A CHARACTERIZATION OF THE MAXIMAL IDEAL IN A FACTOR, II.

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In our previous paper[1] we gave a characterization of the maximal ideal in a factor of the case (II_∞). But, as J. Dixmier[2] pointed out, the statement contains an error, so we shall correct it here. Moreover, in the previous paper, we treated only the separable Hilbert space, but the same proof remains true for any factor which contains a projection relatively smaller than the identity. So that we obtain a final form of such a characterization.

1. Let M be a factor on a (not necessarily separable) Hilbert space H. It is well-known that any factor of the finite case is simple (for example [6]).[3] Moreover, if H is separable then a factor of the case (III) is also simple. But, if M of case (III) is countably decomposable (i.e. any collection of mutually disjoint projections in M is at most countable), then any projection in M is mutually equivalent ([5]; Lemma 7.2.2.). Therefore, we know that M is simple by the same proof with [6]. However, M may be not countably decomposable. In this case, we obtain

Lemma 1. If a factor M is not countably decomposable, then M contains at least one ideal.

Proof. Let x be any element of H and $[M'x]$ be the closed linear manifold generated by (ax; $a \in M'$), then it is well-known that $[M'x]$ belongs to M and countably decomposable. So that, it is clear that $[M'x] \precneqq H$.

Let J be the set of all operators $A \in M$ such that $[R(A)]$ is contained in a countably decomposable manifold, then it is easily seen that J is a non trivial ideal in M.

(Here we denote by R(A) the range of A, and by $[R(A)]$ its closure.)

2. Let us now correct the theorem in the previous paper in the following form, which is valid in the case of (I_∞) and (III).

Theorem 1. Any factor M of the infinite case, except the countably decomposable (III) case, has the unique maximal ideal. This ideal consists of such operators $A \in M$ that every spectral projection $E(\Delta)$, for $|A| = (A^*A)^{1/2}$, contained in $R(|A|)$ has a relative dimension smaller than that of the whole space (in the sense that $E(\Delta) \precneqq H$).

In particular, if H is separable, this condition is equivalent to the condition that every spectral projection $E(\Delta)$ contained in $R(|A|)$ has a finite relative dimension, and moreover, in the case of the total operator ring, A is a completely continuous operator.

Proof. First we remark that for projections P, $Q \in M$, if $P, Q \precneqq H$ then $P \vee Q \precneqq H$. This fact is proved by the similar manner to the proof of the fact that if P, Q are finite then $P \vee Q$ is finite. Now let J be the set of all $A \in M$ such that $[R(A)] \precneqq H$, then J is an ideal in M as is proved in the previous paper. Let $\bar{J}$ be its uniform closure, then it is well-known that $\bar{J}$ is also an ideal in M. On the other hand, any operator $A \in M$ such as described in the theorem is contained in $\bar{J}$. In fact, let $A = U|A|$ be the polar decomposition of A ($|A| = (A^*A)^{1/2}$ is positive-definite, U is a partial isometry in M from $[R(|A|)]$ to $[R(A)]$), then $|A|$ is a uniform limit of the operators of type $\sum \lambda_i E(\Delta_i)$, where $E(\cdot)$ is a spectral measure for $|A|$. Since $|A|$ is positive definite, we can choose the intervals Δ_i as not contain the zero. So that $E(\Delta_i) \subset R(|A|)$, hence $\sum \lambda_i E(\Delta_i) \in J$. Thus we obtain that $|A| \in \bar{J}$, and $A \in \bar{J}$. This implies that there exists an ideal containing all those operators stated in the theorem.

However, if A is an element of an ideal K, then A has the property stated in the theorem. This fact follows from the second part of the proof in the previous paper with slight modifications. That is, if $A \in K$ and let $A = U|A|$ be the polar decomposition, then $|A| = U^* A \in K$. Now suppose that $|A|$ has a spectral projection $E(\Delta) \subset R(|A|)$ such that $E(\Delta) \sim H$. It is clear that $E(\Delta)|A|E(\Delta)$ is the one-to-one transformation on the $E(\Delta)H$ onto itself. Let X be the partial isometry $H \sim E(\Delta)$. Then $C = X^*|A|X \in K$ is a one-to-one transformation on the whole space H onto itself, so that there exists the inverse $C^{-1} \in M$. This implies $I = C^{-1} C \in K$, which contradicts the assumption that K is an ideal. Thus we see that the set of all those operators stated in the theorem is $\bar{J}$ and it is the maximal ideal in M.

The case of the total operator ring on a separable Hilbert space is reduced to the result of Calkin.

Corollary. Any factor M of the infinite case, except the countably decomposable (III) case, has the unique maximal ideal, which coincides with the uniform closure of the set of all operators A such that $[R(A)] \precneqq H$.

3. By Lemma 1 and the above theorem, we obtain

Theorem 2. A ring of operators is simple if and only if it is a countably decomposable factor of the finite case, or of case (III).

(It is well-known that a finite factor is countably decomposable.)

Proof. If a ring of operators is not a factor, then there exists a central projection Z in M such that $0 \lneqq Z \lneqq I$. Let $J = (A \in M;\ ZA = A)$, then it is easily seen that J is a non trivial ideal so that M is not simple.

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1) These Seminar Reports 6, (1954) p.7.

2) Mathematical Reviews, 15(1954).

3) Numbers in brackets refer to the bibliography at the end of the previous paper.

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