ON CENTER-TYPE SINGULAR POINTS

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1. We consider a system of differential equations

$$(1)\quad \begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = -x + P(x,y), \end{cases}$$

where $P(x,y)$ is a polynomial of x and y with real coefficients lacking constant and linear terms. As is well-known, the singular point $x=y=0$ of this system is generally a focus. But when the coefficients of $P(x,y)$ satisfy certain conditions, it becomes a center. Therefore it is desirable to find necessary and sufficient conditions imposed on the coefficients of $P(x,y)$ for the origin $x=y=0$ to be a center.

In general, this problem is a very difficult one, and it is probable that such conditions, if stated explicitly, may take a very complicated form. However, there may exist some special cases in which these conditions can be stated in a comparatively simple form. In this paper, we will give several examples of such cases.

2. First we state some well-known theorems without proof. These theorems will be used freely in our following discussions.

Theorem 1. In order that the singular point $x=y=0$ of the system (1) should be a center, it is necessary and sufficient that there exists a formal power series

$$(2)\quad F(x,y) = x^2+y^2+F_3(x,y)+F_4(x,y)+\cdots$$

where $F_k(x,y)$ are polynomials of degree k homogeneous in x and y, formally satisfying the equation

$$(3)\quad \frac{\partial F}{\partial x}\cdot y + \frac{\partial F}{\partial y}\left(-x+P(x,y)\right) = 0.$$

Theorem 2. In order that the singular point $x=y=0$ of the system (1) should be a center, it is necessary and sufficient that there exists a formal power series

$$(4)\quad M(x,y) = 1 + M_1(x,y) + M_2(x,y) + \cdots$$

where $M_k(x,y)$ are polynomials of degree k homogeneous in x and y, formally satisfying the equation

$$(5)\quad \frac{\partial}{\partial x}(M\cdot y) + \frac{\partial}{\partial y}\left\{M\cdot(-x+P(x,y))\right\} = 0.$$

Theorem 3. If

$$P(x,-y) = P(x,y),$$

the singular point $x=y=0$ of the system (1) is a center.

Theorem 4. If

$$P(-x,y) = -P(x,y),$$

the singular point $x=y=0$ of the system (1) is a center.

3. Proposition 1. When

$$P(x,y) = y^p \sum_{k=n}^{N} a_k x^k, \quad n+p \geqq 2,$$

$x=y=0$ is a center if and only if one of the following conditions is satisfied:

(1) p is an even number;
(2) p is an odd number and $a_k = 0$ for every even k.

Proof. As the sufficiency of the condition is obvious from Theorem 3 and Theorem 4, we have only to prove that, if $x=y=0$ is a center and p is an odd number, every a_k with even k must vanish.

If $x=y=0$ is a center, according to Theorem 1, there exists a formal power series (2) satisfying (3). Introducing polar coordinates

$$x=r\cos\theta, \quad y=r\sin\theta,$$

we rewrite the equation (3) in the following form:

$$(3') \quad \begin{aligned}\sum_{m=3} r^m \frac{dF_m(\theta)}{d\theta} &= \sum_{m=2} m r^{m-1} F_m(\theta) \times \sum_{k=n}^{N} a_k r^{k+p} \sin^{p+1}\theta \cos^k\theta \\ &+ \sum_{m=3} r^m \frac{dF_m(\theta)}{d\theta} \sum_{k=n}^{N} a_k r^{k+p-1} \sin^p\theta \cos^{k+1}\theta,\end{aligned}$$

where $F_2(\theta)=1$ and $F_k(\theta)=F_k(\cos\theta, \sin\theta)$ for $k \geqq 3$.

Equating the terms of the same degree in r on both sides of the above equation, we can successively determine all $F_k(\theta)$. Since $F_k(x,y)$ are polynomials of x and y, all $F_k(\theta)$ thus determined should be free from secular terms.

First we suppose that n is an even number. By comparing the terms of the degree not greater than $n+p$ in r on both sides of (3'), we obtain

$$\frac{dF_m(\theta)}{d\theta} = 0 \qquad \text{for} \quad m \leqq n+p.$$

Hence

$$F_m(\theta) = 0 \qquad \text{for} \quad m \leqq n+p.$$

(It is known that, in determining $F_k(\theta)$ from (3'), values of integration constants are insignificant, and thus can be chosen arbitrarily.)

$F_{n+p+1}(\theta)$ is determined from the equation

$$\frac{dF_{n+p+1}(\theta)}{d\theta} = 2a_n \sin^{p+1}\theta \cos^n\theta.$$

As p is supposed to be odd and n is even, $n+p+1$ is an even number. Consequently

$$\int_0^{2\pi} \sin^{p+1}\theta \cos^n\theta \, d\theta \neq 0.$$

Hence, in order that $F_{n+p+1}(\theta)$ can be determined without secular terms, a_n must be equal to zero.

So we have

$$\begin{aligned}\frac{dF_{n+p+2}(\theta)}{d\theta} &= 2a_{n+1}\sin^{p+1}\theta\cos^{n+1}\theta \\ &+ 3a_n F_3(\theta)\sin^{p+1}\theta\cos^n\theta + a_n\frac{dF_3(\theta)}{d\theta}\sin^p\theta\cos^{n+1}\theta \\ &= 2a_{n+1}\sin^{p+1}\theta\cos^{n+1}\theta.\end{aligned}$$

$n+1$ being an odd number, $F_{n+p+2}(\theta)$ is determined as a polynomial of $\sin\theta$ only.

Then suppose that

$$a_n = a_{n+2} = \cdots = a_{n+2k-2} = 0,$$

and $F_m(\theta)$ have been determined up to $m = n+2k+p$ as polynomials of $\sin\theta$ only. Then the equation determining $F_{n+2k+p+1}(\theta)$ is given by

$$\begin{aligned}\frac{dF_{n+2k+p+1}(\theta)}{d\theta} &= 2a_{n+2k}\sin^{p+1}\theta\cos^{n+2k}\theta \\ &+ 3a_{n+2k-1}F_3(\theta)\sin^{p+1}\theta\cos^{n+2k-1}\theta \\ &+ 5a_{n+2k-3}F_5(\theta)\sin^{p+1}\theta\cos^{n+2k-3}\theta \\ &+ \cdots + (2k+1)a_{n+1}F_{2k+1}(\theta)\cos^{n+1}\theta \\ &+ a_{n+2k-1}\frac{dF_3(\theta)}{d\theta}\sin^p\theta\cos^{n+2k}\theta \\ &+ a_{n+2k-3}\frac{dF_5(\theta)}{d\theta}\sin^p\theta\cos^{n+2k-2}\theta \\ &+ \cdots + a_{n+1}\frac{dF_{2k+1}(\theta)}{d\theta}\cos^{n+2}\theta.\end{aligned}$$

From the assumption of induction, all the terms on the right-hand side, except the first one, have the form

(polynomial of $\sin\theta$) $\times$ (odd power of $\cos\theta$).

Therefore

$$\int_0^{2\pi}\frac{dF_{n+2k+p+1}(\theta)}{d\theta}\,d\theta = 2a_{n+2k}\int_0^{2\pi}\sin^{p+1}\theta\cos^{n+2k}\theta\,d\theta.$$

Since $F_{n+2k+p+1}(\theta)$ should be free from secular terms, and

$$\int_0^{2\pi}\sin^{p+1}\theta\cos^{n+2k}\theta\,d\theta \neq 0,$$

a_{n+2k} must be equal to zero. Hence $F_{n+2k+p+1}(\theta)$ is determined as a polynomial of $\sin\theta$ only.

$F_{n+2k+p+2}(\theta)$ is determined from the equation

$$\frac{dF_{n+2k+p+2}(\theta)}{d\theta} = 2a_{n+2k+1}\sin^{p+1}\theta\cos^{n+2k+1}\theta$$
$$+ 4a_{n+2k-1}F_4(\theta)\sin^{p+1}\theta\cos^{n+2k-1}\theta$$
$$+ \cdots + (2k+2)\,a_{n+1}F_{2k+2}(\theta)\sin^{p+1}\theta\cos^{n+1}\theta$$
$$+ a_{n+2k-1}\frac{dF_4(\theta)}{d\theta}\sin^{p}\theta\cos^{n+2k}\theta$$
$$+ \cdots + a_{n+1}\frac{dF_{2k+2}(\theta)}{d\theta}\sin^{p}\theta\cos^{n+2}\theta.$$

From the assumption of induction and that $F_{n+2k+p+1}(\theta)$ is a polynomial of $\sin\theta$ only, all the terms on the right-hand side of the above equation have the form

(polynomial of $\sin\theta$) $\times$ (odd power of $\cos\theta$).

Therefore $F_{n+2k+p+2}(\theta)$ is also a polynomial of $\sin\theta$ only.
Thus we have completed the proof for even n .

When n is an odd number, the proof can be carried out analogously.

In a similar way, we can prove the following

Proposition 2. When

$$P(x,y) = x^p\sum_{k=n}^{N} a_k y^k, \quad n+p \geqq 2,$$

$x=y=0$ is a center if and only if one of the following conditions is satisfied:

(1) p is an odd number;
(2) p is an even number and $a_k = 0$ for every odd k.

4. Next we consider the case when $P(x,y)$ is a homogeneous polynomial of degree $n+1$, $n \geqq 1$.

For $x=y=0$ to be a center, according to Theorem 2, it is necessary and sufficient that there exists a formal power series (4) satisfying (5).

In polar coordinates, the equation (5) will be written as

$$(5') \quad \begin{aligned} & r^{n+1}\sin\theta\cdot P(\theta)\cdot\sum_{m=1} m r^{m-1} M_m(\theta) \\ & + (-1 + r^n\cos\theta\cdot P(\theta)) \\ & \times \sum_{m=1} r^m \frac{dM_m(\theta)}{d\theta} \\ & + r^n\Big\{(n+1)\sin\theta\cdot P(\theta) + \cos\theta\cdot \\ & \cdot \times\frac{dP(\theta)}{d\theta}\Big\}\cdot\sum_{m=0} r^m M_m(\theta) = 0. \end{aligned}$$

where $P(\theta) = P(\cos\theta, \sin\theta)$, $M_0(\theta) = 1$ and $M_k(\theta) = M_k(\cos\theta, \sin\theta)$ for $k \geqq 1$.

Equating to zero the coefficients of successive powers of r on the left-hand side of the equation (5'), we obtain following system of equations:

$$(6) \left\{ \begin{aligned} & M_m(\theta) = 0 \quad \text{for} \quad m \leqq n-1, \\ & \frac{dM_n(\theta)}{d\theta} = \frac{d}{d\theta}(\cos\theta\cdot P(\theta)) \\ & \qquad + (n+2)\sin\theta\cdot P(\theta), \\ & M_m(\theta) = 0 \quad \text{for} \quad n+1 \leqq m \leqq 2n-1, \\ & \frac{dM_{2n}(\theta)}{d\theta} = \frac{d}{d\theta}(M_n(\theta)\cdot\cos\theta\cdot P(\theta)) \\ & \qquad + (2n+2)M_n(\theta)\sin\theta\cdot P(\theta), \\ & \cdots\cdots\cdots\cdots \\ & \frac{dM_{kn}(\theta)}{d\theta} = \frac{d}{d\theta}(M_{(k-1)n}(\theta)\cos\theta\cdot P(\theta)) \\ & \qquad + (kn+2)M_{(k-1)n}(\theta)\sin\theta\cdot P(\theta), \\ & \cdots\cdots\cdots\cdots \end{aligned} \right.$$

$M_k(\theta)$ are determined successively from these equations. For $x=y=0$ to be a center, $M_k(\theta)$ should be free from secular terms. Whence we get

$$
(7)\quad \begin{cases} \int_0^{2\pi} \sin\theta \cdot P(\theta)\, d\theta = 0, \\ \int_0^{2\pi} M_n(\theta) \sin\theta \cdot P(\theta)\, d\theta = 0, \\ \cdots\cdots \\ \int_0^{2\pi} M_{kn}(\theta) \sin\theta \cdot P(\theta)\, d\theta = 0, \\ \cdots\cdots \end{cases}
$$

Now we put

$$
P(\theta) = \sum_{m=-\infty}^{\infty} a_m e^{im\theta},
$$

$$
\alpha_m = a_{m-1} + a_{m+1},
$$

$$
\beta_m = a_{m-1} - a_{m+1},
$$

$$
m = 0, \pm 1, \pm 2, \cdots .
$$

Obviously we have

$$
a_m = 0 \quad \text{for } m \geqq n+2 \text{ or } m \leqq -n-2
$$

$$
a_{2k} = 0, \quad k = 0, \pm 1, \pm 2, \cdots \quad \text{if } n \text{ is even},
$$

$$
a_{2k+1} = 0, \quad k = 0, \pm 1, \pm 2, \cdots \quad \text{if } n \text{ is odd},
$$

and

$$
(8)\quad \begin{cases} \cos\theta \cdot P(\theta) = \frac{1}{2} \sum_{-\infty}^{\infty} \alpha_m e^{im\theta}, \\ \sin\theta \cdot P(\theta) = \frac{1}{2i} \sum_{-\infty}^{\infty} \beta_m e^{im\theta}. \end{cases}
$$

If conditions (7) are all satisfied, $M_{kn}(\theta)$ are free from secular terms. Therefore we can write

$$
(9)\quad M_{kn}(\theta) = \sum_s C_{kn,s}\, e^{is\theta}, \quad k = 1, 2, \cdots .
$$

Then, from (8), the condition

$$
\int_0^{2\pi} M_{kn}(\theta) \sin\theta \cdot P(\theta)\, d\theta = 0
$$

is equivalent to the relation

$$
(10)\quad \sum_s \beta_{-s}\, C_{kn,s} = 0 .
$$

As $M_{(k+1)n}(\theta)$ is determined from the equation

$$
\frac{d M_{(k+1)n}(\theta)}{d\theta} = \frac{d}{d\theta}\left(M_{kn}(\theta) \cos\theta \cdot P(\theta) \right) + \{ (k+1)n + 2 \} M_{kn}(\theta) \sin\theta \cdot P(\theta),
$$

we have, from (8), (9) and (10),

$$
(11)\quad \begin{cases} M_{(k+1)n}(\theta) = \sum_s C_{(k+1)n,s}\, e^{is\theta}, \\ C_{(k+1)n,s} = \sum_{m=-\infty}^{\infty} \left(\frac{1}{2} \alpha_{s-m} - \frac{(k+1)n+2}{2s} \beta_{s-m} \right) C_{kn,m}, \text{ for } s \neq 0, \\ C_{(k+1)n,0} = \sum_{m=-\infty}^{\infty} \frac{1}{2} \alpha_{-m} C_{kn,m} . \end{cases}
$$

Using (10) and (11) for $k = 0, 1, 2, \cdots$ successively, conditions (7) imposed on the coefficients of $P(\theta)$ can be written down explicitly as follows:

$$
(12,1)\quad \beta_0 = 0,
$$

$$
(12,2)\quad \sum_{m_1+m_2=0} \left(\alpha_{m_1} - \frac{n+2}{m_1} \beta_{m_1} \right) \beta_{m_2} = 0,
$$

$$
(12,3)\quad \sum_{m_1+m_2+m_3=0} \left(\alpha_{m_1} - \frac{2n+2}{m_1+m_2} \beta_{m_1} \right) \times \left(\alpha_{m_2} - \frac{n+2}{m_2} \beta_{m_2} \right) \beta_{m_3} = 0,
$$

$$
\cdots\cdots\cdots
$$

$$
(12,k)\quad \sum_{m_1+m_2+\cdots+m_k=0} \left(\alpha_{m_1} - \frac{(k-1)n+2}{m_1+m_2+\cdots+m_{k-1}} \beta_{m_1} \right) \times \left(\alpha_{m_2} - \frac{(k-2)n+2}{m_2+\cdots+m_{k-1}} \beta_{m_2} \right) \times \cdots \cdots \times \left(\alpha_{m_{k-1}} - \frac{n+2}{m_{k-1}} \beta_{m_{k-1}} \right) \beta_{m_k} = 0,
$$

$$
\cdots\cdots
$$

where we always adopt the convention that every term with vanishing denominator should be identified with zero.

Thus we have

Proposition 3. When $P(x,y)$ is a homogeneous polynomial of degree $n+1$ $(n \geq 1)$, $x=y=0$ is a center if and only if the coefficients of $P(x,y)$ satisfy the conditions (12,1), (12,2), ... simultaneously.

5. Now we apply the conditions just obtained to several simple cases. In what follows, we always adopt the same notations as in the preceding section.

Proposition 4. When

$$P(x,y) = Ax^2 + Bxy + Cy^2$$

$x=y=0$ is a center if and only if

$$B(A+C) = 0.$$

Proof. 1) Necessity: In this case

$$n = 1,$$

and

$$a_2 = \bar{a}_{-2} = \frac{1}{4}\{(A-C) - iB\},$$

$$a_0 = \frac{1}{2}(A+C),$$

other a_m being all zero. Therefore condition (12,1) is an identity. From (12,2) we obtain the condition

$$a_0(a_{-2} - a_2) = 0,$$

which can be rewritten as

$$B(A+C) = 0.$$

2) Sufficiency: If $B=0$, $x=y=0$ is a center according to Theorem 3.

If $A+C=0$, we can write

$$P(x,y) = Ax^2 + Bxy + Cy^2$$

$$= A(x+\omega y)(x - \frac{1}{\omega}y),$$

where

$$\omega = \frac{1}{2}\left(-\frac{B}{A} + \sqrt{\frac{B^2}{A^2}+4}\right).$$

Put

$$x + \omega y = u,$$

$$x - \frac{1}{\omega}y = v.$$

Then the system (1) will be written as

$$\begin{cases} \dfrac{du}{dt} = -\omega v + A\omega uv, \\ \dfrac{dv}{dt} = \dfrac{u}{\omega} - \dfrac{A}{\omega}uv. \end{cases}$$

This system can easily be solved by quadrature, and has an integral

$$(1-Au)(1-Av)^{\omega^2} e^{Au+\omega^2 Av} = K = \text{const.}$$

Therefore we have

$$(1-Ax-A\omega y)(1-Ax+\frac{A}{\omega}y)^{\omega^2} e^{A(1+\omega^2)x} = K.$$

Since the left-hand side is homomorphic at $x=y=0$, we can expand it in power series of x and y. Then we get

$$x^2+y^2+\cdots = \frac{2(1-K)}{A^2(1+\omega^2)} = \text{const.}$$

This shows that $x=y=0$ is a center.

Proposition 5. When

$$P(x,y) = (x^2+y^2)^N (Ax^2+Bxy+Cy^2),$$
$$N > 1,$$

$x=y=0$ is a center if and only if one of the following conditions is satisfied:

(1) $B = 0$;

(2) $A = C = 0$.

Proof. 1) Necessity: In this case

$$n = 2N+1,$$

and the coefficients a_m are same as in Proposition 4. Therefore every $(12,k)$ with an odd k is an identity. From (12,2) we obtain, as in Proposition 4,

$$a_0(a_{-2} - a_2) = 0.$$

If $a_{-2} - a_2 = 0$, we have $B = 0$.

If $a_{-2} - a_2 \neq 0$, we must have $a_0 = 0$, and

$$(13) \begin{cases} \alpha_3 = \beta_3 = \alpha_1 = -\beta_1 = a_2, \\ \alpha_{-3} = -\beta_{-3} = \alpha_{-1} = \beta_{-1} = a_{-2}. \end{cases}$$

From (12,4) and (13), we obtain

$$N(N-1)a_2 a_{-2}(a_2 + a_{-2})(a_2 - a_{-2}) = 0.$$

Since $N > 1$ and $a_2 - a_{-2} \neq 0$, we must have

$$a_2 + a_{-2} = 0.$$

Combining this relation with $a_0 = 0$, we get

$$A = C = 0.$$

2) Sufficiency: Obvious from Theorem 3 and Theorem 4.

6. Proposition 6. When

$$P(x,y) = (x^2+y^2)^N(Ax^3 + Bx^2y + Cxy^2 + Dy^3), \quad N \neq 1,$$

$x = y = 0$ is a center if and only if

$$B = D = 0.$$

Proof. 1) Necessity: In this case

$$n = 2N + 2$$

and

$$a_3 = \bar{a}_{-3} = \frac{1}{8}\{(A-C) - i(B-D)\},$$

$$a_1 = \bar{a}_{-1} = \frac{1}{8}\{(3A+C) - i(B+3D)\},$$

other a_m being all zero. From (12,1), we have

$$(14) \qquad a_1 = a_{-1}.$$

From (12,2) and (14),

$$a_1(a_3 - a_{-3}) = 0.$$

If

$$(15) \qquad a_3 - a_{-3} = 0,$$

combining (14) and (15), we obtain

$$B = D = 0.$$

If $a_3 - a_{-3} \neq 0$, we must have

$$(16) \qquad a_1 = a_{-1} = 0.$$

From (12,3) and (16), we get

$$(N-1)a_3 a_{-3}(a_3 - a_{-3}) = 0.$$

Since this relation can never hold for $N \neq 1$ and $a_3 \neq a_{-3}$, $x = y = 0$ cannot be a center.

Hence we must have

$$B = D = 0.$$

2) Sufficiency: Obvious from Theorem 3.

7. Proposition 7. When $P(\theta)$ is of the form

$$a_{-2s}e^{-2is\theta} + a_{2s}e^{2is\theta}, \quad s \geq 1,$$

$x = y = 0$ is a center if and only if

$$(1) \qquad a_{-2s} = a_{2s},$$

or

$$(2) \qquad a_{-2s} = -a_{2s},$$

save for some exceptional values of n.

Proof. 1) Necessity: In this case

$$\alpha_{-2s-1} = -\beta_{-2s-1} = \alpha_{-2s+1} = \beta_{-2s+1} = a_{-2s},$$

$$\alpha_{2s+1} = \beta_{2s+1} = \alpha_{2s-1} = -\beta_{2s-1} = a_{2s},$$

and other α_m, β_m are all zero. Consequently, the condition (12,k) is an identity for every odd k.

First we notice that the left-hand side of every (12,k) should be purely imaginary. This can easily be seen from

$$\sum\left(\alpha_{m_1}-\frac{(k-1)n+2}{m_1+\cdots+m_{k-1}}\beta_{m_1}\right)\times\cdots$$
$$\cdots\times\left(\alpha_{m_{k-1}}-\frac{n+2}{m_{k-1}}\beta_{m_{k-1}}\right)\beta_{m_k}$$
$$=\sum\left(\alpha_{-m_1}+\frac{(k-1)n+2}{m_1+\cdots+m_{k-1}}\beta_{-m_1}\right)\times\cdots$$
$$\cdots\times\left(\alpha_{-m_{k-1}}+\frac{n+2}{m_{k-1}}\beta_{-m_{k-1}}\right)\beta_{-m_k}$$
$$=-\sum\left(\bar\alpha_{m_1}-\frac{(k-1)n+2}{m_1+\cdots+m_{k-1}}\bar\beta_{m_1}\right)\times\cdots$$
$$\cdots\times\left(\bar\alpha_{m_{k-1}}-\frac{n+2}{m_{k-1}}\bar\beta_{m_{k-1}}\right)\bar\beta_{m_k}$$
$$=-\sum\overline{\left(\alpha_{m_1}-\frac{(k-1)n+2}{m_1+\cdots+m_{k-1}}\beta_{m_1}\right)\times\cdots}$$
$$\overline{\cdots\times\left(\alpha_{m_{k-1}}-\frac{n+2}{m_{k-1}}\beta_{m_{k-1}}\right)\beta_{m_k}},$$

in which we used the fact that

$$\alpha_{-m}=\bar\alpha_m,\quad \beta_{-m}=-\bar\beta_m.$$

Now we consider (12,$2p$) for $2p \leqq 4s-2$.

Each summand of the left-hand side of (12,$2p$) has the form

$$\left(\alpha_{m_1}-\frac{(k-1)n+2}{m_1+\cdots+m_{k-1}}\beta_{m_1}\right)\times\cdots$$
$$\cdots\times\left(\alpha_{m_{2p-1}}-\frac{n+2}{m_{2p-1}}\beta_{m_{2p-1}}\right)\beta_{m_{2p}}$$

with $m_1+\cdots+m_{2p}=0$, and each m_k is equal to one of the numbers $2s+1$, $2s-1$, $-2s+1$, $-2s-1$.

Suppose that, among these $2p$ numbers, $m_{i_1}, \ldots, m_{i_l}$ are positive (i.e. equal to $2s+1$ or $2s-1$). Then l cannot exceed p.

In fact, if $l=p+q$, $q\geqq 1$,

$$m_1+\cdots+m_{2p}\geqq(p+q)(2s-1)$$
$$-(p-q)(2s+1)$$
$$=4sq-2p\geqq 4s-2p\geqq 2.$$

Hence, $m_1+\cdots+m_{2p}$ cannot be zero if $l>p$. In the same way, we can show that l cannot be less than p.

Therefore, among $2p$ numbers $m_1, \cdots, m_{2p}$, there are p positive ones and p negative ones.

On the other hand,

$$\alpha_{m_k}-\frac{(2p-k)n+2}{m_k+\cdots+m_{2p-1}}\beta_{m_k}=\begin{cases}\lambda_k a_{2s}, & \text{if } m_k>0,\\ \mu_k a_{-2s}, & \text{if } m_k<0,\end{cases}$$

where λ_k, μ_k are real numbers. So we can write

$$\left(\alpha_{m_1}-\frac{(2p-1)n+2}{m_1+\cdots+m_{2p-1}}\beta_{m_1}\right)\times\cdots$$
$$\cdots\times\left(\alpha_{m_{2p-1}}-\frac{n+2}{m_{2p-1}}\beta_{m_{2p-1}}\right)\beta_{m_{2p}}$$
$$=\Lambda\cdot(a_{2s}a_{-2s})^p,$$

Λ being real.

Hence the left-hand side of each (12,$2p$), $2p\leqq 4s-2$, is a real number.

However, as we have remarked first, this should also be a purely imaginary number. Therefore the left-hand side of each (12,$2p$), $2p\leqq 4s-2$ must identically vanish.

Making use of this fact, the left-hand side of (12,$4s$) can be expressed in the following form:

$$f(n)\,(a_{2s}a_{-2s})^{2s-1}(a_{-2s}^2-a_{2s}^2),$$

$$f(n)=\sum\left(1-\frac{(4s-1)n+2}{m_1+\cdots+m_{4s-1}}\right)\times\cdots$$
$$\cdots\times\left(1-\frac{n+2}{m_{4s-1}}\right),$$

where $m_1, \cdots, m_{4s}$ are equal to either $2s+1$ or $-2s+1$, and $m_1+\cdots+m_{4s}=0$.

Since $f(n)$ is a polynomial of degree $4s-1$ in n, it can only vanish for at most $4s-1$ values of n. Therefore, if n is not equal to one of these exceptional values, we must have

$$a_{-2s} = a_{2s},$$

or

$$a_{-2s} = -a_{2s}.$$

2) Sufficiency: Obvious from Theorem 3 and Theorem 4.

(The exceptional case stated above can really occur. For example, suppose that

$$P(\theta) = a_{-2}e^{-2i\theta} + a_2 e^{2i\theta}.$$

In this case $s=1$, and the condition (12,4) becomes, as we have calculated in the proof of Proposition 5,

$$(n-1)(n-3)\, a_2 a_{-2} (a_{-2}^2 - a_2^2) = 0.$$

$n=1$ and $n=3$ correspond to exceptional cases.)

Similarly we can prove

Proposition 8. When $P(\theta)$ is of the form

$$a_{-2s-1}e^{-(2s+1)i\theta} + a_{2s+1}e^{(2s+1)i\theta}, \quad s \geqq 1,$$

$x=y=0$ is a center if and only if

$$a_{-2s-1} = a_{2s+1},$$

save for some exceptional values of n.

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