THE PERMUTABILITY IN A CERTAIN ORTHOCOMPLEMENTED LATTICE

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1. In an orthocomplemented lattice[1] L, two elements a and b will be called *permutable* in the sense of Maeda and Sasaki, symbolically $a \backsim b$, if they satisfy

$$\text{(P)} \qquad a = (a \cap b) \cup (a \cap b^{\perp}).$$

It is clear that the permutability satisfies $a \backsim a^{\perp}$ and

$$\text{(Q)} \qquad a \leqq b \quad \textit{implies} \quad a \backsim b.$$

For, $a \leqq b$ implies $a = a \cup (a \cap b^{\perp}) = (a \cap b) \cup (a \cap b^{\perp})$.

In general cases, the permutability is not symmetric. However, we have[2]

Theorem 1. *The permutability of an orthocomplemented lattice is symmetric if and only if the lattice satisfies*

$$\text{(V)} \qquad a \leqq b \quad \textit{implies} \quad b = a \cup (a^{\perp} \cap b).$$

Proof. If the permutability is symmetric, then (Q) implies $b \backsim a$ when $a \leqq b$, that is, $b = (a \cap b) \cup (a^{\perp} \cap b) = a \cup (a^{\perp} \cap b)$ which is (V).

If $a \backsim b$, i.e., (P) is true for a and b, then $a^{\perp} = (a \cap b)^{\perp} \cap (a^{\perp} \cup b)$, whence

$$b \cap a^{\perp} = (a \cap b)^{\perp} \cap (a^{\perp} \cup b) \cap b = (a \cap b)^{\perp} \cap b.$$

Therefore, (V) implies

$$b = (a \cap b) \cup (b \cap (a \cap b)^{\perp}) = (a \cap b) \cup (a^{\perp} \cap b),$$

which shows $b \backsim a$. This completes the proof.

Since the symmetric permutability is characteristic for the lattices satisfying (V), we shall call them *symmetric lattices*. In the present note, we shall extend the permutablility theorem of Sasaki [3; Theorem 5.2] for a general symmetric lattice.

2. In a symmetric lattice L, the *Sasaki projection on a* is defined by

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1) The standard terminologies of G. Birkhoff [1] will be used without any explanation. u/v indicates the interval between u and v.

2) The condition (V) is taken from U. Sasaki [3] and H. L. Loomis [2]. The dual of (V) will be referred to as (V'). The second part of Theorem 1 has been already proved by Sasaki [3; Lemma 5.5].

(S) $$x \to x^a = (x \cup a^\perp) \cap a.$$

The product of two Sasaki projections will be defined as usual by $x^{ab} = (x^a)^b$. They will be called *permutable*, symbolically $ab = ba$, if $x^{ab} = x^{ba}$ for all x. A typical example for permutable projections is the following

(T) $$a \leqq b \quad \textit{implies} \quad ab = ba = a.$$

For, we have $x^{ab} = (x^a \cup b^\perp) \cap b = x^a$ by (V) and using (V′)

$$x^{ba} = (((x \cup b^\perp) \cap b) \cup a^\perp) \cap a = ((x \cup b^\perp) \cup a^\perp) \cap a = (x \cup a^\perp) \cap a = x^a.$$

THEOREM 2. *In a symmetric lattice L, a unary operation $x \to x^*$ into itself is a Sasaki projection on a if and only if it is a Nagao*[3] *operation, i.e.,*

(I) *idempotent*: $x^{**} = x^*$,

(II) *join-endomorphic*: $(x \cup y)^* = x^* \cup y^*$,

(III) *it carries $1/a^\perp$ onto $a/0$ isomorphically.*

Necessity. It is clear by the monotonity of the lattice polynomials [1; 19] that the projection on a preserves the order and carries $1/a^\perp$ into $a/0$. We shall show that the mapping is one-to-one. If $x \cap a = y \cap a$ and $x, y \geqq a^\perp$, then by (V), we have $x = (x \cap a) \cup a^\perp = (y \cap a) \cup a^\perp = y$. Furthermore, the mapping is onto. If not, there is an x such as $0 < x < a$ and $x \neq y \cap a$ for all $y \geqq a^\perp$. By (V′), $x = a \cap (a^\perp \cup x)$ becomes a contradiction. Therefore, the Sasaki projection on a maps $1/a^\perp$ onto $a/0$ in order-preserving and one-to-one way, whence it is an isomorphism. This proves (III). Obviously the projection keeps $a/0$ element-wise, by (V′), whence it satisfies (I). Since $x \to x \cup a^\perp$ is join-endomorphic and $1/a^\perp$ is its range, the first half of the present proof shows (II).

Sufficiency. Let Δ be the isomorphism indicated in (III) and ∇ be its inverse. If $x' = x^{*\nabla}$ then $x \to x'$ is an idempotent join-endomorphism of L onto $1/a^\perp$, whence

$$a^\perp \leqq x \cup a^\perp = (x \cup a^\perp)' = x' \cup a^{\perp\prime} = x'.$$

This shows that ∇ acts on $a/0$ as the converse of the Sasaki projection on a: $x^\nabla = x \cup a^\perp$ if $x \leqq a$. Therefore $x^* = (x \cup a^\perp) \cap a$.

3. Sasaki's permutability theorem [3; Theorem 5.2] will be now extended in the following

THEOREM 3. *In a symmetric lattice, the permutabilities of projections and elements are equivalent, that is, symbolically*

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3) The operation considered in Theorem 2 has been originally introduced by A. Nagao, Zenkoku Sizyo Sugakudanwakwai (in Japanese), 2nd ser., No. 4 (1947), 49—58, for a finite-dimensional modular lattice in connection with the Remak-Schmidt Theorem. The corresponding theorem for a modular lattice has been proved by the author, ibid., No. 5 (1947), 115—117.

(W) $\qquad a \backsim b$ *if and only if* $ab = ba = a \cap b$.

Necessity. By Theorem 2 it is sufficient to show that ab is a Nagao operation having the range $a \cap b/0$. Clearly, ab satisfies (II) since it is the product of two join-endomorphisms. It is also obvious that ab preserves $a \cap b/0$ element-wise since a and b keep $a/0$ and $b/0$ element-wise, respectively. The permutability of the elements implies

$$a^b = ((a \cap b) \cup (a \cap b^\perp))^b = (a \cap b)^b \cup (a \cap b^\perp)^b = (a \cap b)^b = a \cap b,$$

whence $x^a \leqq a$ implies $x^{ab} \leqq a \cap b$, and so $abab = ab$, that is, ab satisfies (I). Thus it remains to show that ab satisfies (III). If $x \geqq a^\perp \cup b^\perp$, then

$$x^a = x \cap a \leqq (x \cap a) \cup b^\perp \leqq (x \cap a) \cup x = x$$

implies $x \cap a \cap b \leqq x^{ab} \leqq x \cap b$. Therefore we have $x^{ab} = x \cap a \cap b$ since we have proved $x^{ab} \leqq a \cap b$.

Sufficiency. Since $(a^\perp \cup b^\perp)^{ab} = 0$ implies

$$b \cap (b^\perp \cup (a \cap (b^\perp \cup a^\perp))) = b \cap (b^\perp \cup (a \cap (a^\perp \cup b^\perp \cup a^\perp))) = 0,$$

and since $x \to x \cap b$ is an isomorphism between $1/b^\perp$ and $b/0$, we have $b^\perp = b^\perp \cup (a \cap (a^\perp \cup b^\perp))$ or $(a^\perp \cup b^\perp) \cap a \leqq b^\perp$. On the other hand, we have clearly $(a^\perp \cup b^\perp) \cap a \leqq a$, whence we have $(a^\perp \cup b^\perp) \cap a \leqq a \cap b^\perp$. Using (V), we finally have

$$a = (a \cap b) \cup ((a^\perp \cup b^\perp) \cap a) \leqq (a \cap b) \cup (a \cap b^\perp) \leqq a,$$

which proves the theorem.

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