infty]d\mu$ be the conditional expectation of $\xi \in L(X)$ conditioned by $\tilde{\mathfrak{F}}_\infty$, then it coincides with $[\xi/\tilde{\mathfrak{F}}_\infty]$ $(=\xi_{\mathfrak{F}}$, say) over $\tilde{\mathfrak{F}}_\infty$ and

$$H_0(\xi_{\mathfrak{F}}, \mathfrak{F}, S) = H(\xi, \mathfrak{F}, S)$$

holds for every $\xi \in L(X, S)$ by the definition of entropy, cf. (4)～(6). Therefore for every $\xi \in L(X, S)$

$$\begin{aligned}
\int E[h(\cdot, \mathfrak{F}, S) \mid \tilde{\mathfrak{F}}_\infty] d\xi_{\mathfrak{F}}(x) &= \int E[h(\cdot, \mathfrak{F}, S) \mid \tilde{\mathfrak{F}}_\infty] d\xi(x) \\
&= \int h(x, \mathfrak{F}, S) E\left[\frac{d\xi}{d\mu} \middle| \tilde{\mathfrak{F}}_\infty\right](x) d\mu(x) \\
&= H\left(E\left[\frac{d\xi}{d\mu} \middle| \tilde{\mathfrak{F}}_\infty\right] d\mu, \mathfrak{F}, S\right) \\
&= \int h_0(x, \mathfrak{F}, S) E\left[\frac{d\xi}{d\mu} \middle| \tilde{\mathfrak{F}}_\infty\right](x) d\mu_{\mathfrak{F}}(x) \\
&= \int h_0(x, \mathfrak{F}, S) d\xi_{\mathfrak{F}}
\end{aligned}$$

and hence

$$h_0(x, \mathfrak{F}, S) = E[h(\cdot, \mathfrak{F}, S) \mid \tilde{\mathfrak{F}}_\infty](x)$$

a.e. $x \in X$ holds.

REFERENCES

[1] ABRAMOV, L. M., On the entropy of a flow. Dokl. Akad. Nauk SSSR **129** (1959), 873–875. (in Russian)

[2] BREIMAN, L., On achieving channel capacity in finite-memory channels, Ill. J. Math. **4** (1960), 246–252.

[3] DAVIS, C., Operator-valued entropy of a quantum mechanical measurement. Proc. Japan Acad. **37** (1961), 533–538.

[4] DOOB, J. L., Stochastic processes. New York, John Wiley (1953).

[5] ECHIGO, M., AND M. Nakamura, A remark of the concept of channels. Proc. Japan Acad., **38** (1962), 307–309.

[6] FEINSTEIN, A., Foundations of information theory. New York, McGraw-Hill, (1958).

[7] HALMOS, P. R., Entropy in ergodic theory, University of Chicago (1959).

[8] KHINCHIN, A. I., Mathematical foundations of information theory. New York, Dover Pub. (1957).

[9] NAKAMURA, M., AND H. UMEGAKI, A note on the entropy for operator algebras. Proc. Japan Acad. **37** (1961), 149–154.

[10] NAKAMURA, M., AND H. UMEGAKI, On von Neumann's theory of measurements in quantum statistics. Math. Jap. **7** (1962), 151–157.

[11] PARTHASARATHY, K. R., On the integral representation of the rate of transmission of a stationary channel. Ill. J. Math. **5** (1961), 299–305.

[12] ROKHLIN, V. A., New progress in the theory of translations with invariant measure, Uspekhi Mat. Nauk **15** (1960), 3–26; Russian Math. Survey, London Math. Soc. **15** (1960), 3–22.

[13] TAKANO, K., On the basic theorems of information theory. Ann. Inst. Statist. Math. Tokyo **9** (1958), 53–77.

[14] TSAREGRADSKY, I. P., On the capacity of a stationary channel with finite memory, Teor. Veroyatnost. Primenen **3** (1958), 84–94. (in Russian)

[15] UMEGAKI, H., Conditional expectation in an operator algebra, IV (Entropy and information). Kōdai Math. Sem. Rep. **14** (1962). 59–85.

[16] UMEGAKI, H., Entropy functionals in stationary channels. Proc. Japan Acad. **38** (1962), 668–672.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.