A NOTE ON THE DIRECT PRODUCT OF OPERATOR ALGEBRAS

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In [10], [11] and [2], Turumaru and Misonou have introduced the notion of the direct product of C^*-algebras and that of W^*-algebras, respectively. They are defined as follows.

Let $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ be C^*-algebras on Hilbert spaces H_1 and H_2 respectively. Then the C^*-direct porduct of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ is the uniform closures of the algebraical direct product $\boldsymbol{M}_1 \odot \boldsymbol{M}_2$ on the direct product Hilbert space $H_1 \otimes H_2$ and denoted by $\boldsymbol{M}_1 \hat{\otimes}_\alpha M_2$ in [9]. If $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are W^*-algebras on Hilbert spaces H_1 and H_2, their W^*-direct product is the weak closure of $\boldsymbol{M}_1 \odot \boldsymbol{M}_2$ on $H_1 \otimes H_2$ and denoted by $M_1 \otimes M_2$ in [2]. These two notions generally do not coincide each other. Precisely speaking, it can be shown that *if $\boldsymbol{M}_1$ and M_2 are W^*-algebras whose C^*-direct product $\boldsymbol{M}_1 \hat{\otimes}_\alpha \boldsymbol{M}_2$ coincides with the W^*-direct product $\boldsymbol{M}_1 \otimes \boldsymbol{M}_2$, then either $\boldsymbol{M}_1$ or $\boldsymbol{M}_2$ is finite dimensional matrix algebra*[1]. In the following we shall prove this result under slightly general conditions.

In the C^*-direct product $\boldsymbol{M}_1 \hat{\otimes}_\alpha \boldsymbol{M}_2$, the cross-norm α of $\sum_{i=1}^m a_i \otimes b_i$ is given in [10] as follows

$$\alpha(\textstyle\sum_{i=1}^m a_i \otimes b_i)$$
$$= \sup \frac{<(\sum_{j=1}^m x_j^* \otimes y_j^*)(\sum_{i=1}^n a_i^* \otimes b_i^*)(\sum_{i=1}^n a_i \otimes b_i)(\sum_{j=1}^m x_j \otimes y_j), \varphi \otimes \psi>}{<(\sum_{j=1}^m x_j^* \otimes y_j^*)(\sum_{j=1}^m x_j \otimes y_j), \varphi \otimes \psi>}$$

where φ and ψ run over all states of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ respectively and $\sum_{j=1}^m x_j \otimes y_j$ runs over all non-zero elements of $\boldsymbol{M}_1 \odot \boldsymbol{M}_2$[2]. On the other hand, Schatten [4] defined the cross-norm λ of the direct product of Banach spaces E and F as follows

$$\lambda(\textstyle\sum_{i=1}^m a_i \otimes b_i) = \sup\{|\sum_{i=1}^m <a_i, \varphi><b_i, \psi>|;\ \varphi \in E^*,\ \|\varphi\| \leqq 1,\ \psi \in F^*,\ \|\psi\| \leqq 1\}.$$

In particular, when E and F are C^*-algebras $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$, we have generally $\lambda \leqq \alpha$. The necessary and sufficient condition that α-norm coincides with λ-norm is that either $\boldsymbol{M}_1$ or $\boldsymbol{M}_2$ is commutative (cf. [5]).

Now we state our main theorem.

Theorem. *If $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are C^*-algebras whose C^*-direct product $\boldsymbol{M}_1 \hat{\otimes}_\alpha \boldsymbol{M}_2$*

Received June 4, 1959.

1) The author expresses his hearty thanks for Prof. M. Nakamura who has suggested this result to him.

2) In the following, we write always $\|\sum_{i=1}^n x_i \otimes y_i\|$ in stead of $\alpha(\sum_{i=1}^n x_i \otimes y_i)$.

is an AW^*-algebra, then both $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are AW^*-algebras and either $\boldsymbol{M}_1$ or $\boldsymbol{M}_2$ satisfies the finite chain condition, i.e. one of them is finite dimensional matrix algebra over the complex number field.

Proof. For an arbitrary state ψ_0 of $\boldsymbol{M}_2$, put the mapping θ_{ψ_0} from $\boldsymbol{M}_1 \odot \boldsymbol{M}_2$ to $\boldsymbol{M}_1$ such as $\theta_{\psi_0}(\sum_{i=1}^m x_i \otimes y_i) = \sum_{i=1}^m <y_i, \psi_0> x_i$ for every $\sum_{i=1}^m x_i \otimes y_i \in \boldsymbol{M}_1 \odot \boldsymbol{M}_2$. Then we have

$$\begin{aligned}
\| \theta_{\psi_0}(\textstyle\sum_{i=1}^m x_i \otimes y_i) \| &= \| \textstyle\sum_{i=1}^m <y_i, \psi_0> x_i \| \\
&= \sup\{ | \textstyle\sum_{i=1}^m <y_i, \psi_0><x_i, \varphi_0> |;\ \varphi \in \boldsymbol{M}_1^*,\ \| \varphi \| \leqq 1 \} \\
&\leqq \sup\{ | \textstyle\sum_{i=1}^m <x_i, \varphi><y_i, \varphi> |;\ \varphi \in \boldsymbol{M}_1^*,\ \| \varphi \| \leqq 1,\ \psi \in \boldsymbol{M}_2^*,\ \| \psi \| \leqq 1 \} \\
&= \lambda(\textstyle\sum_{i=1}^m x_i \otimes y_i) \leqq \| \textstyle\sum_{i=1}^m x_i \otimes y_i \|
\end{aligned}$$

so that θ_{ψ_0} is uniformly continuous on $\boldsymbol{M}_1 \odot \boldsymbol{M}_2$. Hence θ_{ψ_0} is extended to the mapping from $\boldsymbol{M}_1 \hat{\otimes}_\alpha \boldsymbol{M}_2$ to $\boldsymbol{M}_1$. Furthermore, if I is the unit of $\boldsymbol{M}_1 \hat{\otimes}_\alpha \boldsymbol{M}_2$, then $<\theta_{\psi_0}(I), \varphi> = <I, \varphi \otimes \psi_0> = 1$ for every state φ of $\boldsymbol{M}_1$, for $\varphi \otimes \psi_0$ is a state of $\boldsymbol{M}_1 \hat{\otimes}_\alpha \boldsymbol{M}_2$. Hence the state space of $\boldsymbol{M}_1$ is $\sigma(\boldsymbol{M}_1^*, \boldsymbol{M}_1)$-compact, so that $\boldsymbol{M}_1$ has the unit which coincides with $\theta_{\psi_0}(I)$ by [5; Theorem 1]. Similarly $\boldsymbol{M}_2$ has a unit. Thus, identifying $\boldsymbol{M}_1$ and $\boldsymbol{M}_1 \otimes I$, $\boldsymbol{M}_1$ is considered to be a subalgebra of $\boldsymbol{M}_1 \hat{\otimes}_\alpha \boldsymbol{M}_2$ and θ_{ψ_0} an expectation from $\boldsymbol{M}_1 \hat{\otimes}_\alpha \boldsymbol{M}_2$ to $\boldsymbol{M}_1$ in the sense of [3] (projection of norm one in the sense of [8]). Therefore $\boldsymbol{M}_1$ is an AW^*-algebra by [8; Theorem 5]. Similarly $\boldsymbol{M}_2$ is also an AW^*-algebra. This completes the first part of our demonstrations.

Next we assume that both $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ do not satisfy the finite chain condition, i.e. there exist infinite families $\{e_n\}$ and $\{f_n\}$ of orthogonal projections in $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ respectively. Let $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be AW^*-subalgebras of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ generated by $\{e_n\}$ and $\{f_n\}$, there exist expectations θ_1 and θ_2 from $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ onto $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ respectively by [1; Theorem 2] and [8; Theorem 1]. Assuming the following two lemmas, we shall meet a contradiction.

LEMMA 1. *There exists an expectation θ from $\boldsymbol{M}_1 \hat{\otimes}_\alpha \boldsymbol{M}_2$ onto $\boldsymbol{A}_1 \hat{\otimes}_\alpha \boldsymbol{A}_2$ such that $\theta(x \otimes y) = \theta_1(x) \otimes \theta_2(y)$.*

LEMMA 2. *$\boldsymbol{A}_1 \hat{\otimes}_\alpha \boldsymbol{A}_2$ is not AW^*-algebra.*

By Lemma 1 and [8; Theorem 5] $\boldsymbol{A}_1 \hat{\otimes}_\alpha \boldsymbol{A}_2$ is an AW^*-algebra, but this is impossible by Lemma 2. Therefore either $\boldsymbol{M}_1$ or $\boldsymbol{M}_2$ satisfies finite chain condition. This finishes the proof of the theorem.

Now we shall prove the lemmas.

Proof of Lemma 1. Put $\theta(\sum_{i=1}^m x_i \otimes y_i) = \sum_{i=1}^m \theta_1(x_i) \otimes \theta_2(y_i)$ for $\sum_{i=1} x_i \otimes y_i \in \boldsymbol{M}_1 \odot \boldsymbol{M}_2$. By [7; Proposition 2] the cross-norm α on $\boldsymbol{A}_1 \hat{\otimes}_\alpha \boldsymbol{A}_2$ coincides with λ-norm. Hence we get

$$\begin{aligned}
\| \theta(\textstyle\sum_{i=1}^m x_i \otimes y_i) \| &= \| \textstyle\sum_{i=1}^m \theta_1(x_i) \otimes \theta_2(y_i) \| = \lambda(\textstyle\sum_{i=1}^m \theta_1(x_i) \otimes \theta_2(y_i)) \\
&= \sup\{ | \textstyle\sum_{i=1} <\theta_1(x_i), \varphi><\theta_2(y_i), \psi> |;\ \varphi \in A_1^*,\ \| \varphi \| \leqq 1,\ \psi \in A_2^*,\ \| \psi \| \leqq 1 \} \\
&= \sup\{ | \textstyle\sum_{i=1}^m <x_i, {}^t\theta_1(\varphi)><y_i, {}^t\theta_2(\psi)> |;\ \varphi \in A_1^*,\ \| \varphi \| \leqq 1,\ \psi \in A_2^*,\ \| \psi \| \leqq 1 \}
\end{aligned}$$

$$\leqq \sup\{|\textstyle\sum_{i=1}^m <x_i, \varphi><y_i, \psi>|;\ \varphi \in M_1^*,\ \|\varphi\| \leqq 1,\ \psi \in M_2^*,\ \|\psi\| \leqq 1\}$$
$$= \lambda(\textstyle\sum_{i=1}^m x_i \otimes y_i) \leqq \|\sum_{i=1}^m x_i \otimes y_i\|$$

for every $\sum_{i=1}^m x_i \otimes y_i \in \boldsymbol{M}_1 \odot \boldsymbol{M}_2$. That is, θ is uniformly continuous and of norm one. Therefore θ is extended to the projection of norm one from $\boldsymbol{M}_1 \hat{\otimes}_\alpha \boldsymbol{M}_2$ onto $\boldsymbol{A}_1 \hat{\otimes}_\alpha \boldsymbol{A}_2$ such as $\theta(x \otimes y) = \theta(x) \otimes \theta_2(y)$. By [8; Theorem 1] θ is an expectation from $\boldsymbol{M}_1 \hat{\otimes}_\alpha \boldsymbol{M}_2$ onto $\boldsymbol{A}_1 \hat{\otimes}_\alpha \boldsymbol{A}_2$. This completes the proof.

REMARK. If either $\boldsymbol{A}_1$ or $\boldsymbol{A}_2$ is commutative, then general bounded linear mappings θ_i from $\boldsymbol{M}_i$ to $\boldsymbol{A}_i$ $(i = 1, 2)$ have common extension θ from $\boldsymbol{M}_1 \hat{\otimes}_\alpha \boldsymbol{M}_2$ to $\boldsymbol{A}_1 \hat{\otimes}_\alpha \boldsymbol{A}_2$ such as $\theta(x \otimes y) = \theta_1(x) \otimes \theta_2(y)$ with the bound $\|\theta_1\| \cdot \|\theta_2\|$ by the above arguments. Moreover in the case of $\boldsymbol{M}_i$ and $\boldsymbol{A}_i$ to be W^*-algebras $(i = 1, 2)$ and θ_i to be σ-weakly continuous, considering their conjugate spaces and transposed mappings, the above mapping θ becomes σ-weakly continuous mapping from W^*-product $\boldsymbol{M}_1 \otimes \boldsymbol{M}_2$ to $\boldsymbol{A}_1 \otimes \boldsymbol{A}_2$.

Recently, J. Tomiyama has proved Lemma 2 without the assumption of commutativity for $\boldsymbol{A}_1$ or $\boldsymbol{A}_2$.

Proof of Lemma 2. The spectrum space of $\boldsymbol{A}_1$ is the Čech's compactification of the set of all integers and by Ω. By the argument in [7; Proposition 2] $\boldsymbol{A}_1 \hat{\otimes}_\alpha \boldsymbol{A}_2$ is the algebra of all $\boldsymbol{A}_2$-valued continuous functions on Ω, i.e. $\boldsymbol{A}_1 \hat{\otimes}_\alpha \boldsymbol{A}_2 \cong C_{\boldsymbol{A}_2}(\Omega)$. Put $p_n = \sum_{i=1}^n e_i \otimes f_n$, then $\{p_n\}$ is a family of monotone increasing sequence of projections in $\boldsymbol{A}_1 \hat{\otimes}_\alpha \boldsymbol{A}_2$. If $\boldsymbol{A}_1 \hat{\otimes}_\alpha \boldsymbol{A}_2$ is an AW^*-algebra, then $\{p_n\}$ has the least upper bound projection p in $\boldsymbol{A}_1 \hat{\otimes}_\alpha \boldsymbol{A}_2$. Considering the function-representations of p_n and p in $C_{\boldsymbol{A}_2}(\Omega)$ as

$$p_n = \{f_1, f_2, \cdots, f_n, 0, 0, \cdots\}$$

and

$$p = \{a_1, a_2, \cdots, a_n, \cdots\},$$

the equality $p = \text{l.u.b. } p_n$ implies $a_n = f_n$, $n = 1, 2, \cdots$. Hence the bounded function $\{f_1, f_2, \cdots, f_n, \cdots\}$ on the space of integers is necessarily extended to the whole space Ω preserving its continuity. That is, for any $\varepsilon > 0$ and $t \in \Omega$, there exists a neighborhood U of t such that $\|p(t_1) - p(t_2)\| < \varepsilon$ for every pair $t_1, t_2 \in U$. But this is impossible because, choosing an ideal point of compactification as t and $\varepsilon = 1/2$, every neighborhood U of t contains two distinct integers n_1 and n_2, so that $\|p(n_1) - p(n_2)\| = \|f_{n_1} - f_{n_2}\| > \varepsilon$. Therefore $\{p_n\}$ has not the least upper bound in $\boldsymbol{A}_1 \hat{\otimes}_\alpha \boldsymbol{A}_2$, that is, $\boldsymbol{A}_1 \hat{\otimes}_\alpha \boldsymbol{A}_2$ is not an AW^*-algebra. This completes the proof.

Our theorem, in particular, induces the following Wada's Theorem [13] as

COROLLARY. *If the cartesian product space of two locally compact spaces Ω_1 and Ω_2 is a stonian space, then Ω_1 and Ω_2 are both stonian spaces and either Ω_1 or Ω_2 is a finite set.*

Proof. Assuming $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ in our theorem to be commutative, one can easily see that the conclusion of this Corollary is nothing but the change of the statement in our theorem by the terminology of those spectrum spaces of C^*-algebras $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$.

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